Trends in Mathematics
Information Center for Mathematical Sciences
Volume 9, Number 1, June, 2006, Pages 31–34

SURVEY OF GROSS'S CONJECTURE AND ITS REFINEMENTS

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Abstract. Let K/k be an abelian extension of global fields (i.e. number fields or function fields of curves defined over finite field) with Galois group G. In [8], B. Gross has conjectured a congruence relation which relates the Stickelberger element in $\mathbb{Z}[G]$ with the class number of k and the generalized regulator. The relation may be viewed as a generalization of the classical class number formula which describes the leading term of the Taylor expansion of $\zeta_k(s)$ at $s = 0$ in terms of the class number and the regulator of k. This conjecture has been verified to be true in many important special cases but yet it remains to be proved in general.

Tate has discovered that both sides of the conjecture vanishes under mild restriction when G is a cyclic group of prime power order, and he conjectured that a finer version of the congruence relation would hold. This idea has been developed independently by Burns [5] and Aoki, Lee and Tan [4] to cover the case when G is an arbitrary finite abelian group with no restriction.

In this paper, we review the various conjectures mentioned above and discuss their relationships.

1. Gross's conjecture

Let K/k be an abelian extension of global fields with Galois group G. Let S be a finite non-empty set of places of k which contains all archimedean places and all places ramified in K, and let T be a finite non-empty set of places of k which is disjoint from S. We choose T so that $U_{S,T}$, the group of S-units in k which are congruent to 1 (mod v) for all $v \in T$, is a free abelian group of rank $n = |S| - 1$.

For a complex character $\chi \in \widehat{G} = \mathrm{Hom}(G, \mathbb{C}^*)$, the associated modified L-function is defined as

$$L_{S,T}(\chi, s) = \prod_{v \in T} (1 - \chi(g_v)\boldsymbol{N}v^{1-s}) \prod_{v \notin S} (1 - \chi(g_v)\boldsymbol{N}v^{-s})^{-1},$$

where $g_v \in G$ is the Frobenius element for v. The Stickelberger element $\theta_G \in \mathbb{C}[G]$ is the unique element that satisfies

$$\chi(\theta_G) = L_{S,T}(\chi, 0)$$

for all $\chi \in \widehat{G}$. In fact, $\theta_G \in \mathbb{Z}[G]$ which is a deep theorem of Deligne-Ribet(cf. [7]).

2000 *Mathematics Subject Classification.* 11R42.
Key words and phrases. Stickelberger element, Abelian L-functions, class number formula.

Let I_G be the augmentation ideal of $\mathbb{Z}[G]$, which is defined via the following short exact sequence

$$0 \to I_G \to \mathbb{Z}[G] \to \mathbb{Z} \to 0,$$

where the map from $\mathbb{Z}[G]$ to $\mathbb{Z}$ is given by trivial character of G extended to $\mathbb{Z}[G]$ by linearity. The sequence may be compared with

$$0 \to (s) \to \mathbb{C}[[s]] \to \mathbb{C} \to 0,$$

where the map from $\mathbb{C}[[s]]$ to $\mathbb{C}$ is, of course, the evaluation map at $s = 0$.

Choose an ordered basis $\{u_1, \ldots, u_n\}$ of $U_{S,T}$. Pick a place $v_0 \in S$, and for each $v_i \in S \setminus \{v_0\}$, we let $f_i : k^* \to G$ denote the homomorphism induced from local reciprocity map for v_i. We set

$$R_G := \det_{1\le i,j\le n}(f_i(u_j) - 1).$$

Gross has conjectured (cf. [8])

Conjecture 1.

$$\theta_G \equiv \pm h_{S,T} R_G \pmod{I_G^{n+1}}.$$

Here, the integer $h_{S,T}$ is defined by

$$h_{S,T} = h_S \cdot \frac{\prod_{v\in T}(Nv - 1)}{(U_S : U_{S,T})}$$

where h_S is the S-class number of k and U_S is the set of S-units. The $\pm$ sign is determined by the (S, T)-version of the analytic class number formula.

Note that Conjecture 1 implies the following weaker congruence relation

$$\theta_G \in I_G^n$$

which may be viewed as the "order of vanishing" part of the conjecture. Loosely speaking, Conjecture 1 predicts the leading term of the Taylor expansion of equivariant L-function at the trivial character.

It is known (cf. [12]) that if $G = H_1 \oplus H_2$ where H_1, H_2 are finite abelian groups of coprime orders, then $I_G^n/I_G^{n+1} \cong I_{H_1}^n/I_{H_1}^{n+1} \oplus I_{H_2}^n/I_{H_2}^{n+1}$ for all $n \ge 1$. Hence we may assume that G is an l-group for some prime number l. Also, when S contains a place that splits completely in K, then Conjecture 1 holds for trivial reason – both sides of the conjecture are 0 in that case. This enables us to assume, when k is a number field, that $l = 2$, k is totally real and that K is totally imaginary.

Several people have proved that Conjecture 1 holds in various special cases ([1, 5, 6, 8, 9, 11, 13, 14, 16, 17]). In number field case, The first author has proved the conjecture when $k = \mathbb{Q}$ in [1], and Tate's result in the next section shows that when $k \neq \mathbb{Q}$ and G is cyclic, then the conjecture holds. When k has characteristic $p \neq 0$, Burns proved the conjecture when $l \neq p$ in [5] and Tan proved the conjecture when $l = p$ in [14], so the conjecture holds in full generality.

2. Tate's refinement

Suppose G is cyclic of order l^m. If S contains a place which splits completely in K, then Conjecture 1 holds as mentioned before. Tate has considered the next case, namely when S contains a place whose decomposition group has order l (Tate calls such a place an "almost splitting place"). Note that the archimedean place either splits completely or almost splits.

For each $v \in S$, let G_v denote the decomposition group of v in G. We fix the ordering of the elements of $S = \{v_0, v_1, \dots, v_n\}$ so that

$$G_{v_0} \supseteq G_{v_1} \supseteq \cdots \supseteq G_{v_n}.$$

Let $l^{m_i} = (G : G_{v_i})$ for $i = 0, \dots, n$. Thus $m_0 \leq m_1 \leq \ldots \leq m_n = m - 1$. Let $N = l^{m_0} + \cdots + l^{m_{n-1}}$.

Tate has shown that both θ_G and $h_{S,T}R_G$ belong to I_G^N and conjectured (cf. [16])

Conjecture 2. *Suppose $m_0 = 0$. Then*

$$\theta_G \equiv \pm h_{S,T} R_G \pmod{I_G^{N+1}}.$$

Clearly $N \geq n$, and $N = n$ if and only if $m_{n-1} = 0$. Therefore Conjecture 2 is stronger than Conjecture 1 in general. When $m_0 \neq 0$ (i.e. S has no place with full decomposition group), there are cases when Conjecture 2 does not hold, as shown in [10].

The first author [2] has proved that Conjecture 2 holds when k is a number field, and that in general Conjecture 2 holds up to a unit modulo $|G|$. Tan [15] proved Conjecture 2 when k is a function field and $l = \mathrm{char}(k)$. A result of Burns [5] implies that Conjecture 2 holds if k is a function field and $l \neq \mathrm{char}(k)$.

3. Generalization of Tate's refinement

Now let G be an arbitrary finite abelian group. For each subgroup H of G, define $I_H = \mathrm{Ker}(\mathbb{Z}[G] \to \mathbb{Z}[G/H])$. Note that when $H = G$, this definition coincides with the augmentation ideal of $\mathbb{Z}[G]$.

Conjecture 3. *For an arbitrary enumeration $v_0, v_1, \dots, v_n$ of the places of S,*

$$\theta_G \equiv \pm h_{S,T} R_G \pmod{I_G \prod_{i=1}^{n} I_{G_{v_i}}}.$$

This conjecture is a natural generalization of Conjecture 2 in the following sense. Let G be a cyclic group of order l^m, and assume that S contains an almost splitting place. If v_0 has full decomposition group, Conjecture 3 is equivalent to Conjecture 2. On the other hand, if v_0 does not have full decomposition group, it follows from Tate's result that both θ_G and $h_{S,T}R_G$ are in $I_G \prod_{i=1}^{n} I_{G_{v_i}}$, hence Conjecture 3 still holds.

In [3], the first author proved Conjecture 3 when G is an elementary abelian 2-group. When k is a function field, work of Burns [5] proves Conjecture 3 if $l \neq \mathrm{char}(k)$, and the joint work of the authors with Tan [4] proves Conjecture 3 if $l = \mathrm{char}(k)$. It is worthwhile to note that Burns's work is very general — his "leading term conjecture" provides a unifying approach to Conjecture 3 together with conjectures of Stark, Rubin, Darmon and and their refinements. When $\mathrm{char}(k) \nmid [K : k]$, the leading term conjecture is proved by Burns and therefore many conjectures including Conjecture 3 is settled in that case.

Acknowledgements The authors would like to thank David Burns, Benedict Gross, Anthony Hayward, Cristian Popescu, Michael Reid, Karl Rubin, Ki-Seng Tan and John Tate for many insightful comments and warm encouragements over the years, and the organizers of Korea-Japan number theory seminar for the hospitality.

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