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PUNCTURED DISTRIBUTIONS OF GLOBAL FUNCTION FIELDS

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1. Introduction

The theory of distributions plays an important role in the study of number theory. Distributions on general global fields are introduced by Yin([Y2]). Yin proved that the universal ordinary distributions on global function fields are torsion free, and conjectured that this is true for all global fields. Later Belliard and Oukhaba [BO] gave detailed information about the torsion subgroups in the case of number fields, and constructed a counterexample of Yin's conjecture on imaginary quadratic number fields. The determination of the structures of various kinds of universal distributions on a global field k will describe the arithmetic of the ray class fields of k. The ranks of the level groups of various kinds of universal distributions are given by Yin([Y2],[Y3]). Anderson constructed a resolution of the universal ordinary distribution and then used the double complex method to compute the sign cohomology groups of the universal ordinary distributions on the rational number field and global function fields([A], Appendix of [O]). Those sign cohomology groups are just the torsion subgroups of the even or odd universal ordinary distributions in the case of the rational number field and global function fields ([BGY] Proposition 2.4). The structures of the torsion subgroups of the level groups of these three distributions were known in the case of the rational number field and the rational function fields ([B],[Sc]).

We first introduce two specific ordinary punctured distributions on a global function field, one is even and the other odd. We use them to obtain some criteria for elements in the universal ordinary punctured even or odd distributions to be torsion, which are analogous to those in [E] and [Sc] in the case of the rational number field. These criteria are different from those in [Y3]. Using these criteria we generalize the methods in [B] and [Sc] to compute the torsion subgroups of the level groups of these three universal ordinary punctured distributions on global function fields. We omit most of the details. Interested reader can consult [B2] for details.

Notations

- $\mathbb{F}_q$:= the finite field with q elements of characteristic p.

We assume $q > 2$.
- k := a global function field with the field of constants $\mathbb{F}_q$.
- ∞ := a fixed place of k with degree 1.
- $\mathbb{A}$:= the ring of functions of k which are regular away from ∞.
- k_∞ := the completion of k at ∞.
- $\mathfrak{e}$:= the unit ideal of $\mathbb{A}$.
- $K_\mathfrak{e}$:= the Hilbert class field of (k, ∞).

For each nonzero integral ideal $\mathfrak{m}$ of $\mathbb{A}$,

- $K_{\mathfrak{m}}$:= the cyclotomic function field of the triple $(k, \mathfrak{m}, \infty)$ with conductor $\mathfrak{m}$.
- $G_{\mathfrak{m}} := \mathrm{Gal}(K_{\mathfrak{m}}/k)$ and $H_{\mathfrak{m}} := \mathrm{Gal}(K_{\mathfrak{m}}/K_{\mathfrak{e}})$.
- $N_{\mathfrak{m}}$:= the subgroup of $G_{\mathfrak{e}}$ generated by $\tau_{\mathfrak{p}}$'s for all prime divisors $\mathfrak{p}$ of $\mathfrak{m}$, the Artin automorphism associated to $\mathfrak{p}$.
- $|A|$:= the cardinality of a set A.
- $h := |G_{\mathfrak{e}}|$ = the class number of k.
- $\phi(\mathfrak{m}) := |(\mathbb{A}/\mathfrak{m})^*|$ = the number of units in $\mathbb{A}/\mathfrak{m}$.
- J := the inertia group at ∞ in $H_{\mathfrak{m}}$, which we call the sign group.

Note that J is naturally isomorphic to $\mathbb{F}_q^*$.
- γ := a fixed generator of J.

For a subset A of $G_{\mathfrak{m}}$,
- $s(A) := \sum_{\sigma \in A} \sigma \in \mathbb{Z}[G_{\mathfrak{m}}]$.

2. The universal ordinary punctured distribution

We briefly recall some basic facts about the universal ordinary punctured distributions on global function fields from [Y2]. The notations are changed slightly from those of [Y2].

Fix a sign function $sgn : k_\infty \longrightarrow \mathbb{F}_\infty = \mathbb{F}_q$ with $sgn(0) = 0$, where $\mathbb{F}_\infty$ is the residue field at ∞. We call $x \in k$ *positive* or *monic* if $sgn(x) = 1$. Let T be the set of all nonzero fractional ideals of $\mathbb{A}$ and $T_{\mathfrak{e}}$ the set of all nonzero integral ideals of $\mathbb{A}$. For $\mathfrak{m} \in T_{\mathfrak{e}}$, let $T_{\mathfrak{m}}$ be the set of all nonzero fractional ideals which can be written as $\mathfrak{a}\mathfrak{m}^{-1}$ for some $\mathfrak{a} \in T_{\mathfrak{e}}$, and let $T'_{\mathfrak{m}}$ be the set of all nonzero fractional ideals which can be written as $\mathfrak{a}\mathfrak{m}^{-1}$ for some $\mathfrak{a} \in T_{\mathfrak{e}}$ prime to $\mathfrak{m}$. We see that T is the union of $T_{\mathfrak{m}}$ with $\mathfrak{m} \in T_{\mathfrak{e}}$, and $T_{\mathfrak{m}}$ is the disjoint union of $T'_{\mathfrak{n}}$ for all ideals $\mathfrak{n} \mid \mathfrak{m}$.

We define an equivalence relation $\sim$ on T: for $\mathfrak{u}$ and $\mathfrak{v}$ in T, $\mathfrak{u} \sim \mathfrak{v}$ if and only if $\mathfrak{v} = (1+x)\mathfrak{u}$ for some $x \in \mathfrak{u}^{-1}$ with $1+x$ positive. It is known that if $\mathfrak{u} \sim \mathfrak{v}$ and $\mathfrak{u} \in T'_{\mathfrak{m}}$, then $\mathfrak{v} \in T'_{\mathfrak{m}}$ ([Y2], Lemma 1.2).

From now on we fix a non-unit integral ideal $\mathfrak{m}$. Let

$$\bar{T} := T/\sim, \quad \bar{T}_{\mathfrak{m}} := T_{\mathfrak{m}}/\sim, \quad \bar{T}'_{\mathfrak{m}} := T'_{\mathfrak{m}}/\sim,$$

and

$$T^* = T \setminus T_{\mathfrak{e}}, \quad T^*_{\mathfrak{m}} = T_{\mathfrak{m}} \setminus T_{\mathfrak{e}}, \quad \bar{T}^* = T^*/\sim, \quad \bar{T}^*_{\mathfrak{m}} = T^*_{\mathfrak{m}}/\sim .$$

Note that $\bar{T}_{(m)} \simeq \frac{1}{m}\mathbb{Z}/\mathbb{Z}$, when $k = \mathbb{Q}$ and $m \in \mathbb{Z}$, and that $\bar{T}_{(m)} \simeq \frac{1}{m}\mathbb{F}_q[T]/\mathbb{F}_q[T]$, when $k = \mathbb{F}_q(T)$ and $m \in \mathbb{F}_q[T]$.

For $\mathfrak{f} \in T$ we denote by $\{\mathfrak{f}\}$ the image of $\mathfrak{f}$ in $\bar{T}$. $T_{\mathfrak{e}}$ acts on $\bar{T}$ by $\mathfrak{n}\{\mathfrak{f}\} = \{\mathfrak{n}\mathfrak{f}\}$. $G_{\mathfrak{m}}$ acts on $\bar{T}_{\mathfrak{m}}$ by $\sigma\{\mathfrak{f}\} = \{\mathfrak{a}\mathfrak{f}\}$ where $\sigma = \sigma_{\mathfrak{a}}$ is the element in $G_{\mathfrak{m}}$ associated to the ideal $\mathfrak{a}$ with $(\mathfrak{a}, \mathfrak{m}) = \mathfrak{e}$ via the Artin map. Since $\bar{T} = \cup_{\mathfrak{m}} \bar{T}_{\mathfrak{m}}$, $\bar{T}$ becomes a G-set, where $G = \varprojlim G_{\mathfrak{m}}$. In this way every $a \in \bar{T}'_{\mathfrak{n}}$ can be uniquely written as $\sigma\{\mathfrak{n}^{-1}\}$ for some $\sigma \in G_{\mathfrak{n}}$. This σ will be denoted by σ_a.

For $a \in \bar{T}$, there exists a unique integral ideal $\mathfrak{d}_a$ such that $a \in \bar{T}'_{\mathfrak{d}_a}$. Then one may express $a \in \bar{T}$ uniquely as $\sigma_a\{\mathfrak{d}_a^{-1}\}$ for $\sigma_a \in G_{\mathfrak{d}_a}$, which can be thought as the analogue of $\frac{r}{s}$ with $r, s \in \mathbb{Z}$, $(r, s) = 1$ in the case of rational number field. In this sense, one may call σ_a the *'numerator'* of a and $\mathfrak{d}_a$ the *'denominator'* of a.

Let $\mathcal{A}$ be the free abelian group generated by the symbols $[a]$ for $a \in \bar{T}^*$, and $\mathcal{A}_{\mathfrak{m}}$ the subgroup of $\mathcal{A}$ generated by the symbols $[a]$ for $a \in \bar{T}^*_{\mathfrak{m}}$. Then $G_{\mathfrak{m}}$ acts on $\mathcal{A}_{\mathfrak{m}}$ by $\sigma[a] = [\sigma a]$, and hence G acts on $\mathcal{A}$.

Let $\mathcal{U}$ be the subgroup of $\mathcal{A}$ generated by the elements of the form

$$E(\mathfrak{n}, a) = [a] - \sum_{\mathfrak{n}b=a} [b], \quad \text{for} \quad \mathfrak{n} \in T_{\mathfrak{e}}, \quad a, b \in \bar{T}^*,$$

and let $\mathcal{U}_{\mathfrak{m}} = \mathcal{U} \cap \mathcal{A}_{\mathfrak{m}}$. Note that, for $a \in \bar{T}'_{\mathfrak{d}}$ and a prime ideal $\mathfrak{p}$, we have

$$E(\mathfrak{p}, a) = \begin{cases} [a] - \mathrm{Fr}_{\mathfrak{p}}^{-1}[a] - \sum_{\sigma \in G(K_{\mathfrak{d}\mathfrak{p}}/K_{\mathfrak{d}})} \sigma[\mathfrak{p}^{-1}a], & \text{if } \mathfrak{p} \nmid \mathfrak{d} \\ [a] - \sum_{\sigma \in G(K_{\mathfrak{d}\mathfrak{p}}/K_{\mathfrak{d}})} \sigma[\mathfrak{p}^{-1}a], & \text{if } \mathfrak{p} \mid \mathfrak{d}. \end{cases}$$

Here $\mathrm{Fr}_{\mathfrak{p}}$ is the Frobenius map at $\mathfrak{p}$ in $G(K_{\mathfrak{d}}/k)$ and $\mathfrak{p}^{-1}a = \tilde{\sigma_a}\{\mathfrak{p}^{-1}\mathfrak{d}_a^{-1}\}$, where $\tilde{\sigma_a}$ is any extension of σ_a to $K_{\mathfrak{p}\mathfrak{d}_a}$. The quotient groups $\mathbb{U} = \mathcal{A}/\mathcal{U}$ and $\mathbb{U}_{\mathfrak{m}} = \mathcal{A}_{\mathfrak{m}}/\mathcal{U}_{\mathfrak{m}}$ are called the *universal ordinary punctured distribution* and the *universal ordinary punctured distribution of level* $\mathfrak{m}$, respectively.

Let $\mathcal{R}^+$ be the subgroup of $\mathcal{A}$ generated by $\mathcal{U}$ and the elements of the form $[a] - \gamma[a]$ for $a \in \bar{T}^*$, and $\mathcal{R}^+_{\mathfrak{m}} = \mathcal{R}^+ \cap \mathcal{A}_{\mathfrak{m}}$. The quotient groups $\mathbb{U}^+ = \mathcal{A}/\mathcal{R}^+$ and $\mathbb{U}^+_{\mathfrak{m}} = \mathcal{A}_{\mathfrak{m}}/\mathcal{R}^+_{\mathfrak{m}}$ are called the *universal ordinary punctured even distribution* and the *universal ordinary punctured even distribution of level* $\mathfrak{m}$, respectively.

Let $\mathcal{R}^-$ be the subgroup of $\mathcal{A}$ generated by $\mathcal{U}$ and the elements $s(J)[a]$ for $a \in \bar{T}^*$, and $\mathcal{R}^-_{\mathfrak{m}} = \mathcal{R}^- \cap \mathcal{A}_{\mathfrak{m}}$. The quotient groups $\mathbb{U}^- = \mathcal{A}/\mathcal{R}^-$ and $\mathbb{U}^-_{\mathfrak{m}} = \mathcal{A}_{\mathfrak{m}}/\mathcal{R}^-_{\mathfrak{m}}$ are called the *universal ordinary punctured odd distribution* and the *universal ordinary punctured odd distribution of level* $\mathfrak{m}$, respectively. The following result is shown in [Y3], §4.

Theorem 1.1. *Let r be the number of distinct prime divisors of $\mathfrak{m}$ and $t := |G_{\mathfrak{e}}/N_{\mathfrak{m}}|$. Then*
a) $\operatorname{rank} \mathbb{U}_{\mathfrak{m}} = h\phi(\mathfrak{m}) + r - t$.
b) $\operatorname{rank} \mathbb{U}^+_{\mathfrak{m}} = h\frac{\phi(\mathfrak{m})}{q-1} + r - t$.
c) $\operatorname{rank} \mathbb{U}^-_{\mathfrak{m}} = \frac{q-2}{q-1}h\phi(\mathfrak{m})$.
d) $\mathbb{U}_{\mathfrak{m}}$ *has no p-torsion for $p \nmid h$.* □

An ordinary punctured distribution (even, odd, respectively) with values in an abelian group V is defined to be a homomorphism from $\mathbb{U}$ ($\mathbb{U}^+$, $\mathbb{U}^-$, respectively) to V. Distributions with level structures are similarly defined. In the following we introduce two important punctured distributions.

For a nonzero fractional ideal $\mathfrak{u}$ of $\mathbb{A}$, let $\xi(\mathfrak{u})$ be the ξ-invariant associated to $\mathfrak{u}$, which is determined up to the roots of unity. We fix $\xi(\mathfrak{u})$ as in [H1]. Let $e_{\mathfrak{u}}$ be the Drinfeld lattice function associated to the lattice $\mathfrak{u}$. For a non-unit integral divisor $\mathfrak{n}$ of $\mathfrak{m}$, let $\lambda_{\mathfrak{n}} = \xi(\mathfrak{n})e_{\mathfrak{n}}(1)$, a primitive $\mathfrak{n}$-torsion point of an appropriate sign-normalized rank 1 Drinfeld module. For $a \in \bar{T}'_{\mathfrak{n}}$, we define λ_a to be $\lambda_{\mathfrak{n}}^{\sigma_a} = \sigma_a(\lambda_{\mathfrak{n}})$, i.e., $\lambda_a = \sigma_a(\lambda_{\mathfrak{d}_a})$.

Let $P_{\mathfrak{m}}$ be the group of cyclotomic numbers of level $\mathfrak{m}$, that is, the $G_{\mathfrak{m}}$-submodule of $K^*_{\mathfrak{m}}$ generated by $\lambda_{\mathfrak{n}}$ for all non-unit integral divisors $\mathfrak{n}$ of $\mathfrak{m}$. Define a homomorphism

$$\Phi : \mathcal{A}_{\mathfrak{m}} \longrightarrow P_{\mathfrak{m}}/\mathbb{F}_q^*$$

by $\Phi([a]) = \lambda_a$. Let $\mathcal{R}'_{\mathfrak{m}}$ be the kernel of Φ. It is shown in [BGY],§7 that Φ is an even punctured distribution. Thus $\mathcal{R}^+_{\mathfrak{m}} \subset \mathcal{R}'_{\mathfrak{m}}$.

Let $f^-([a])$ be the first nonzero coefficient in the Taylor expansion of the partial zeta function $\zeta_{\mathfrak{n}}^-(s,\mathfrak{a})$ of the narrow ray class of $\mathfrak{a}$ modulo $\mathfrak{n}$ at $s=0$, where $a=\{\mathfrak{a}\mathfrak{n}^{-1}\}$ with $(\mathfrak{a},\mathfrak{n})=\mathfrak{e}$. For $a\in\bar{T}_{\mathfrak{n}}^*$, define the *Stickelberger distribution* $\Theta_{\mathfrak{n}}$ by

$$\Theta_{\mathfrak{n}}([a])=\sum_{\sigma\in G_{\mathfrak{n}}} f^-(\sigma[a])\sigma^{-1},$$

which is an odd distribution ([BGY], §3). Let $\mathcal{R}_{\mathfrak{m}}$ be the kernel of $\Theta_{\mathfrak{m}}$. Then $\mathcal{R}_{\mathfrak{m}}^-\subset\mathcal{R}_{\mathfrak{m}}$. Write $\Theta_{\mathfrak{n}}(1)=\Theta_{\mathfrak{n}}([\{\mathfrak{n}^{-1}\}])$ for simplicity.

3. Criteria for elements to lie in $\mathcal{U}_{\mathfrak{m}}$, $\mathcal{R}_{\mathfrak{m}}$ and $\mathcal{R}'_{\mathfrak{m}}$

Let v_∞ be an extension of the normalized valuation on k at ∞ to $K_{\mathfrak{m}}$. For each prime divisor $\mathfrak{p}$ of $\mathfrak{m}$, let $v_{\mathfrak{p}}$ be an extension of the normalized valuation on k at $\mathfrak{p}$ to $K_{\mathfrak{m}}$.

Recall that a character is a homomorphism $\chi: G_{\mathfrak{m}}\longrightarrow\mathbb{C}^*$. The conductor $\mathfrak{f}_\chi$ of a character χ is the smallest integral ideal $\mathfrak{n}$, where χ factors through $G_{\mathfrak{n}}$. We denote by χ_1 the trivial character. A character χ is called *even* if χ is trivial on J, and *odd*, if otherwise. χ is said to be *unramified* if $\mathfrak{f}_\chi=\mathfrak{e}$.

Let $R=\sum_{a\in\bar{T}_{\mathfrak{m}}^*} n_a[a]\in\mathcal{A}_{\mathfrak{m}}$. For a character χ and a non-unit integral ideal $\mathfrak{d}$ such that $\mathfrak{f}_\chi\mid\mathfrak{d}\mid\mathfrak{m}$, set

$$T(\chi,\mathfrak{d},R)=\sum_{a\in\bar{T}'_{\mathfrak{d}}}\chi(\sigma_a)n_a.$$

For $\chi\neq\chi_1$, set

$$Y(\chi,R)=\sum_{\mathfrak{f}_\chi|\mathfrak{d}|\mathfrak{m}}\frac{1}{\phi(\mathfrak{d})}\prod_{\mathfrak{p}|\mathfrak{d}}(1-\bar{\chi}(\sigma_{\mathfrak{p}}))T(\chi,\mathfrak{d},R),$$

where $\sigma_{\mathfrak{p}}\in G_{\mathfrak{m}}$ is the Artin automorphism at $\mathfrak{p}$. For a prime divisor $\mathfrak{p}$ of $\mathfrak{m}$, set

$$Y_{\mathfrak{p}}(R)=\sum n_a v_{\mathfrak{p}}(\lambda_a).$$

Note that

$$T(\chi,\mathfrak{d},\sigma R)=\chi(\sigma)T(\chi,\mathfrak{d},R)\quad\text{and}\quad Y(\chi,\sigma R)=\chi(\sigma)Y(\chi,R)$$

for every $\sigma\in G_{\mathfrak{m}}$.

Theorem 2.1. *$R\in\mathcal{R}'_{\mathfrak{m}}$ if and only if*

$$Y(\chi,R)=0\quad\textit{for every even character }\chi\neq\chi_1$$

and

$$Y_{\mathfrak{p}}(R)=0\quad\textit{for every prime }\mathfrak{p}\mid\mathfrak{m}.\qquad\square$$

Theorem 2.2. *$R\in\mathcal{U}_{\mathfrak{m}}$ if and only if*

$$Y(\chi,R)=0\quad\textit{for every character }\chi\neq\chi_1$$

and

$$Y_{\mathfrak{p}}(R)=0\quad\textit{for every prime }\mathfrak{p}\mid\mathfrak{m}.\qquad\square$$

Theorem 2.3. *$R\in\mathcal{R}_{\mathfrak{m}}$ if and only if $Y(\chi,R)=0$ for every odd character χ.* $\square$

4. Torsion of punctured distirbutions

Lemma 3.1. i) If $R \in \mathcal{R}'_{\mathfrak{m}}$, then $(q-1)R \in \mathcal{R}^{+}_{\mathfrak{m}}$.
ii) If $R \in \mathcal{R}_{\mathfrak{m}}$, then $(q-1)R \in \mathcal{R}^{-}_{\mathfrak{m}}$.

Theorem 3.2. i) $Tor(\mathbb{U}_{\mathfrak{m}}) = 0$.
ii) $H^0(J, \mathbb{U}_{\mathfrak{m}}) = Tor(\mathbb{U}^{-}_{\mathfrak{m}}) = \mathcal{R}_{\mathfrak{m}}/\mathcal{R}^{-}_{\mathfrak{m}}$.
iii) $H^{-1}(J, \mathbb{U}_{\mathfrak{m}}) = Tor(\mathbb{U}^{+}_{\mathfrak{m}}) = \mathcal{R}'_{\mathfrak{m}}/\mathcal{R}^{+}_{\mathfrak{m}}$.

To determine the structures of torsion subgroups of the universal punctured distributions, we have to determine $H^0(J, \mathbb{U}_{\mathfrak{m}})$ and $H^{-1}(J, \mathbb{U}_{\mathfrak{m}})$

For each integral divisor $\mathfrak{f}$ of $\mathfrak{m}$, let

$$\alpha_{\mathfrak{f}} = s(I_{\mathfrak{f}}) \prod_{\mathfrak{p}|\mathfrak{f}} (1 - \bar{\sigma}_{\mathfrak{p}}),$$

where $I_{\mathfrak{f}} = Gal(K_{\mathfrak{m}}/K_{\mathfrak{f}})$ and $\bar{\sigma}_{\mathfrak{p}} = \mathrm{Fr}_{\mathfrak{p}}^{-1} s(T_{\mathfrak{p}})/|T_{\mathfrak{p}}|$. Here $T_{\mathfrak{p}}$ is the inertia group at $\mathfrak{p}$. Note that $I_{\mathfrak{e}} = H_{\mathfrak{m}}$. Let $\mathfrak{A}$ be the $\mathbb{Z}[G_{\mathfrak{m}}]$-submodule of $\mathbb{Q}[G_{\mathfrak{m}}]$ generated by $\alpha_{\mathfrak{f}}$ with $\mathfrak{f} \mid \mathfrak{m}$. This module $\mathfrak{A}$ was first introduced by Sinnott([Si]), and we call it the *Sinnott module*. Note that $\mathfrak{A}$ is $G_{\mathfrak{m}}$-equivariantly isomorphic to the universal ordinary (non-punctured) distribution of level $\mathfrak{m}$([A], §5).

For each character χ of $G_{\mathfrak{m}}$, let

$$e_{\chi} = \frac{1}{|G_{\mathfrak{m}}|} \sum_{\sigma \in G_{\mathfrak{m}}} \chi(\sigma^{-1})\sigma$$

be the idempotent associated to χ. Define an element ω in $\mathcal{A}_{\mathfrak{m}}$ by

$$\omega = \sum_{\mathfrak{f}_{\chi} \neq \mathfrak{e}} \phi(\mathfrak{f}_{\chi}) e_{\chi} [\{\mathfrak{f}_{\chi}^{-1}\}].$$

Let

$$e_0 = \frac{1}{\phi(\mathfrak{m})} s(I_{\mathfrak{e}}) = \frac{1}{\phi(\mathfrak{m})} \alpha_{\mathfrak{e}}.$$

Lemma 3.3. *The $G_{\mathfrak{m}}$-modules $(|I_{\mathfrak{e}}| - s(I_{\mathfrak{e}}))\mathbb{U}_{\mathfrak{m}}$ and $(1 - e_0)\mathfrak{A}$ are isomorphic.*

From the exact sequence

$$0 \to \mathbb{U}_{\mathfrak{m}}^{I_{\mathfrak{e}}} \to \mathbb{U}_{\mathfrak{m}} \to (|I_{\mathfrak{e}}| - s(I_{\mathfrak{e}}))\mathbb{U}_{\mathfrak{m}} \to 0$$

we have the following exact sequence of cohomology groups

(*)

$$\begin{array}{ccccc}
 & H^0(J, \mathbb{U}_{\mathfrak{m}}) & \longrightarrow & H^0(J, (|I_{\mathfrak{e}}| - s(I_{\mathfrak{e}}))\mathbb{U}_{\mathfrak{m}}) & \\
\overset{\eta}{\nearrow} & & & & \searrow \\
H^0(J, \mathbb{U}_{\mathfrak{m}}^{I_{\mathfrak{e}}}) & & & & H^{-1}(J, \mathbb{U}_{\mathfrak{m}}^{I_{\mathfrak{e}}}). \\
\nwarrow & & & & \swarrow \\
 & H^{-1}(J, (|I_{\mathfrak{e}}| - s(I_{\mathfrak{e}}))\mathbb{U}_{\mathfrak{m}}) & \longleftarrow & H^{-1}(J, \mathbb{U}_{\mathfrak{m}}) &
\end{array}$$

One can prove the following.

Lemma 3.4. *We have*

a) $\mathbb{U}_{\mathfrak{m}}^{I_{\mathfrak{e}}}$ *is a free abelian group of rank* $r+h-t$.

b) $Im(\eta) = \mathbb{U}_{\mathfrak{m}}^{I_{\mathfrak{e}}}/(s(J)\mathbb{U}_{\mathfrak{m}})^{I_{\mathfrak{e}}} = 0$.

Corollary 3.5. *We have*

$$H^{-1}(J, \mathbb{U}_{\mathfrak{m}}^{I_{\mathfrak{e}}}) = 0 \quad \textit{and} \quad H^0(J, \mathbb{U}_{\mathfrak{m}}^{I_{\mathfrak{e}}}) = (\mathbb{Z}/(q-1))^{r+h-t}. \qquad \square$$

From the exact sequence (*), in order to compute $H^i(J, \mathbb{U}_{\mathfrak{m}})$, we need to compute $H^i(J, (|I_{\mathfrak{e}}| - s(I_{\mathfrak{e}}))\mathbb{U}_{\mathfrak{m}}) \simeq H^i(J, (1-e_0)\mathfrak{A})$.

From the exact sequence of $G_{\mathfrak{m}}$-modules

$$0 \to \mathfrak{A}^{I_{\mathfrak{e}}} \to \mathfrak{A} \to (1-e_0)\mathfrak{A} \to 0$$

we have the following exact sequence of cohomology groups

(**)

$$\begin{array}{ccccc}
 & & H^0(J,\mathfrak{A}) \longrightarrow H^0(J,(1-e_0)\mathfrak{A}) & & \\
 & \overset{\mu}{\nearrow} & & \searrow & \\
H^0(J,\mathfrak{A}^{I_{\mathfrak{e}}}) & & & & H^{-1}(J,\mathfrak{A}^{I_{\mathfrak{e}}}). \\
 & \nwarrow & & \swarrow & \\
 & & H^{-1}(J,(1-e_0)\mathfrak{A}) \longleftarrow H^{-1}(J,\mathfrak{A}) & &
\end{array}$$

It is shown in [A] (Theorem 5.2.3) that

$$H^*(J, \mathfrak{A}) = \mathbb{Z}/(q-1)[G_{\mathfrak{e}}/N_{\mathfrak{m}}]^{2^{r-1}}.$$

Let $\tilde{B}$ be the $\mathbb{Z}[G_{\mathfrak{m}}]$-submodule of $\mathbb{Q}[G_{\mathfrak{m}}]$ generated by $\alpha_{\mathfrak{f}}$, with $\mathfrak{e} \neq \mathfrak{f} \mid \mathfrak{m}$. Let $B = \tilde{B}^{I_{\mathfrak{e}}}$ and $C = \mathbb{Z}[G_{\mathfrak{m}}]\alpha_{\mathfrak{e}}$. Then $\mathfrak{A}^{I_{\mathfrak{e}}} = B + C$. By some tedious computations, we can show that if $a \in B$, then

$$a = \sum_{\mathfrak{p}} a_{\mathfrak{p}} s(H_{\mathfrak{p}^{e_p}})\alpha_{\mathfrak{p}^{e_{\mathfrak{p}}}}$$

with $a_{\mathfrak{p}} \in \mathbb{Z}[G_{\mathfrak{m}}]$. In particular, $a \in s(J)\mathfrak{A}$. Therefore we have,

Proposition 3.6. *We have*

i) $(\mathfrak{A})^{I_{\mathfrak{e}}} = \mathbb{Z}[G_{\mathfrak{e}}]\alpha_{\mathfrak{e}}$.

ii) $(s(J)\mathfrak{A})^{I_{\mathfrak{e}}} = B + (q-1)C$.

iii) $B = < 1 - \tau_{\mathfrak{p}} : \mathfrak{p} \mid \mathfrak{m} > \alpha_{\mathfrak{e}}$. $\square$

From Proposition 3.6, i) we see that

$$H^{-1}(J, \mathfrak{A}^{I_{\mathfrak{e}}}) = 0, \quad \text{and} \quad H^0(J, \mathfrak{A}^{I_{\mathfrak{e}}}) \simeq (\mathbb{Z}/(q-1))^h.$$

Since

$$Im(\mu) = \mathfrak{A}^{I_{\mathfrak{e}}}/(s(J)\mathfrak{A})^{I_{\mathfrak{e}}} = (B+C)/(B+(q-1)C) = C/((q-1)C + (B \cap C)),$$

we have

Proposition 3.7. $Im(\mu) = \mathbb{Z}/(q-1)[G_{\mathfrak{e}}/N_{\mathfrak{m}}]$. □

Therefore, from the exact sequence (**), we have the following theorem.

Theorem 3.8. *We have, for* $t = |G_{\mathfrak{e}}/N_{\mathfrak{m}}|$,

$$H^{-1}(J,(1-e_0)\mathfrak{A}) = (\mathbb{Z}/(q-1))^{t(2^{r-1}-1)+h} \quad and \quad H^0(J,(1-e_0)\mathfrak{A}) = (\mathbb{Z}/(q-1))^{t(2^{r-1}-1)}.$$

And so

$$H^{-1}(J,\mathbb{U}_{\mathfrak{m}}) = (\mathbb{Z}/(q-1))^{t2^{r-1}-r} \quad and \quad H^0(J,\mathbb{U}_{\mathfrak{m}}) = (\mathbb{Z}/(q-1))^{t(2^{r-1}-1)}.$$ □

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