Trends in Mathematics
Information Center for Mathematical Sciences
Volume 9, Number 1,June 2006, Pages 7–12

q-EULER NUMBERS AND POLYNOMIALS ASSOCIATED WITH p-ADIC q-INTEGRALS AND BASIC q-ZETA FUNCTION

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Abstract. The purpose of this paper is to introduce the some interesting properties of q-Euler numbers and polynomials, cf. [1, 2, 5]. Finally, we will consider the " sums of products of q-Euler polynomials".

1. Introduction

Let p be a fixed odd prime, and let $\mathbb{C}_p$ denote the p-adic completion of the algebraic closure of $\mathbb{Q}_p$. For d a fixed positive integer with $(p,d)=1$, let

$$X = X_d = \varprojlim_N \mathbb{Z}/dp^N, \quad X_1 = \mathbb{Z}_p,$$

$$X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p,$$

$$a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\},$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$, (cf. [1], [2]).

The p-adic absolute value in $\mathbb{C}_p$ is normalized so that $|p|_p = \frac{1}{p}$. Let q be variously considered as an indeterminate a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, we always assume $|q| < 1$. If $q \in \mathbb{C}_p$, we always assume $|q-1|_p < p^{-\frac{1}{p-1}}$, so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. Throughout this paper, we use the following notation :

$$[x]_q = [x:q] = \frac{1-q^x}{1-q}.$$

We say that f is uniformly differentiable function at a point $a \in \mathbb{Z}_p-$ and denote this property by $f \in UD(\mathbb{Z}_p)-$ if the difference quotients

$$F_f(x,y) = \frac{f(x)-f(y)}{x-y},$$

have a limit $l = f'(a)$ as $(x,y) \to (a,a)$, cf. [1, 11, 12]. For $f \in UD(\mathbb{Z}_p)$, let us start with the expression

$$\frac{1}{[p^N]_q} \sum_{0 \leq j < p^N} q^i f(j) = \sum_{0 \leq j < p^N} f(j)\mu_q(j + p^N\mathbb{Z}_p), \text{ cf. [2, 4]},$$

2000 *Mathematics Subject Classification.* 11S80.
Key words and phrases. Euler numbers, combinatorial analysis, p-adic number field.

representing q-analogue of Riemann sums for f.

The integral of f on $\mathbb{Z}_p$ will be defined as limit ($n \to \infty$) of these sums, when it exists. An invariant p-adic q-integral of a function $f \in \mathrm{UD}(\mathbb{Z}_p)$ on $\mathbb{Z}_p$ is defined by

$$\int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{0\le j<p^N} f(j) q^j.$$

Note that if $f_n \to f$ in $UD(Z_p)$; then

$$\int_{\mathbb{Z}_p} f_n(x) d\mu_q(x) \to \int_{\mathbb{Z}_p} f(x) d\mu_q(x).$$

It was well known that the ordinary Euler numbers are defined by

$$F(t) = \frac{2}{e^t+1} = e^{Et} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!},$$

where we use the technique method notation by replacing E^m by E_m ($m \ge 0$), symbolically, cf.[2, 6]. In this paper, we introduce the definitions and properties of q-Euler numbers and polynomials. Finally we introduce formulae for the " sums of products of q-Euler polynomials" .

§2. q-Euler and Genocchi numbers associated with p-adic q-integrals

The Euler polynomials are defined by means of the following generating function: $\frac{2}{e^t-1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}$. Note that $E_n(0) = E_n$. From these Euler polynomials, we can evaluate the value of the below alternating sums of powers of consecutive integers:

(1)

$$-1^m + 2^m - 3^m + \cdots + (-1)^{m-1}(n-1)^m = \frac{1}{2}\left((-1)^{n+1} E_m(n) - E_m\right), \text{ cf. [1, 2, 3] .}$$

In the meaning of fermionic, we now consider the below p-adic q-integrals:

(2) $$\int_{X_f} [x]_q^k d\mu_{-q}(x) = \int_{\mathbb{Z}_p} [x]_q^k d\mu_{-q}(x) = E_{k,q} \text{ for } k, f \in \mathbb{N}, \text{ cf. [7, 8] .}$$

From the computation of this p-adic q-integral, we derive the below Eq.(3):

(3) $$E_{k,q} = [2]_q \left(\frac{1}{1-q}\right)^k \sum_{l=0}^{k} \binom{k}{l} (-1)^l \frac{1}{1+q^{l+1}}, \text{ cf. [8],}$$

where $\binom{k}{i}$ is the binomial coefficient. Note that $\lim_{q\to 1} E_{k,q} = E_k$. Hence, $E_{k,q}$ is q-extension of Euler numbers which are called q-Euler numbers. Let $F_q(t) = \sum_{n=0}^{\infty} E_{n,q} \frac{t^n}{n!}$ be the generating function of q-Euler numbers. Then we easily see that

(4) $$F_q(t) = e^{\frac{t}{1-q}} \sum_{n=0}^{\infty} \frac{[2]_q}{[2]_{q^{j+1}}} \left(\frac{1}{q-1}\right)^j \frac{t^j}{j!} = [2]_q \sum_{l=0}^{\infty} (-q)^l e^{[l]_q t}, \text{ cf. [6, 8, 9, 10.]}$$

By using an invariant p-adic q-integral on $\mathbb{Z}_p$, we can also consider the q-extension of ordinary Euler polynomials which are called q-Euler polynomials. For $x \in \mathbb{Z}_p$, we define q-Euler polynomials as follows:

$$\int_{\mathbb{Z}_p} [x+y]_q^k d\mu_{-q}(y) = E_{k,q}(x), \text{ cf. [7, 8].} \tag{5}$$

By (5), we easily see that

$$E_{k,q}(x) = \sum_{n=0}^{k} \binom{k}{n} [x]_q^{k-n} q^{nx} E_{n,q}.$$

In Eq.(5), it is easy to see that

$$E_{n,q}(x) = \int_{\mathbb{Z}_p} [x+y]_q^n d\mu_{-q}(y) = [2]_q \left(\frac{1}{1-q}\right)^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k q^{xk} \frac{1}{1+q^{k+1}}.$$

By using the definition of Eq.(5), we will give the distribution of q-Euler polynomials. From the definition of p-adic q-integral , , we derive the below formula:

$$\int_{X_m} [x+y]_q^n d\mu_{-q}(y) = \frac{[m]_q^n}{[m]_{-q}} \sum_{a=0}^{m-1} (-1)^a q^a \int_{\mathbb{Z}_p} [\frac{a+x}{m} + y]_{q^m}^n d\mu_{-q^m}(y), \text{ if } m \text{ is odd.}$$

Thus, if m is the odd integer, then we have

$$E_{n,q}(x) = \frac{[m]_q^n}{[m]_{-q}} \sum_{a=0}^{m-1} (-1)^a q^a E_{n,q^m}(\frac{a+x}{m}).$$

From the definition of the q-Euler polynomials, we note that

$$F_q(x,t) = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!} = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n e^{[n+x]_q t}, \text{ cf. [8].}$$

It is well known that Genocchi numbers are defined by

$$\frac{2t}{e^t+1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}.$$

Thus, we easily see that $G_n = \sum_{l=0}^{n-1} \binom{n}{l} 2^l B_l$, where B_l are ordinary Bernoulli numbers. We now define the q-extension of Genocchi number which are called q-Genocchi numbers as follows:

$$F_q^*(t) = [2]_q t \sum_{l=0}^{\infty} (-1)^n q^n e^{[n]_q t} = \sum_{n=0}^{\infty} G_{n,q} \frac{t^n}{n!}. \tag{6}$$

By Eq.(6), we easily see that

(7)

$$G_{n,q} = n[2]_q \left(\frac{1}{1-q}\right)^{n-1} \sum_{l=0}^{n-1} \binom{n-1}{l} \frac{(-1)^l}{[2]_{q^{l+1}}} = [2]_q n \int_{\mathbb{Z}_p} [x]_q^{n-1} d\mu_q(x), \text{ when } n \text{ is odd .}$$

From Eq.(6), we can also derive the definition of q-Genocchi polynomials as follows:

$$F_q^*(x,t) = [2]_q t \sum_{n=0}^{\infty} (-1)^n q^{n+x} e^{[n+x]_q t} = \sum_{n=0}^{\infty} G_{n,q}(x) \frac{t^n}{n!}, \text{ when } n \text{ is odd.} \tag{8}$$

Let $a_1, a_2, \cdots, a_k, b_1, b_2, \cdots, b_k$ be positive integers . For $w \in \mathbb{Z}_p$, we define multiple Daehee q-Euler polynomials by using the invariant p-adic q-integrals as follows, cf. [7]:

(9)

$$E_n^{(k)}(w, q | a_1, a_2, \cdots, a_k : b_1, b_2, \cdots, b_k) = \int_{\mathbb{Z}_p^k} q^{\sum_{j=1}^k (b_j - 1)x_j} [w + \sum_{j=1}^k a_j x_j]_q^n d\mu_{-q}(x),$$

and

$$E_n^{(k)}(q | a_1, \cdots, a_k : b_1, \cdots, b_k) = \int_{\mathbb{Z}_p^k} q^{\sum_{j=1}^k (b_j - 1)x_j} [\sum_{j=1}^k a_j x_j]_q^n d\mu_{-q}(x),$$

where

$$\int_{\mathbb{Z}_p^k} f(x) d\mu_{-q}(x) = \underbrace{\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k \text{ times}} f(x) d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r).$$

From the Eq.(9), we can derive the below theorem:

Proposition. *Let $a_1, a_2, \cdots, a_k, b_1, b_2, \cdots, b_k$ be positive integers. Then we have*

(10)

$$E_n^{(k)}(w, q | a_1, \cdots, a_k : b_1, \cdots, b_k) = \frac{[2]_q^k}{(1-q)^n} \sum_{r=0}^{n} \binom{n}{r} (-q^w)^r \prod_{j=1}^{k} \left(\frac{1}{[2]_{q^{b_j + r a_j}}} \right).$$

Given elements $\alpha_1, \cdots, \alpha_m \in \mathbb{C}_p$ and positive integers $N_1, \cdots, N_m, n$, it is easy to see that

$$\begin{aligned} &[N_1(x_1 + \alpha_1) + \cdots + N_m(x_m + \alpha_m)]_q^n \\ &= \sum_{\substack{i_1, \cdots, i_m \geq 0 \\ i_1 + \cdots + i_m = n}} \sum_{k_1=0}^{n-i_1} \sum_{k_2=0}^{n-i_1-i_2} \cdots \sum_{k_{m-1}=0}^{n-i_1-\cdots-i_{m-1}} \\ &\quad \times \binom{n}{i_1, \cdots, i_m} \binom{n-i_1}{k_1} \binom{n-i_1-i_2}{k_2} \cdots \binom{n-i_1-i_2-\cdots-i_{m-1}}{k_{m-1}} \\ &\quad \times (q-1)^{k_1 + \cdots + k_{m-1}} [N_1]_q^{i_1+k_1} \cdots [N_{m-1}]_q^{i_{m-1}+k_{m-1}} [N_m]_q^{i_m} \\ &\times [x_1 + \alpha_1 : q^{N_1}]^{k_1+i_1} \cdots [x_{m-1} + \alpha_{m-1} : q^{N_{m-1}}]^{k_{m-1}+i_{m-1}} [x_m + \alpha_m : q^{N_m}]^{i_m}. \end{aligned} \tag{11}$$

Hence, we have

$$\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{m \text{ times}}[N_1(x_1+\alpha_1)+\cdots+N_m(x_m+\alpha_m)]_q^n d\mu_{-q^{N_1}}(x_1)\cdots d\mu_{-q^{N_m}}(x_m)$$

$$\begin{aligned}
&= \sum_{\substack{i_1,\cdots,i_m\geq 0\\ i_1+\cdots+i_m=n}}\sum_{k_1=0}^{n-i_1}\sum_{k_2=0}^{n-i_1-i_2}\cdots\sum_{k_{m-1}=0}^{n-i_1-\cdots-i_{m-1}}\\
&\times\binom{n}{i_1,\cdots,i_m}\binom{n-i_1}{k_1}\binom{n-i_1-i_2}{k_2}\cdots\binom{n-i_1-i_2-\cdots-i_{m-1}}{k_{m-1}}\\
&\times(q-1)^{k_1+\cdots+k_{m-1}}[N_1]_q^{i_1+k_1}\cdots[N_{m-1}]_q^{i_{m-1}+k_{m-1}}[N_m]_q^{i_m}\\
&\times E_{k_1+i_1,q^{N_1}}(\alpha_1)\cdots E_{k_{m-1}+i_{m-1},q^{N_{m-1}}}(\alpha_{m-1})E_{i_m,q^{N_m}}(\alpha_m).
\end{aligned}\tag{12}$$

From (9), (10), (11) and (12), we can derive the below theorem:

Theorem. *(Sums of products of q-Euler polynomials)*
Given elements $\alpha_1,\cdots,\alpha_m\in\mathbb{C}_p$ *and positive integers* $N_1,\cdots,N_m,n$,

$$\begin{aligned}
&\sum_{\substack{i_1,\cdots,i_m\geq 0\\ i_1+\cdots+i_m=n}}\sum_{k_1=0}^{n-i_1}\sum_{k_2=0}^{n-i_1-i_2}\cdots\sum_{k_{m-1}=0}^{n-i_1-\cdots-i_{m-1}}\\
&\times\binom{n}{i_1,\cdots,i_m}\binom{n-i_1}{k_1}\binom{n-i_1-i_2}{k_2}\cdots\binom{n-i_1-i_2-\cdots-i_{m-1}}{k_{m-1}}\\
&\qquad\times(q-1)^{k_1+\cdots+k_{m-1}}[N_1]_q^{i_1+k_1}\cdots[N_{m-1}]_q^{i_{m-1}+k_{m-1}}[N_m]_q^{i_m}\\
&\qquad\quad\times E_{k_1+i_1,q^{N_1}}(\alpha_1)\cdots E_{k_{m-1}+i_{m-1},q^{N_{m-1}}}(\alpha_{m-1})E_{i_m,q^{N_m}}(\alpha_m)\\
&= E_n^{(m)}(N_1\alpha_1+\cdots+N_m\alpha_m,q|N_1,\cdots,N_m:1,1,\cdots,1),
\end{aligned}$$

where $\binom{n}{i_1,\cdots,i_m}$ *is multinomial coefficient.*

§3. Further Remarks and Observations

In this section, we assume that $q\in\mathbb{C}$ with $|q|<1$. Let $\Gamma(s)$ be the ordinary gamma function given by $\Gamma(s)=\int_0^\infty e^{-t}t^{s-1}dt$, $s\in\mathbb{C}$. From (8) and complex integral, we can derive the below formula:

$$\frac{1}{\Gamma(s)}\int_0^\infty t^{s-2}F_q^*(x,-t)dt=[2]_q\sum_{n=0}^\infty\frac{(-1)^{n+1}q^{n+x}}{[n+x]_q},\ \text{for } s\in\mathbb{C}.\tag{13}$$

For $s\in\mathbb{C}$, we define the(Hurwitz's type) q-Genocchi zeta function as follows:

$$\zeta_{q,G}(s,x)=[2]_q\sum_{n=0}^\infty\frac{(-1)^{n+1}q^{x+n}}{[n+x]_q^s},\ \text{where } x\in\mathbb{R} \text{ with } 0<x<1.\tag{14}$$

By (8), (13) and (14), we easily see that
(15)

$$\zeta_{q,G}(s,x) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-2} F_q^*(x,-t)dt = \sum_{n=0}^{\infty} \frac{G_{n,q}(x)}{n!} \left(\frac{1}{\Gamma(s)} \int_0^\infty t^{n+s-2} dt \right).$$

By using the residue theorem in Eq.(15), we easily see that

$$\zeta_{q,G}(1-n,x) = \frac{(-1)^{n-1}}{n} G_{n,q}(x), \ n \in \mathbb{N} .$$

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