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REMARKS ON TWISTED TRACES OF SINGULAR MODULI

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1. Introduction

Let d denote a positive integer congruent to 0 or 3 modulo 4. We denote by $\mathcal{Q}_d$ the set of positive definite binary quadratic forms $Q = [a, b, c] = aX^2 + bXY + cY^2$ $(a, b, c \in \mathbb{Z})$ of discriminant $-d$, with usual action of the modular group $\Gamma = SL_2(\mathbb{Z})$. To each $Q \in \mathcal{Q}_d$, we associate its unique root $\alpha_Q \in \mathfrak{H}$ (=upper half plane). Let $j(\tau)$ $(\tau \in \mathfrak{H})$ be the elliptic modular invariant and $\mathbf{t}(d)$ be the (weighted) trace of a singular modulus of discriminant $-d$ $(= \sum_{Q\in\mathcal{Q}_d/\Gamma} \frac{1}{|\bar\Gamma_Q|}(j(\alpha_Q) - 744))$. Here $\bar\Gamma_Q = \{\gamma \in \bar\Gamma = PSL_2(\mathbb{Z}) \mid Q\circ\gamma = Q\}$. In addition we set $\mathbf{t}(-1) = -1, \mathbf{t}(0) = 2$ and $\mathbf{t}(d) = 0$ for $d < -1$ or $d \equiv 1, 2 \pmod 4$. Zagier showed the series $\sum_{d\in\mathbb{Z}} \mathbf{t}(d)q^d$ $(q = e^{2\pi i\tau}, \tau \in \mathfrak{H})$ is a modular form of weight 3/2 on $\Gamma_0(4) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}), 4|c\}$, holomorphic in $\mathfrak{H}$ and meromorphic at cusps ([14] Theorem 1). The higher level analogue of Zagier's trace formula are described in Theorem 8 of [14] and further properties are developed in [8] and [9]. In [14] §7 when the level N is equal to 1 and $D > 1$, Zagier described the D-th coefficient of $f_{d,1}$ in terms of the relative trace of singular modulus of discriminant $-dD$ from the Hilbert class field of $\mathbb{Q}(\sqrt{-dD})$ to its real quadratic subfield $\mathbb{Q}(\sqrt{D})$. In this article we will investigate the analogue of this in higher level cases and construct Borcherds products (see Theorem 1.1).

Let $\Gamma_0(N)^*$ $(N = 1, 2, 3, \dots)$ be the group generated by $\Gamma_0(N)$ and all Atkin-Lehner involutions W_e for $e||N$. Here W_e is represented by a matrix of the form $\frac{1}{\sqrt{e}}\left(\begin{smallmatrix} ex & y \\ Nz & ew \end{smallmatrix}\right)$ with $x, y, z, w \in \mathbb{Z}$ and $xwe - yzN/e = 1$. There are finitely many values of N for which $\Gamma_0(N)^*$ is of genus 0. Let j_N^* be the corresponding *Hauptmodul*, whose Fourier expansion starts with $q^{-1} + 0 + a_1q + a_2q^2 + \cdots$. It can be described by means of Dedekind eta function or theta functions. For example, if $N - 1$ divides 24, then the Hauptmodul j_N^* is explicitly given by

$$j_N^*(\tau) = \left(\frac{\eta(\tau)}{\eta(N\tau)}\right)^{\frac{24}{N-1}} + \frac{24}{N-1} + N^{\frac{12}{N-1}}\left(\frac{\eta(N\tau)}{\eta(\tau)}\right)^{\frac{24}{N-1}}.$$

Let d be an integer ≥ 0 such that $-d$ is congruent to a square modulo $4N$. We choose an integer $\beta \pmod{2N}$ with $\beta^2 \equiv -d \pmod{4N}$ and consider the set $\mathcal{Q}_{d,N,\beta} = \{[a, b, c] \in \mathcal{Q}_d \mid a \equiv 0 \pmod N,\ b \equiv \beta \pmod{2N}\}$ on which $\Gamma_0(N)$ acts.

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Let D and $-d$ be a positive and a negative discriminant, respectively. For simplicity we suppose them to be fundamental and coprime. Let $K = \mathbb{Q}(\sqrt{-Dd})$. Let $p_1, \ldots, p_r$ be odd primes dividing Dd and put $p_i^* = (-1)^{(p_i-1)/2}p_i$ for $i = 1, \ldots, r$. By class field theory the genus field M of K is expressed as $K(\sqrt{p_1^*}, \ldots, \sqrt{p_r^*})$ and is contained in the Hilbert class field H of K (see [5] Theorem 6.1). We have obvious inclusion $N = K(\sqrt{D}) \subset M \subset H$. The genus character $\chi = \chi_{D,-d}$ assigns to any quadratic form Q of discriminant $-dD$ a value ± 1 defined by $\chi(Q) = \left(\frac{D}{q}\right) = \left(\frac{-d}{q}\right)$ where q is any prime represented by Q and not dividing Dd. This is independent of the choice of q. We further assume D and $-d$ to be congruent to a square modulo $4p$. We define the "twisted trace" $\mathbf{t}^{(p)}(D,d)$ by

$$\mathbf{t}^{(p)}(D,d) = \sum_{Q \in \mathcal{Q}_{dD,p,\beta}/\Gamma_0(p)} \chi(Q) j_p^*(\alpha_Q).$$

Then the definition of $\boldsymbol{t}^{(p)}(D,d)$ is independent of the choice of β and $\frac{1}{\sqrt{D}}\boldsymbol{t}^{(p)}(D,d)$ is a rational integer. Here are some numerical examples with $p = 2$ and $p = 3$:
$\frac{1}{\sqrt{17}}\mathbf{t}^{(2)}(17,4) = \frac{1}{\sqrt{17}}(j_2^*\left(\alpha_{[18,2,1]}\right) + j_2^*\left(\alpha_{[2,2,9]}\right) - j_2^*\left(\alpha_{[6,-2,3]}\right) - j_2^*\left(\alpha_{[6,2,3]}\right)) = -204800$,
$\frac{1}{\sqrt{8}}\mathbf{t}^{(2)}(8,7) = \frac{1}{\sqrt{8}}(j_2^*\left(\alpha_{[14,0,1]}\right) + j_2^*\left(\alpha_{[2,0,7]}\right) - j_2^*\left(\alpha_{[6,4,3]}\right) - j_2^*\left(\alpha_{[10,8,3]}\right)) = 90112$,
and
$\frac{1}{\sqrt{13}}\mathbf{t}^{(3)}(13,3) = \frac{1}{\sqrt{13}}(j_3^*\left(\alpha_{[12,3,1]}\right) + j_3^*\left(\alpha_{[3,3,4]}\right) - j_3^*\left(\alpha_{[6,3,2]}\right) - j_3^*\left(\alpha_{[15,9,2]}\right)) = -378$,
$\frac{1}{\sqrt{13}}\mathbf{t}^{(3)}(13,8) = \frac{1}{\sqrt{13}}(j_3^*\left(\alpha_{[27,2,1]}\right) - j_3^*\left(\alpha_{[21,8,2]}\right) + j_3^*\left(\alpha_{[3,2,9]}\right) + j_3^*\left(\alpha_{[9,2,3]}\right) - j_3^*\left(\alpha_{[6,-4,5]}\right) - j_3^*\left(\alpha_{[15,14,5]}\right)) = -11968$,
$\frac{1}{\sqrt{21}}\mathbf{t}^{(3)}(21,8) = \frac{1}{\sqrt{21}}(j_3^*\left(\alpha_{[42,0,1]}\right) - j_3^*\left(\alpha_{[21,0,2]}\right) + j_3^*\left(\alpha_{[3,0,14]}\right) - j_3^*\left(\alpha_{[6,0,7]}\right)) = 342144$.
We want to express these traces as the coefficients of certain modular forms.

Let $M^!_{k+1/2}(N)$ be the vector space consisting of *nearly holomorphic modular forms* (holomorphic in $\mathfrak{H}$ and meromorphic at cusps) of half-integral weight $k+1/2$ on $\Gamma_0(4N)$ whose n-th Fourier coefficient vanishes unless $(-1)^k n$ is congruent to a square modulo $4N$. Then for every integer $d \geq 0$ such that $-d$ is congruent to a square modulo $4N$, we can find a unique modular form $f_{d,N} \in M^!_{1/2}(N)$ having a Fourier expansion of the form

$$f_{d,N} = q^{-d} + \sum_{D>0} A(D,d) q^D.$$

As explained in the Appendix of [8], $f_{d,p}$ can be found by making use of "Rankin-Cohen bracket" $[\ ,\ ]_n$. Let $f(\tau)$ and $g(\tau)$ denote two modular forms of weight k and l on some group $\Gamma' \subset \Gamma$. We denote by D the differential operator $\frac{1}{2\pi i}\frac{d}{d\tau} = q\frac{d}{dq}$ and use $f', f'', \ldots, f^{(n)}$ instead of $Df, D^2 f, \ldots, D^n f$. The n-th Rankin-Cohen bracket of f and g is defined by the formula

$$[f,g]_n(\tau) = \sum_{r+s=n} (-1)^r \binom{n+k-1}{s}\binom{n+l-1}{r} f^{(r)}(\tau) g^{(s)}(\tau).$$

By [3] §7 (or [13] §1) $[f,g]_n(\tau)$ is a modular form of weight $k+l+2n$ on Γ'. Let us consider the construction of $f_{d,p}$ in the cases $p = 2$ and $p = 3$:

$p = 2$: Let $\theta = \sum_{n\in\mathbb{Z}} q^{n^2}$, $u = ([\theta, E_{10}(8\tau)]_1/\Delta(8\tau) + 2112\theta)/(-40)$ and $v = ([\theta, E_8(8\tau)]_2/\Delta(8\tau) - 11520\theta)/72$. Then it follows that
$f_{4,2} = (v-u)/12 = q^{-4} - 52q + 272q^4 + 2600q^8 - 8244q^9 + 15300q^{12} + 71552q^{16} - 204800q^{17} + 282880q^{20} + \cdots$,
$f_{7,2} = (4u-v)/3 = q^{-7} - 23q - 2048q^4 + 45056q^8 + 252q^9 - 516096q^{12} + 4145152q^{16} - 1771q^{17} - 26378240q^{20} + \cdots$
$p = 3$: Let $u = ([\theta, E_{10}(12\tau)]_1/\Delta(12\tau) + 1584\theta)/(-20)$, $v = ([\theta, E_8(12\tau)]_2/\Delta(12\tau) - 25920\theta)/72$ and $w = ([\theta, E_6(12\tau)]_3/\Delta(12\tau) + 272160\theta)/(-112)$. Then we obtain that
$f_{3,3} = (4u - 5v + w)/360 = q^{-3} - 14q + 40q^4 - 78q^9 + 168q^{12} - 378q^{13} + 688q^{16} - 897q^{21}\cdots$,
$f_{8,3} = (-9u + 10v - w)/60 = q^{-8} - 34q - 188q^4 + 2430q^9 + 8262q^{12} - 11968q^{13} - 34936q^{16} + 171072q^{21}\cdots$,
$f_{11,3} = (36u - 13v + w)/24 = q^{-11} + 22q - 552q^4 - 11178q^9 + 48600q^{12} + 76175q^{13} - 269744q^{16} - 1782891q^{21}\cdots$.

Comparing these coefficients with the examples of twisted traces $\mathbf{t}^{(p)}(D,d)$ we are led to the following result which can be proved in a way analogous to the proof of [14] Theorem 6:

$$\frac{1}{\sqrt{D}}\mathbf{t}^{(p)}(D,d) = A^*(D,d) = -B^*(D,d) \tag{1}$$

with $A^*(D,d) = 2^{s(D,p)}A(D,d)$ and $B^*(D,d) = 2^{s(D,p)}B(D,d)$. We derive the following certain product identities from (1).

Theorem 1.1. [10] *Let $D > 1$ and $-d < 0$ be coprime fundamental discriminants and congruent to a square modulo 4p. We define $\mathcal{H}_{D,d,p}(X) = \prod_{Q\in\mathcal{Q}_{dD,p,\beta}/\Gamma_0(p)}(X - j_p^*(\alpha_Q))^{\chi(Q)}$. Then*

$$\mathcal{H}_{D,d,p}(j_p^*(\tau)) = \prod_{u=1}^{\infty} P_D(q^u)^{A^*(u^2D,d)}$$

where $A^*(D,d) = 2^{s(D,p)}A(D,d)$ and $P_D(t) = \prod_{0<n<D}(1-\zeta_D^n t)^{\left(\frac{D}{n}\right)}$.

Through the article we adopt the following notations:

- β: an element in $\mathbb{Z}/2N\mathbb{Z}$.
- $\bar{\Gamma}'$: the image of Γ' in $PSL_2(\mathbb{R})$.
- $\bar{\Gamma}'_Q$: $= \{\gamma \in \bar{\Gamma}' \mid Q\circ\gamma = Q\}$.
- When we write $\mathcal{Q}_{d,N,\beta}$, β is always assumed to satisfy $\beta^2 \equiv -d \pmod{4N}$.
- $e(x)$: $= \exp(2\pi i x)$.
- ζ_n: $= e(1/n) = \exp(2\pi i/n)$.
- $s(D,N)$: the number of prime factors dividing (D,N).
- U_m: the Hecke operator defined by $\sum_{n\in\mathbb{Z}} c(n)q^n \mid_{U_m} = \sum_{n\in\mathbb{Z}} c(mn)q^n$.
- $E_k(\tau)$: the normalized Eisenstein series of weight k, equal to $1 - (2k/B_k)\sum_{n>0}\sigma_{k-1}(n)q^n$ where B_k is the k-th Bernoulli number defined by $\sum_{k\in\mathbb{Z}} B_k t^k/k! = t/(e^t - 1)$ and $\sigma_{k-1}(n) = \sum_{\substack{d\mid n\\ d>0}} d^{k-1}$.

2. Example

Example 2.1. Let us consider the case $p = 5$ and $D = 5, d = 4$. For $\beta \equiv 0 \pmod{10}$, a set of representatives for $\mathcal{Q}_{20,5,0}/\Gamma_0(5)$ is given by $\{[5,0,1],[15,20,7]\}$. Meanwhile the q-expansion of $f_{4,5}$ is

$$q^{-4} - 8q + q^4 + 10q^5 + 12q^9 - 62q^{16} + 65q^{20} + \cdots + 690q^{45} + \cdots + 8510q^{80} + \cdots .$$

Theorem 1.1 yields the following product formula:

$$\frac{j_5^*(\tau) - j_5^*\left(\alpha_{[5,0,1]}\right)}{j_5^*(\tau) - j_5^*\left(\alpha_{[15,20,7]}\right)} = \prod_{n=1}^{\infty} P_5(q^n)^{2A(5n^2,4)}$$

where $P_5(x) = \frac{(1-\zeta_5 x)(1-\zeta_5^4 x)}{(1-\zeta_5^2 x)(1-\zeta_5^3 x)} = \frac{1-\frac{1+\sqrt{5}}{2}x+x^2}{1-\frac{1-\sqrt{5}}{2}x+x^2}$. Thus we have

$$\begin{aligned}
&\frac{j_5^*(\tau) - (6+10\sqrt{5})}{j_5^*(\tau) - (6-10\sqrt{5})} \\
=&1 - 20\sqrt{5}q + (1000-120\sqrt{5})q^2 + (12000-8040\sqrt{5})q^3 + (340000-136960\sqrt{5})q^4 + \cdots \\
=&\left(\frac{1-\frac{1+\sqrt{5}}{2}q+q^2}{1-\frac{1-\sqrt{5}}{2}q+q^2}\right)^{2\cdot 10}\left(\frac{1-\frac{1+\sqrt{5}}{2}q^2+q^4}{1-\frac{1-\sqrt{5}}{2}q^2+q^4}\right)^{2\cdot 65} \\
&\qquad\left(\frac{1-\frac{1+\sqrt{5}}{2}q^3+q^6}{1-\frac{1-\sqrt{5}}{2}q^3+q^6}\right)^{2\cdot 690}\left(\frac{1-\frac{1+\sqrt{5}}{2}q^4+q^8}{1-\frac{1-\sqrt{5}}{2}q^4+q^8}\right)^{2\cdot 8510}\cdots .
\end{aligned}$$

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