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TRAVELLING WAVES IN CONTINUUM/DISCRETE BISTABLE/MONOSTABLE DYNAMICS

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1. Introduction

Traveling waves play a key role in mathematical models since they characterize observable fundamental phenomena in nature.

A typical traveling wave corresponds to a state which is time independent (in some cases time periodic) in a moving coordinates. In a one space dimensional setting, it depends only on $z := x - ct$ where c is the wave speed and x and t are respectively space and time variables.

Consider an autonomous dynamical system

$$\dot{u} = \mathcal{A}[u]$$

where $u : t \in [0, \infty) \to u(t) = u(t; \cdot) \in C^m(\mathbb{R})$ and $\mathcal{A} : \phi(\cdot) \in C^m(\mathbb{R}) \to \mathcal{A}[\phi](\cdot) \in C(\mathbb{R})$ is an operator. Here autonomous means that $\mathcal{A}$ is **translation invariant**; namely, for any $h \in \mathbb{R}$ and ϕ in the definition domain of $\mathcal{A}$,

$$\text{with } \phi^h(\cdot) := \phi(\cdot + h), \qquad \mathcal{A}[\phi^h](\cdot) = \mathcal{A}[\phi](\cdot + h). \tag{1.1}$$

The following are examples of translation invariant operators:

$$\begin{aligned} \mathcal{A}_1[\phi](x) &:= \phi_{xx}(x) + f(\phi(x)), \\ \mathcal{A}_2[\phi](x) &:= \phi(x+1) + \phi(x-1) - 2\phi(x) + f(\phi(x)), \\ \mathcal{A}_3[\phi](x) &:= J * \phi(x) - b\,\phi(x) + f(\phi(x)), \end{aligned}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a given smooth function and

$$J * \phi(x) := \int_{\mathbb{R}} J(x-y)\phi(y)\,dy, \qquad J \in L^1(\mathbb{R}), \quad b = \int_{\mathbb{R}} J(y)dy.$$

Suppose $\mathcal{A}$ is translation invariant. Then $\mathcal{A}$ maps a constant function to a constant function. Hence, denoting by $\mathbf{1}$ the function defined by $\mathbf{1}(x) = 1$ for all $x \in \mathbb{R}$, there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that

$$\mathcal{A}[s\mathbf{1}] = f(s)\mathbf{1} \qquad \forall s \in \mathbb{R}. \tag{1.2}$$

In studying phase transitions, quite often phase (equilibrium) states are denoted by constant functions. For this we assume that $f(-1) = f(1) = 0$. Then both $-\mathbf{1}$

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and **1** are equilibrium states of the dynamical system $\dot{u} = \mathcal{A}[u]$. A phase transition is quite often represented by a function ϕ satisfying $\phi(\pm\infty) := \lim_{x\to\pm\infty} \phi = \pm 1$.

In the above setting, a traveling wave refers to a pair (c, ϕ), where $c \in \mathbb{R}$ is called **wave speed** and ϕ a function on $\mathbb{R}$ is called **wave profile**, such that $u(t) := u(t; x) = \phi(x - ct)$ is a solution to $\dot{u} = \mathcal{A}[u]$. This renders to the following problem: Find (c, ϕ) such that

$$-c\,\phi' = \mathcal{A}[\phi], \qquad \phi(\pm\infty) = \pm 1, \quad \phi(0) = 0. \tag{1.3}$$

Here the condition $\phi(0) = 0$ is introduced to fix the translation of ϕ.

For $\dot{u} = \mathcal{A}[u]$, the following terminologies are frequently used. Let f be defined as in (1.2).

Bistable: Among all constant functions, $-\mathbf{1}$ and $\mathbf{1}$ are the only stable equilibria; i.e.,

$$f \in C^1(\mathbb{R}), \quad \{s \in \mathbb{R} \mid f(s) = 0\} = \{-1, \alpha, 1\}, \qquad f'(\pm 1) < 0 < f'(\alpha). \tag{1.4}$$

Monostable: Among constant functions valued in $[0, 1]$, $\mathbf{0}$ and $\mathbf{1}$ are the only equilibria; i.e.,

$$f \in C^1([0, 1]), \quad f(0) = f(1) = 0 < f(s) \quad \forall s \in (0, 1). \tag{1.5}$$

Continuum Model: $u(t) = u(t; \cdot)$ is a continuous function on $\mathbb{R}$. For example,

$$\dot{u} = \mathcal{A}_1[u], \qquad \dot{u} = \mathcal{A}_2[u], \qquad \dot{u} = \mathcal{A}_3[u].$$

Discrete Model: $u(t) = \{u_j(t)\}_{j\in\mathbb{Z}}$ is discrete. For example,

$$\dot{u}_j = u_{j+1} + u_{j-1} - 2u_j + f(u_j) \qquad \forall\, j \in \mathbb{Z}. \tag{1.6}$$

One can transfer this discrete model into a continuum model. Denote by $[x]$ the maximum integer no bigger than x and let $s(\cdot)$ be a non-negative periodic function of period 1. Define

$$w(t; x) = u_{[x]}(t + s(x)) \qquad \forall\, x \in \mathbb{R}, t > 0.$$

Then $\dot{w} = \mathcal{A}_2[w]$.

Local Operator: For any functions ϕ and ψ in the definition domain of $\mathcal{A}$,

$$\phi = \psi \text{ in } (z - \delta, z + \delta) \text{ for some } z \in \mathbb{R}, \delta > 0 \quad \Longrightarrow \quad \mathcal{A}[\phi](z) = \mathcal{A}[\psi](z).$$

Non-Local Operator: There are $z \in \mathbb{R}, \delta > 0$, and functions ϕ, ψ such that

$$\phi(x) = \psi(x)\ \forall x \in (z - \delta, z + \delta), \qquad \mathcal{A}[\phi](z) \neq \mathcal{A}[\psi](z).$$

For example, $\mathcal{A}[\phi] := \phi_{xx} + \phi_x^2 + f(\phi)$ is a local operator, whereas for any non-trivial $J \in L^1(\mathbb{R})$, $\mathcal{A}[\phi] := J * \phi$ is a non-local operator.

In studying evolution problems, one of the most used tools is the comparison principle.

Strong Comparison Principle:

$$\begin{cases} \dot{u} - \mathcal{A}[u] \geqslant \dot{v} - \mathcal{A}[v] & \forall\, t > 0, \\ u(0) \gneqq v(0) & \text{at } \ t = 0 \end{cases} \implies \quad u(t) > v(t) \ \ \forall\, t > 0.$$

When the conclusion $u(t) > v(t)$ is replaced by $u(t) \geqslant v(t)$, the comparison is called **weak**.

For example, suppose $\theta \in [0, 1]$ is a constant, J is a non-negative non-trivial $L^1(\mathbb{R})$ function, S is Lipschitz and strictly increasing, and f is Lipschitz. Then for

$$\mathcal{A}[\phi] := \theta\phi_{xx} + (1-\theta)\Big\{ J * S(\phi) - bS(\phi) \Big\} + f(\phi) \tag{1.7}$$

the dynamics $\dot{u} = \mathcal{A}[u]$ satisfies the strong comparison principle.

We remark that the dynamics $\dot{u} = \mathcal{A}_2[u]$ satisfies only a weak comparison principle, since when $\dot{u} - \mathcal{A}_2[u] = \mathbf{0} = \dot{v} - \mathcal{A}_2[v]$ and $u(0; j) = v(0; j)$ for all $j \in \mathbb{Z}$, $u(t; j) \equiv v(t; j)$ for all $j \in \mathbb{Z}$ and $t > 0$.

2. A Bistable Case

Consider the travelling wave problem (1.3). Assume the following:

(1) $\mathcal{A}$ is translation invariant and bistable, i.e., (1.1), (1.2) and (1.4) hold;
(2) $\mathcal{A}$ satisfies the strong comparison principle;
(3) $\mathcal{A}$ has certain regularity (for simplicity, we omit the details).

One example of such an $\mathcal{A}$ is that given in (1.7), where f satisfies (1.4), $S(\cdot)$ is C^1 and non-decreasing, $\theta \in [0, 1]$, $\theta + S'(\cdot) > 0$ and $J(\cdot) \geqslant 0$ on $\mathbb{R}$, $b := \int_{\mathbb{R}} J(z)dz \in (0, \infty)$.

Theorem 1 ([4])**.** *Under the above assumptions, the traveling wave problem* (1.3) *admits a unique solution. In addition, the traveling wave is globally asymptotically exponentially stable; that is, there exists* $\nu > 0$ *such that whenever the initial value* $u(0)$ *to* $\dot{u} = \mathcal{A}[u]$ *is uniformly bounded and*

$$\limsup_{x \to -\infty} u(0; x) < \alpha < \liminf_{x \to \infty} u(0; x),$$

there exist $K > 0$ *and* $s_0 \in \mathbb{R}$ *such that*

$$|u(t; z + s_0 + ct) - \phi(z)| \leq K e^{-\nu t} \quad \forall\, z \in \mathbb{R}, t \geq 0.$$

For stability, a key idea in [4] is the introduction of a distance $d(t)$ between the wave profile ϕ and the point $u(t)$ on the trajectory of $\dot{u} = \mathcal{A}[u]$: For each $t \geq 0$,

$$d(t) := \min_{(s,h,\delta) \in D(t)} \{h + M\delta\},$$

$$D(t) := \Big\{ (s, h, \delta) \,\Big|\, \phi(z - h) - \delta \leqslant u(t; z + s) \leqslant \phi(z + h) + \delta \quad \forall z \in \mathbb{R} \Big\}.$$

By constructing comparison functions, it is shown that there are constants $T > 0$ and $\mu \in (0, 1)$ such that $d(T) \leq 1$ and $d(t + 1) \leq \mu\, d(t)$ for all $t \geqslant T$. This implies the exponential decay $d(t) \le Me^{-\nu t}$ for all $t \ge 0$, where $\nu = -\ln \mu$ does not depend on $u(0)$.

3. A Monostable Discrete Model

Consider a traveling wave problem for (1.6), or $\dot u = \mathcal{A}_2[u]$; i.e., find (c, ϕ) such that

$$\text{(3.1)} \qquad \begin{cases} c\,\phi'(z) = \phi(z+1) + \phi(z-1) - 2\phi(z) + f(\phi(z)) \ \forall\, z \in \mathbb{R}, \\ \phi(-\infty) = 0, \quad \phi(\infty) = 1, \quad \phi(0) = 1/2, \quad 0 \leqslant \phi \leqslant 1 \text{ on } \mathbb{R}. \end{cases}$$

Theorem 2 ([7]). *Assume* (1.5). *There exists* $c_{\min} > 0$ *such that the following holds:*

(1) *Given* $c \in \mathbb{R}$, (3.1) *admits a solution* ϕ *if and only if* $c \geqslant c_{\min}$.
(2) *Given* $c \geqslant c_{\min}$, *the solution* ϕ *to* (3.1) *is unique,* $\phi' > 0$ *on* $\mathbb{R}$ *and*

$$\lim_{x\to-\infty} \frac{\phi'(x)}{\phi(x)} = \lim_{x\to-\infty} \frac{\phi''(x)}{\phi'(x)} = \lambda, \quad \lim_{x\to\infty} \frac{\phi'(x)}{\phi(x)-1} = \lim_{x\to\infty} \frac{\phi''(x)}{\phi'(x)} = \mu$$

where $\mu \leqslant 0 \leqslant \lambda$ *are roots to the* **characteristic equations**

$$c\,\mu = e^{\mu} + e^{-\mu} - 2 + f'(1), \qquad c\,\lambda = e^{\lambda} + e^{-\lambda} - 2 + f'(0),$$

with λ *being the smaller root when* $c > c_{\min}$ *and the large root when* $c = c_{\min}$. *In addition,*

$$\lim_{x\to-\infty} \frac{f(\phi(x))}{\phi'(x))} = \begin{cases} c & \text{if } \lambda = 0, \\ \frac{f'(0)}{\lambda} & \text{otherwise} \end{cases}, \qquad \lim_{x\to\infty} \frac{f(\phi(x))}{\phi'(x))} = \begin{cases} c & \text{if } \mu = 0, \\ \frac{f'(1)}{\mu} & \text{otherwise.} \end{cases}$$

Under some regularity assumption on f, one can show the following.

Theorem 3 ([7]). *The wave profile has the following asymptotic behavior as* $x \to -\infty$*:*

(i) *When* $c = c_{\min}$ *and* λ *is a double root, for some* $x_0 \in \mathbb{R}$,

$$\text{either } \ \phi(x+x_0) \sim e^{\lambda x} \qquad \text{or } \ \phi(x+x_0) \sim |x| e^{\lambda x};$$

(ii) *When* $c = c_{\min}$ *and* λ *is not a double root or* (iii) *when* $c > c_{\min}$ *and* $f'(0) > 0$,

$$\phi(x+x_0) \sim e^{\lambda x};$$

(iv) *When* $c > c_{\min}$ *and* $f'(0) = 0$,

$$\lim_{x\to-\infty} \left\{ \int_{1/2}^{\phi(x)} \frac{ds}{f(s)[1+f'(s)/c^2]} - \frac{x+x_0}{c} \right\} = 0.$$

For example, when $f(s) = ks^2(1-s)^p$ where $k > 0$ and $p \ge 1$ are constants,

$$\phi(x+x_0) = \frac{c}{k|x| + (pc - 2k/c)\ln|x|} + \frac{o(1)}{|x|^2}, \quad \lim_{x\to-\infty} o(1) = 0.$$

When $\mathcal{A}$ is non-degenerately monostable, i.e., $f'(0)f'(1) < 0$, the theorem is established in [6]; see also [30, 14, 5]. The general case is treated in [7], in which one key idea is the following observation.

For each $z(x)$ being one of the following functions

$$\frac{\phi'(x)}{\phi(x)}, \qquad \frac{\phi'(x)}{\phi(x)-1}, \qquad \frac{\phi''(x)}{\phi'(x)},$$

$$c\, z(x) = \exp(\textstyle\int_x^{x+1} z(s)ds) + \exp(-\int_{x-1}^{x} z(s)ds) + B(x)$$

where $B(x)$ is uniformly continuous and bounded. One can show that any solution z to this equation has the following properties:

(1) For $m = \|B\|_{L^\infty(\mathbb{R})}/c$, $-m < z(x) < m + 4ce^m + e^m/c \quad \forall x \in \mathbb{R}$.

(2) Assume that $B^\pm := \lim_{x\to\pm\infty} B(x)$ exits. Then there exists $\lim_{x\to\pm\infty} z(x) = \lambda$ which is a root to the characteristic equation $c\,\lambda = e^\lambda + e^{-\lambda} + B^\pm$.

Another technical difficulty is to show the uniqueness and monotonicity of the wave profile in the degenerate case $f'(0)f'(1) = 0$. For this, we introduced the following function

$$W(x,\xi) = \int_{\psi(x+\xi)}^{\phi(x)} \frac{ds}{f(s)}.$$

This function magnifies the difference between functions $\phi(x)$ and $\psi(x+\xi)$.

For more material on traveling waves, see the following references.

References

[1] D.G. Aronson & H.F. Weinberger, *Nonlinear diffusion in population genetics, combustion and nerve propagation,* in "Partial Differential Equations and Related Topics", 5–49, Lecture Notes in Mathematics **446**, Springer, New York, 1975.

[2] P.W. Bates, Xinfu Chen, & A. Chmaj, *Traveling waves of bistable dynamics on a lattice* SIAM J. Math. Anal. **35** (2003), 520-546.

[3] M. Bramson, Convergence of Solutions of the Kolmogorov Equation to Traveling Waves, Memoirs Amer. Math. Soc. **44**, 1983.

[4] Xinfu Chen, *Existence, uniqueness, and asymptotic stability of traveling waves in nonlocal evolution equations*, Advances in Differential Equations **2** (1997), 125–160.

[5] Xinfu Chen & J.-S. Guo, *Existence and asymptotic stability of traveling waves of a discrete monostable equation*, J. Diff. Eqns. **184** (2002), 1137-1149.

[6] Xinfu Chen & J.-S. Guo, *Uniqueness and existence of traveling waves for discrete quasilinear monostable dynamics*, Math. Ann. **326** (2003), 123-146.

[7] Xinfu Chen, S-C Fu, & J-S Guo, *Uniqueness and Asymptotics of Traveling Waves of Monostable Dynamics on Lattices*, preprint.

[8] S.-N. Chow, J. Mallet-Paret, & W. Shen, *Traveling waves in lattice dynamical systems,* J. Diff. Eqns. **149** (1998), 248–291.

[9] A. De Pablo & J.L. Vazquez, *Traveling waves and finite propagation in a reaction-diffusion equation,* J. Diff. Eqns. **93** (1991), 19–61.

[10] U. Ebert & W. van Saarloos, *Front propagation into unstable states: universal algebraic convergence towards uniformly translating pulled fronts,* Phy. D, **146** (2000),1–99.

[11] P.C. Fife, Mathematical Aspect of Reacting and Diffusing Systems, Lecture Notes in Biomathematics, **28**, Springer Verlag, 1979.

[12] R.A. Fisher, *The advance of advantageous genes,* Ann. Eugenics **7** (1937), 355–369.

[13] P.C. Fife & J.B. McLeod, *A phase plane discussion of convergence to traveling fronts for nonlinear diffusion,* Archiev Rat. Mech. Mech. **75** (1981), 281–314.

[14] S.-C. Fu, J.-S. Guo, & S.-Y Shieh, *Traveling wave solutions for some discrete quasilinear parabolic equations*, Nonl. Anal. **48** (2002), 1137-1149.

[15] F. Hamel & N. Nadirashvili, *Travelling fronts and entire solutions of the Fisher-KPP equation in $\mathbb{R}^N$,* Arch. Rat. Mech. Anal. **157** (2001), 91-163.

[16] W. Hudson & B. Zinner, *Existence of traveling waves for a generalized discrete Fisher's equation*, Comm. Appl. Nonlinear Anal. **1** (1994), 23–46.

[17] Ya.I. Kanel', *On the stabilization of Cauchy problem for equations arising in the theory of combustion,* Mat. Sbornik **59** (1962), 245-288.

[18] A.L. Kay, J.A. Sherratt, & J.B. McLeod, *Comparison theorems and variable speed waves for a scalar reaction-diffusion equation*, Proc. Royal Soc. Edinburgh, **131A** (2001), 1133-1161.

[19] A.N. Kolmogorov, I.G. Petrovsky, & N.S. Piskunov, *Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un probléme biologique*, Bull. Univ. Moskov. Ser. Internat., Sect. **A 1** (1937), 1-25. also in Dynamics of Curved Fronts (P. Peclé Ed.), Academic Press, San Diego, 1988 [Translated from Bulletin de l'Université d'État á Moscou, Ser. Int., Sect. **A. 1** (1937)]

[20] S.J.A. Malham & M. Oliver, *Accelerating fronts in autocatalysis*, Proc. R. Soc. Lond. A **456** (2000), 1609-1624.

[21] D.J. Needham & A.N. Barnes, *Reaction-diffusion and phase waves occurring in a class of scalar reaction-diffusion equations*, Nonlinearity **12** (1999), 41-58.

[22] N. Shigesada & K. Kawasaki, Biological invasions: theory and practice, Oxford Series in Ecology and Evolution, Oxford, Oxford University Press, 1997.

[23] J.A. Sherratt & B.P. Marchant, *Algebraic decay and variable speeds in wavefront solutions of a scalar reaction-diffusion equation*, IMA J. Appl. Math. **56** (1996), 289-302.

[24] B. Shorrocks & I.R. Swingland, Living in a Patch Environment, Oxford Univ. Press, New York, 1990.

[25] K. Uchiyama, *The behavior of solutions of some diffusion equation for large time*, J. Math. Kyoto Univ., **18** (1978), 453-508.

[26] H.F. Weinberger, Long-time behavior of a class of biological models, *SIAM J. Math. Anal.* **13** (1982), 353-396.

[27] J. Wu & X. Zou, *Asymptotic and periodic boundary values problems of mixed PDEs and wave solutions of lattice differential equations*, J. Diff. Eqns. **135** (1997), 315–357.

[28] B. Zinner, *Stability of traveling wavefronts for the discrete Nagumo equations,* SIAM J. Math. Anal. **22** (1991), 1016–1020.

[29] B. Zinner, *Existence of traveling wavefronts for the discrete Nagumo equations,* J. Diff. Eqns. **96** (1992), 1–27.

[30] B. Zinner, G. Harris, & W. Hudson, *Traveling wavefronts for the discrete Fisher's equation*, J. Diff. Eqns. **105** (1993), 46–62.

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