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A DIAGRAM REALIZATION OF COMPLEX REFLECTION GROUPS AND SCHUR-WEYL DUALITIES

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Abstract. We describe elements of the complex reflection group $G(r,1,n)$ using r-decorated n-diagrams. We also construct a representation of $G(r,1,n)$ on a tensor product space which commutes with the action of Lie group $G = GL(m_0)\times\cdots\times GL(m_{r-1})$. We also give a Schur-Weyl duality for the complex reflection group $G(r,1,n)$ and G on the tensor product space.

1. Coxeter groups and Complex reflection groups

A (*real*) *reflection* is a linear transformation of $\mathbb{R}^n$ which is a reflection in some hyperplane. A *Coxeter group* may be defined as a group generated by (real) reflections in Euclidean space. More formally we define a Coxeter group G with a presentation by a system of generators $\{s_1, \dots, s_n\}$ and relations

$$s_i^2 = 1 \qquad \text{for } 1 \le i \le n \ , \tag{1.1}$$

$$(s_i s_j)^{m_{ij}} = 1 \qquad \text{for } 1 \le i \ne j \le n, \tag{1.2}$$

where each m_{ij} is either ∞ or a positive integer greater than 2. The following theorem, due to Coxeter, is well-known and the two definitions of Coxeter groups are equivalent.

Theorem 1.3 (See, for example, [2]). *A group is a finite Coxeter group if and only if it is a finite group generated by real reflections.*

Let G be a Coxeter group, and A be an $n\times n$ symmetric matrix whose (i,j)-entry is m_{ij} in (1.2) with 1's on the diagonal. We call A is the *Coxeter matrix* associated to G. A Coxeter matrix is also encoded as a *Coxeter graph* with n vertices. The i and j vertices are connected by an edge if $m_{ij} \ge 3$, and the edge is labeled with the value of m_{ij}.

A finite Coxeter group G is *irreducible* if it cannot be written as a direct product of finite Coxeter groups. A Coxeter group is irreducible if and only if its Coxeter graph is indecomposable, i.e., the graph does not have two or more connected components.

Every Weyl group of a finite dimensional simple Lie algebra is a Coxeter group since it is generated by reflections in a Euclidean space. Moreover a Coxeter graph

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can be obtained from a Dynkin diagram by replacing every double edge with an edge labeled 4 and every triple edged by an edge labeled 6. More details including a complete classifications about Weyl groups and Dynkin diagrams can be found, for example, in [4].

Proposition 1.4 ([4]). *An indecomposable Dynkin diagram correspond to one of the following types;*

$$A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2.$$

The following is well-known classification theorem of (irreducible) finite Coxeter groups.

Theorem 1.5 (See for example, [2] or [5]). *An irreducible finite Coxeter group corresponds to one of the following types;*

$$A_n, B_n, D_n, E_6, E_7, E_8, F_4, H_3, H_4, I_2(m).$$

Note the Dynkin diagrams of type B_n and C_n give the same Coxeter graph. And the Dynkin diagram of type G_2 is associated to the Coxeter graph of type $I_2(3)$, which is the only Coxeter graph obtained from a Dynkin diagram among those of type $I_2(m)$.

The irreducible finite Coxeter groups of type A_n, B_n and D_n are of *classical type* and the others are of *exceptional type*. The Coxeter group of type A_{n-1} is the symmetric group S_n, and the group of type B_n is the hyperoctahedral group $\mathbb{Z}_2 \wr S_n$, the wreath product of the group $\mathbb{Z}_2$ and the symmetric group S_n. And the Coxeter group of type D_n is a subgroup of index 2 in the Coxeter group of type B_n. The Coxeter group of type $I_2(m)$ is a dihedral group of order $2m$.

The Coxeter groups play eminent roles in many mathematical subjects such as geometry, representation theory, number theory, knot theory and combinatorics.

A *complex reflection* or *pseudo-reflection* is an invertible linear transformation of $\mathbb{C}^n$ which acts trivially on a hyperplane of $\mathbb{C}^n$. More precisely a complex reflection is an invertible linear transformation in $GL(\mathbb{C}^n)$ of finite order which has exactly one eigenvalue that is not 1. A *complex reflection group* is a group generated by complex reflections in $\mathbb{C}^n$. Any finite Coxeter group can naturally be thought of as a complex reflection group, simply by complexifying (tensoring with the complex numbers $\mathbb{C}$) the vector space on which the real reflection acts. But there are many complex reflection groups which do not arise in this way.

Irreducible complex reflection groups have been classified by Shephard and Todd (see [7] or [3]). They contain one infinite family $G(r,p,n)$ depending on 3 parameters r, p and n, and 34 *exceptional* groups (which have been given by Shephard and Todd a number which actually varies from 4 to 37, and covers also the exceptional Coxeter groups, e.g., the Coxeter group E_8 is the group of Shephard-Todd number 37).

Now let us give a definition of the complex reflection group $G(r,p,n)$.

Definition 1.6. Let r, p, d and n be positive integers such that $pd = r$. The complex reflection group $G(r,p,n)$ is the set of $n \times n$ matrices such that

(a) the entries are either 0 or rth roots of unity,

(b) there is exactly one nonzero entry in each row and each column,
(c) the dth power of the product for the nonzero entries is 1.

By an $n \times n$ permutation matrix, we mean an $n \times n$ matrix whose entries are either 0 or 1 with exactly one nonzero entry in each row and column. The following proposition provides an alternative realization of $G(r, p, n)$.

Proposition 1.7 (See, for example, [3]). *Assume r, p, d and n are positive integers with $pd = r$. Let Π_n be the group of all $n \times n$ permutation matrices. We also let $A(r, p, n)$ be the group of all $n \times n$-matrices $(a_{i,j})$ such that $a_{ij} = \theta_i \delta_{ij}$, where $\theta_i^r = 1$ for each i, and $(\det(a_{i,j}))^d = 1$. Then Π_n normalizes $A(r, p, n)$ and*

$$G(r, p, n) = A(r, p, n)\Pi_n.$$

The following theorem is also well-known.

Theorem 1.8 (See [3]). *The group $G(r, p, n)$ is a normal subgroup of $G(r, 1, n)$ of index p and*

$$\mid G(r, p, n) \mid = dr^{n-1}n!.$$

Moreover

(a) *$G(1, 1, n)$ is isomorphic to S_n, the Weyl group of type A_{n-1},*
(b) *$G(2, 1, n)$ is isomorphic to $\mathbb{Z}_2 \wr S_n$, the Weyl group of type B_n,*
(c) *$G(r, 1, n)$ is isomorphic to $\mathbb{Z}_r \wr S_n$,*
(d) *$G(2, 2, n)$ is isomorphic to the Weyl group of type D_n.*

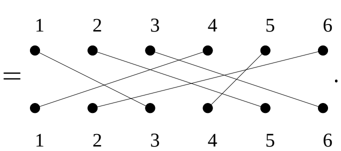
2. Symmetric groups and diagrams

In many circumstances, it is convenient to represent permutations of the symmetric group S_n, the Coxeter group of type A_{n-1}, by n-diagrams. We give a little explanation of n-diagram in this section.

Consider a graph with two rows of n vertices each, one above the other, and n edges such that each vertex in the top row is incident to precisely one vertex in the bottom row. There is a natural one-to-one correspondence between such n-diagrams and elements of the symmetric group S_n, which is illustrated by the following example;

Example 2.1.

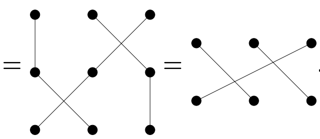
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 1 & 4 & 2 \end{pmatrix} =$$



Notice that the ith vertex in top row is incident to the $\sigma(i)$th vertex in bottom row.

Now we describe the multiplication of the group using diagrams. Let d_1 and d_2 be the diagrams corresponding to permutations σ_1 and σ_2 respectively. Place d_1 below d_2 and identify the vertices in the bottom row of d_2 with the corresponding vertices in the top row of d_1. The resulting diagram corresponds to the product $\sigma_1\sigma_2$. For example,

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (12)(23) =$$



Note that we stack the left element of the product on the bottom of the diagram and the right element on the top.

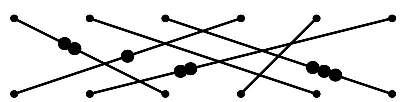
There are some benefits from our identification of a permutation $\sigma \in S_n$ with its n-diagram. For example, the *length* $\ell(\sigma)$ of a permutation σ is the number of crossings of edges in the n-diagram identified with σ. And an expression $\sigma = s_{i_1} \cdots s_{i_j}$ of $\sigma \in S_n$ is reduced if $j = \ell(\sigma)$, where s_k denotes the transposition $(k\ k+1)$. For example, the n-diagram shown in Example 2.1 has 9 edge crossings, and so $\ell(\sigma) = 9$, and

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 1 & 4 & 2 \end{pmatrix} = s_2 s_1 s_3 s_4 s_3 s_2 s_5 s_4 s_3,$$

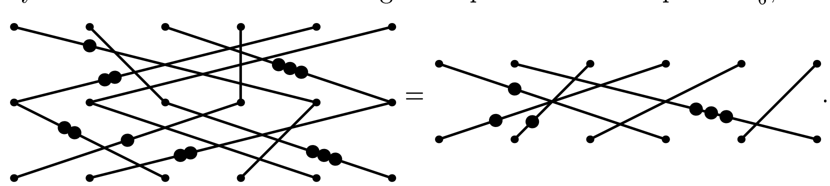
where the product on the right is a reduced expression for σ.

3. Complex reflection group $G(r,1,n)$ and diagrams

In this section, we describe the complex reflection group $G(r,1,n)$ using r-decorated n-diagrams. The r-decorated n-diagrams are n-diagrams of permutations with each edge is decorated by i dots, where $0 \leq i \leq r-1$. The following is a typical r-decorated 6-diagram for $r \geq 4$.



We let $\mathcal{A}_n^r$ be an associative algebra over $\mathbb{C}$ whose basis consists of r-decorated n-diagrams. The product $d_1 d_2$ of two r-decorated n-diagrams d_1 and d_2 are obtained as follows: Place d_1 below d_2 and identify the vertices in the bottom row of d_2 with the corresponding vertices in the top row of d_1. The resulting diagram corresponds to the product $d_1 d_2$. The decoration of an edge on the resulting diagram is obtained as follows; we count all decoration dots on two edges in d_1 and d_2 which result the edge in the product $d_1 d_2$. Say there are s dots total. If $0 \leq s \leq r-1$, then decorate the edge in the resulting diagram with s dots. On the other hand, if $s \geq r$, then use only $s-r$ dots to decorate the edge. We provide an example in $\mathcal{A}_6^4$;



The following proposition on the dimension of $\mathcal{A}_n^r$ is clear from its construction.

Proposition 3.1. *The dimension of $\mathcal{A}_n^r$ over $\mathbb{C}$ is $r^n|S_n| = (n!)r^n$.*

Now we give a presentation of $\mathcal{A}_n^r$.

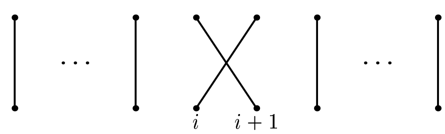
Proposition 3.2. *The algebra $\mathcal{A}_n^r$ has a presentation as a unital associative algebra over $\mathbb{C}$ generated by $\mathfrak{s}_1, \ldots, \mathfrak{s}_{n-1}$ and $\mathfrak{g}_1, \ldots, \mathfrak{g}_n$ and relations*

(3.3a)	$\mathfrak{s}_i^2 = 1,$	*for* $i = 1, \ldots, n-1,$
(3.3b)	$\mathfrak{s}_i\mathfrak{s}_{i+1}\mathfrak{s}_i = \mathfrak{s}_{i+1}\mathfrak{s}_i\mathfrak{s}_{i+1},$	*for* $i = 1, \ldots, n-2,$
(3.3c)	$\mathfrak{s}_i\mathfrak{s}_j = \mathfrak{s}_j\mathfrak{s}_i,$	*if* $\lvert i-j\rvert \geq 2,$
(3.3d)	$\mathfrak{g}_i^r = 1,$	*for* $i = 1, \ldots, n,$
(3.3e)	$\mathfrak{g}_i\mathfrak{s}_i = \mathfrak{s}_i\mathfrak{g}_{i+1},$	*for* $i = 1, \ldots, n-1,$
(3.3f)	$\mathfrak{g}_{i+1}\mathfrak{s}_i = \mathfrak{s}_i\mathfrak{g}_i,$	*for* $i = 1, \ldots, n-1,$
(3.3g)	$\mathfrak{g}_i\mathfrak{s}_j = \mathfrak{s}_j\mathfrak{g}_i,$	*if* $\lvert i-j\rvert \geq 2$ *or* $i = j-1,$
(3.3h)	$\mathfrak{g}_i\mathfrak{g}_j = \mathfrak{g}_j\mathfrak{g}_i,$	*for* $i, j = 1, \ldots, n,$
(3.3i)	$\mathfrak{g}_1\mathfrak{s}_1\mathfrak{g}_1\mathfrak{s}_1 = \mathfrak{s}_1\mathfrak{g}_1\mathfrak{s}_1\mathfrak{g}_1.$	

Note that the generators $\mathfrak{s}_i$, for $i = 1, \ldots, n-1$, will generate a subalgebra which is isomorphic to the group algebra $\mathbb{C}[S_n]$ of the symmetric group S_n.

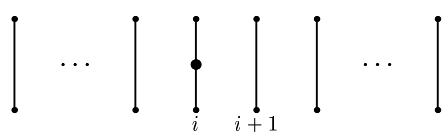
Proof. Let $\mathcal{F}$ be the free algebra generated by $\mathfrak{s}_1, \ldots, \mathfrak{s}_{n-1}$ and $\mathfrak{g}_1, \ldots, \mathfrak{g}_n$. We let $\mathcal{B}$ be the quotient algebra of $\mathcal{F}$ by equations (3.3).

We identify the following element



$\in \mathcal{A}_n^r$

with $\mathfrak{s}_i$, for $i = 1, \ldots, n-1$, and



$\in \mathcal{A}_n^r$

with $\mathfrak{g}_i$, for $i = 1, \ldots, n$. Then we can easily verify that equations in (3.3) are satisfied by these elements. Because a r-decorated n-diagram is a product of these elements, we have that $\dim \mathcal{A}_n^r$ is less than equal to $\dim \mathcal{B}$.

Now it is sufficient to show that

$$\dim \mathcal{B} \leq \dim \mathcal{A}_n^r = (n!)r^n.$$

Because of the equations (3.3e), (3.3f) and (3.3g), every word on $\mathfrak{s}_1, \ldots, \mathfrak{s}_{n-1}$ and $\mathfrak{g}_1, \ldots, \mathfrak{g}_n$ in $\mathcal{B}$ may be reduced to a

$$\mathfrak{g}_{i_1} \cdots \mathfrak{g}_{i_p}\mathfrak{s}_{j_1} \cdots \mathfrak{s}_{j_q} \tag{3.4}$$

for some nonnegative integers p and q, where $1 \leq i_s \leq n$ for $s = 1, \ldots, p$, and $1 \leq j_t \leq n-1$ for $t = 1, \ldots, q$. Then because of the equations (3.3d), and (3.3h), the word of the form (3.4) may be reduced to a

$$\mathfrak{g}_1^{\ell_1} \cdots \mathfrak{g}_n^{\ell_n}\mathfrak{s}_{j_1} \cdots \mathfrak{s}_{j_q},$$

where $0 \leq \ell_i < r$ for $1 \leq i \leq n$. Then, since $\mathfrak{s}_1, \ldots, \mathfrak{s}_{n-1}$ generates $\mathbb{C}[S_n]$, we have

$$\dim \mathcal{B} \leq\mid S_n \mid r^n = (n!)r^n.$$

This completes the proof. ◻

Now the following theorem gives a diagram realization of the complex reflection group $G(r,1,n)$.

Proposition 3.5. *The group algebra $\mathbb{C}[G(r,1,n)]$ of $G(r,1,n)$ and $\mathcal{A}_n^r$ are isomorphic as associative algebras.*

Proof. We recall subsets Π_n and $A(r,1,n)$ in $G(r,1,n)$ from Proposition 1.7. Let s_i the permutation matrix in Π_n corresponding to the transposition $(i\,i+1)$ for $i = 1, \ldots, n-1$. We let $\omega \in \mathbb{C}$ be a primitive rth root of unity, and g_i, $i = 1, \ldots, n$, be the diagonal matrix in $A(r,1,n)$ whose ℓth diagonal entry is 1 for $\ell \neq i$ and ω for $\ell = i$.

Now we define a linear map $f : \mathcal{A}_n^r \longrightarrow \mathbb{C}[G(r,1,n)]$ by

$$\mathfrak{g}_i \mapsto g_i, \quad \text{and } \mathfrak{s}_i \mapsto s_i,$$

and extending it by linearity. Then it is not difficult to show that f preserves relations (3.3), and f is a homomorphism of algebras. And f is surjective since g_i generates $A(r,1,n)$ and s_i generates Π_n. Then because $\dim A_n^r = \dim [G(r,1,n)] = (n!)r^n$, f is injective also. Therefore f is an isomorphism between algebras. ◻

4. Schur-Weyl Duality of $G(r,1,n)$

In this section, we obtain a representation of $G(r,1,n)$ on a tensor product space and show a Schur-Weyl duality theorem on the space.

First we obtain a representation of $\mathcal{A}_n^r$ on the tensor product space. Let V be a vector space over $\mathbb{C}$ such that $V = V_0 \oplus \cdots \oplus V_{r-1}$ where each V_i is an m_i-dimensional $\mathbb{C}$-vector space so that $\dim V = m = m_0 + \cdots + m_{r-1}$.

Fix a primitive rth root of unity $\omega \in \mathbb{C}$. Let $g : V \longrightarrow V$ be a linear map such that $g(v) = \omega^\ell v$ if $v \in V_\ell$ for $\ell = 0, 1, \ldots, r-1$. We also let $s : V^{\otimes 2} \longrightarrow V^{\otimes 2}$ be a map such that $s(w_1 \otimes w_2) = w_2 \otimes w_1$. Now let $\mathfrak{s}_i \in \mathcal{A}_r^n$ act on $V^{\otimes n}$ by a place permutation $s_i = 1^{\otimes(i-1)} \otimes s \otimes 1^{(n-i-1)}$, for $i = 1, \ldots, n-1$, and $\mathfrak{g}_i \in \mathcal{A}_n^r$ by $g_i = 1^{\otimes(i-1)} \otimes g \otimes 1^{\otimes(n-i)}$, for $i = 1, \ldots, n$. It is not hard to show that the actions s_i's and g_i's satisfy equations (3.3) on $V^{\otimes n}$, and this action determines a representation of $\mathcal{A}_n^r$ on $V^{\otimes n}$.

Proposition 4.1. *For $i = 0, \ldots, r-1$, let V_i be an m_i-dimensional vector space over $\mathbb{C}$. Write $V = V_0 \oplus V_1 \oplus \cdots \oplus V_{r-1}$. Then there exists a representation $\Psi : \mathcal{A}_n^r \longrightarrow \mathrm{End}(V^{\otimes n})$ of $\mathcal{A}_n^r$ on $V^{\otimes n}$ given by*

$$\Psi(\mathfrak{s}_i) = 1^{\otimes(i-1)} \otimes s \otimes 1^{(n-i-1)},$$

and

$$\Psi(\mathfrak{g}_i) = 1^{\otimes(i-1)} \otimes g \otimes 1^{\otimes(n-i)}.$$

The Lie group $GL(V)$ also acts on V by matrix multiplication, and this action may extend to an action on $V^{\otimes n}$. As a subgroup $G = GL(m_0) \times \cdots \times GL(m_{r-1}) < GL(V)$ also acts on V and its tensor product space $V^{\otimes n}$. Note each V_ℓ is an G-invariant subspace of V, and the subspaces $V_{\ell_1} \otimes \cdots \otimes V_{\ell_k}$, where $\ell_i \in \{0, 1, \ldots, r-1\}$, of $V^{\otimes n}$ are invariant under the action of G on $V^{\otimes n}$. Because $\mathfrak{g}_i$ acts as a scalar multiplication on each $V_{\ell_1} \otimes \cdots \otimes V_{\ell_k}$, for $\ell_i \in \{0, 1, \ldots, r-1\}$, the action of G commutes with the action of $\mathfrak{g}_i$ on $V^{\otimes n}$ for each $i = 1, \ldots, n$.

Furthermore, it is well-known that the action of $GL(V)$ on $V^{\otimes n}$ commutes with the action of s_i on $V^{\otimes n}$ for each $i = 1, \ldots, n-1$. Thus the whole action of $\mathcal{A}_n^r$ commutes with the action of G on $V^{\otimes n}$, and we have

$$\Phi\left(\mathbb{C}[G]\right) \subseteq \operatorname{End}_{\mathcal{A}_n^r}(V^{\otimes n}) = \{x \in \operatorname{End}(V^{\otimes n}) \mid x\Psi(y) = \Psi(y)x, \quad \forall y \in \mathcal{A}_n^r\}, \tag{4.2}$$

and

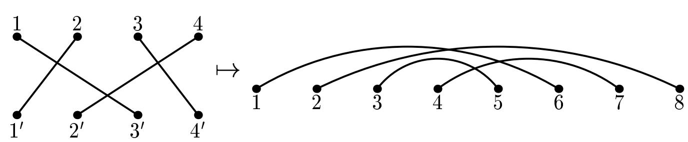
$$\Psi\left(\mathcal{A}_n^r\right) \subseteq \operatorname{End}_G(V^{\otimes n}) = \{x \in \operatorname{End}(V^{\otimes n}) \mid x\Phi(y) = \Phi(y)x, \quad \forall y \in \mathbb{C}[G]\}, \tag{4.3}$$

where $\Phi : \mathbb{C}[G] \longrightarrow \operatorname{End}(V^{\otimes n})$ is the representation of the group algebra $\mathbb{C}[G]$ on $V^{\otimes n}$ induced by the action of G. In the rest of this paper, we will show that $\mathbb{C}[G]$ and $\mathcal{A}_n^r$ actually determine the full centralizer of each other.

We let $\{e_i^\ell\}_{i=1,\ldots,m_\ell}$ be a basis of V_ℓ for $\ell = 0, \ldots, r-1$. The dual space V^* of V is also a G-module, and we let $\{\left(e_i^\ell\right)^*\}_{i=1,\ldots,m_\ell}$ be a basis of V^* which is dual to $\{e_i^\ell\}_{i=1,\ldots,m_\ell}$ for each $\ell = 0, \ldots, r-1$. From now on, we assume that *we have enough basis vectors* $\{e_i^\ell\}_{i=1,\ldots,m_\ell}$*, for each* $\ell = 0, \ldots, r-1$*, so that* Φ *and* Ψ *are faithful representations.*

A $2n$-*one-factor* is a graph with one row of $2n$ vertices and n edges such that each vertex is incident to precisely one edge. We assume the vertices are numbered 1 to $2n$ from left to right and often represent $2n$-one-factor d as a sequence of pairs $d = \big((l_1, r_1), \ldots, (l_n, r_n)\big)$, where $l_i, r_i \in \{1, \ldots, 2n\}$ give the left and right vertex respectively of each edge. Among $2n$-one-factors, a $2n$*-one-factor of an* n*-diagram* is the one with $1 \le l_i \le n$ and $1 + n \le r_i \le 2n$ for each $i = 1, \ldots, n$. There is a bijection between a set consisting of all n-diagrams and a set of all $2n$-one-factors of n-diagrams. The bijection u is called as the *unfolding map*, and is explained by the following example;

(4.4) $u :$



The unfolding map converts an n-diagram into a $2n$-one-factor by relabeling and repositioning the vertices so that the sequence $\{1, 2, \ldots, n, n', (n-1)', \ldots, 1'\}$ becomes $\{1, \ldots, 2n\}$. More details about $2n$-one-factor and the unfolding map can be found in [1].

Now we consider the isomorphism $\operatorname{End}_G(V^{\otimes n}) \cong \left((V^*)^{\otimes n} \otimes V^{\otimes n}\right)^G$ between spaces of G-invariants. The classical Schur-Weyl duality says there exists a basis of

$$\left((V_\ell^*)^{\otimes n} \otimes (V_\ell)^{\otimes n}\right)^{GL(m_\ell)}$$

indexed by $2n$-one-factors of n-diagrams for each $\ell = 0, \ldots, r-1$. For example, $(V_\ell^* \otimes V_\ell)^{GL(m_\ell)}$ is one dimensional which is spanned by a $GL(m_\ell)$-invariant $\sum_{i=1}^{m_\ell} \left(e_i^\ell\right)^* \otimes e_i^\ell$.

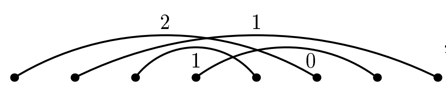
Moreover we claim that there is a linear basis of $\left((V^*)^{\otimes n} \otimes V^{\otimes n}\right)^G$ indexed by $2n$-one-factors of n-diagrams whose edges have colors from r-different colors $\{0, 1, \ldots, r-1\}$, which we call *r-colored $2n$-one factors of n-diagrams.* For an r-colored $2n$-one-factor d of an n-diagram, we associate a G-invariant in $(V^*)^{\otimes n} \otimes V^{\otimes n}$ as follows: Let $d = \big((l_1, r_1), \ldots, (l_n, r_n)\big)$ be a $2n$-one-factor of an n-diagram such that $1 \le l_t \le n$ and $n+1 \le r_t \le 2n$, for $t = 1, \ldots, n$, and whose ith edge (l_i, r_i) has a color $\ell_i \in \{0, 1, \ldots r-1\}$. Then associated to d is a tensor

$$w_d = \sum_{\substack{i_1, \ldots, i_n \\ 1 \le i_t \le m_{\ell_t} \text{ for } t = 1, \ldots, n}} w_{i_1, \ldots, i_n} \in \left((V^*)^{\otimes n} \otimes V^{\otimes n}\right)^G,$$

where $w_{i_1, \ldots, i_n} = v_1 \otimes \cdots \otimes v_{2n}$ is a simple tensor in $(V^*)^{\otimes n} \otimes V^{\otimes n}$ such that

$$v_j = \begin{cases} e_{i_t}^{\ell_t} & \text{if } j = l_t \le n, \\ \left(e_{i_t}^{\ell_t}\right)^* & \text{if } j = r_t \ge n+1. \end{cases}$$

As an example, for a 3-colored 8-one-factor



where colors of its edges are indicated by the number on them, the associated G-invariant is

$$\sum_{i_1=1}^{m_2} \sum_{i_2=1}^{m_1} \sum_{i_3=1}^{m_1} \sum_{i_4=1}^{m_0} e_{i_1}^2 \otimes e_{i_2}^1 \otimes e_{i_3}^1 \otimes e_{i_4}^0 \otimes \left(e_{i_3}^1\right)^* \otimes \left(e_{i_1}^2\right)^* \left(e_{i_4}^0\right)^* \left(e_{i_2}^1\right)^*.$$

Thus the dimension of $\mathrm{End}_G(V^{\otimes n}) \cong \left((V^*)^{\otimes k} \otimes V^{\otimes n}\right)^G$ is equal to the number of r-colored $2n$-one-factors of n-diagrams, and

$$\dim \mathrm{End}_G(V^{\otimes n}) = (n!) r^n,$$

because the number of $2n$-one-factors of n-diagrams is $n!$. Note the dimension of $\mathcal{A}_n^r$ is $(n!)r^n$ also. Thus, from (4.2) and (4.3), we have the following Schur-Weyl reciprocity for $\mathcal{A}_n^r$ (or for $G(r,1,n)$) and $G = GL(m_0) \times \cdots \times GL(m_{r-1})$.

Theorem 4.5. *For $i = 0, \ldots, r-1$, we let V_i be an m_i-dimensional vector space over $\mathbb{C}$. Let $V = V_0 \oplus V_1 \oplus \cdots \oplus V_{r-1}$. Then*

$$\Phi\left(\mathbb{C}[GL(m_0) \times \cdots \times GL(m_{r-1})]\right) = \mathrm{End}_{\mathcal{A}_n^r}(V^{\otimes n}),$$

and

$$\Psi\left(\mathcal{A}_n^r\right) = \mathrm{End}_{GL(m_0) \times \cdots \times GL(m_{r-1})}(V^{\otimes n}).$$

A quantum version of Theorem 4.5 was proved by Sakamoto and Shoji in [6]. But we were not able to find a proof of Theorem 4.5 in the literature, and we present it in this paper.

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