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ON THE DEAD-CORE PROBLEM

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ABSTRACT. We study the dead-core problem for the semilinear heat equation with strong absorption. We find that the dead-core rate is faster than the one given by the corresponding ODE. This stands in sharp contrast with known results for the related extinction, quenching and blow-up problems. Some applications of this result are given for blow-up problems with perturbations. This work is based on a joint work with Philippe Souplet (Math. Ann. **331** (2005), pp. 651-667).

1. INTRODUCTION

Consider the following initial boundeary value problem:

$$u_t = u_{xx} + f(u),\ t > 0,\ -1 < x < 1, \tag{1}$$

$$u(t, \pm 1) = k,\ t > 0, \tag{2}$$

$$u(0, x) = u_0(x),\ -1 \le x \le 1, \tag{3}$$

where $f(u) = \pm u^p$ with $p \in \mathbf{R}$, $u_0 \ge 0$ in $[-1, 1]$, and $k \ge 0$.

For the solution of (1)-(3), the following situations can happen:

(a) **blow-up:** the solution becomes unbounded (hence singular) in finite time (e.g., $f(u) = u^p$ with $p > 1$ and $k \ge 0$);

(b) **quenching:** the solution reaches zero and its time derivative becomes unbounded (hence singular) in finite time (e.g., $f(u) = -u^p$ with $p < 0$, $u_0 > 0$ in $[-1, 1]$, and $k > 0$);

(c) **dead-core:** the solution reaches zero in finite time, but stays regular for all time (e.g., $f(u) = -u^p$ with $0 < p < 1$, $u_0 > 0$ in $[-1, 1]$, and $k > 0$).

We are interested in the case of dead-core (cf. [15, 13, 1, 19]):

$$\begin{cases} u_t = u_{xx} - u^p, \quad t > 0, \quad -1 < x < 1, \\ u(t, \pm 1) = k, \quad t > 0, \\ u(0, x) = u_0(x), \quad -1 \le x \le 1, \end{cases} \tag{4}$$

where $0 < p < 1$, $k > 0$, and u_0 is assumed to satisfy

$$u_0 \in C([-1, 1]),\ 0 < u_0 \le k \text{ in } [-1, 1],\ u_0(\pm 1) = k.$$

Note that problem (4) admits a unique, globally defined classical solution $u \ge 0$ for all u_0. It is well-known that a region without reactant (i.e. $u = 0$), called

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dead-core, appears in finite time for suitably small initial data. Moreover, this occurs for all u_0 if k is small enough (cf. [1]). The intuitive explanation of this phenomenon is that the reaction (with rate u^{p-1}) is strong enough for small values of u, to prevent the diffusion from drawing the reactant from the boundary.

Denote by $T = T(u_0)$ the **dead-core time**, i.e.

$$T := \min\{t > 0 \mid m(t) = 0\}, \quad m(t) := \min_{|x|\le 1} u(t,x)$$

Our main interest is to investigate the **dead-core rate** of u, i.e. the behavior of the function $m(t)$ as $t \to T^-$.

Solving the ODE $v' = -v^p$, a special solution with dead-core time T is given by:

$$v(t) = \kappa(T-t)^\alpha,\ t < T, \quad \kappa := \alpha^{-\alpha},\ \alpha := 1/(1-p).$$

Since the reaction dominate near $t = T$ for small values of u, one might be tempted to conjecture that

$$m(t) \sim \kappa(T-t)^\alpha, \quad t \to T^-. \tag{5}$$

Indeed, such guesses are true for the blow-up rate, in which

$$\|u(t)\|_\infty := \|u(\cdot,t)\|_\infty \sim (p-1)^{-1/(p-1)}(T-t)^{-1/(p-1)},$$

(cf. [20, 6, 7, 16]), and the quenching rate (i.e., (5) holds, cf. [8, 3, 9, 14]).

Also, for another closely related problem, namely, the **extinction** problem:

$$\begin{cases} u_t = u_{xx} - u^p, \quad t > 0, \quad x \in \mathbf{R}, \\ u(0,x) = u_0(x), \quad x \in \mathbf{R}, \end{cases}$$

with $0 < p < 1$ and $u_0 \ge 0$ with compact support, it is well-known that the solution vanishes identically after a finite time T_e. Moreover, if u_0 is even and non-increasing in $|x|$, then

$$\|u(t)\|_\infty \sim \kappa(T_e - t)^\alpha, \quad t \to T_e^-.$$

See, e.g., [5, 18, 11].

Surprisingly, this expected ODE behavior fails for the dead-core problem (4). In [10], we obtain the following result of fast dead-core rate.

Theorem 1. *Assume that*

(6) u_0 *is even and non-decreasing in* $|x|$ *and that* $T(u_0) < \infty$.

Then

$$\lim_{t\to T^-} (T-t)^{-\alpha} m(t) = 0.$$

We remark that the exact rate of vanishing of $m(t)$ is still unknown. We conjecture that there should not exist a single rate for all solutions.

We also remark that a fast blow-up rate occurs for the equation $u_t = \Delta u + u^p$ with certain super-critical p in the higher spatial dimension (cf. [12, 4, 17]).

2. Applications to Blow-up Problems

We first consider the following problem

$$(7)\qquad \begin{cases} w_t = w_{xx} + w^q - \lambda w_x^2/w, \quad t>0, \quad -1<x<1, \\ w(t,\pm 1) = 1, \quad t>0, \\ w(0,x) = w_0(x), \quad -1 \le x \le 1, \end{cases}$$

where $q>1$, $\lambda \ge 0$, $w_0 \in C^1([-1,1])$, $w_0 \ge 1$ in $[-1,1]$, and $w_0(\pm 1) = 1$.

It is known that the solution of (7) blows up in a finite time T, i.e., $\lim_{t\to T^-} \|w(t)\|_\infty = \infty$, if w_0 is suitably large.

Using Theorem 1, we can show that there is a threshold value of λ, namely $\lambda = q$, above which the blow-up rate becomes faster.

Theorem 2 ([10]). *Suppose that the solution w of (7) blows up in a finite time T.*
(i) If $0 \le \lambda \le q$ and $w_t \ge 0$, then

$$C_1 \le (T-t)^{1/(q-1)} \|w(t)\|_\infty \le C_2, \quad 0<t<T,$$

for some constants $C_1, C_2 > 0$.
(ii) If $\lambda > q$ and $w_0(x)$ is even and non-increasing in $|x|$, then

$$\|w(t)\|_\infty (T-t)^{1/(q-1)} \to \infty \quad \text{as} \quad t \to T^-.$$

Proof. The proof can be divided into two cases by different transformations as follows.
Case $\lambda \neq 1$: we take

$$w = au^m, \ m = 1/(1-\lambda), \ a = |m|^{1/(q-1)}, \ p = (\lambda - q)/(\lambda - 1).$$

Then $u_t = u_{xx} + u^p$ with $p>1$, if $\lambda<1$; whereas $u_t = u_{xx} - u^p$ with $p<0$, if $1<\lambda<q$; with $p=0$, if $\lambda = q$; with $0<p<1$, if $\lambda > q$. Hence the results of [6, 8, 3, 9, 10] can be applied.
Case $\lambda = 1$: we take $u(t,x) = a \ln w(t/a, x/\sqrt{a})$, $a = q-1$, then $u_t = u_{xx} + e^u$ and the result of [6] can be applied. □

Next, we consider the problem

$$(8)\qquad \begin{cases} w_t = w_{xx} + e^w - \lambda |w_x|^2, \quad t>0, \quad -1<x<1, \\ w(t,\pm 1) = 0, \quad t>0, \\ w(0,x) = w_0(x), \quad -1 \le x \le 1, \end{cases}$$

where $\lambda \ge 0$, $w_0 \in C_0^1([-1,1])$ with $w_0 \ge 0$.

Similarly, the solution w of (8) blows up in a finite time $T = T(w_0)$, if w_0 is suitably large. Also, we find a similar threshold phenomenon in terms of λ for this problem.

Theorem 3 ([10]). *Suppose that the solution w of (8) blows up in a finite time T.*
(i) If $0 \le \lambda \le 1$ and $w_t \ge 0$, then

$$C_1 \le \|w(t)\|_\infty + \log(T-t) \le C_2, \quad 0<t<T,$$

for some constants C_1, C_2.
(ii) If $\lambda > 1$ *and* $w_0(x)$ *is even and non-increasing in* $|x|$, *then*

$$\|w(t)\|_\infty + \log(T-t) \to \infty \quad \text{as } t \to T^-.$$

Proof. Consider $J := w_t - \epsilon e^w$ for $\lambda \le 1$ and apply the maximum principle.

Consider $u(t,x) := ke^{-\lambda w(t,x)}$, $k := \lambda^{-\lambda}$, if $\lambda > 1$. Then u is the solution of problem (4) with $p := 1 - 1/\lambda \in (0,1)$, $k := \lambda^{-\lambda}$, and $u_0(x) := ke^{-\lambda w_0(x)}$. □

3. The proof of Theorem 1

First, we recall the following proposition from [10]. It shows a single-point dead-core and describes the dead-core profile near $x = 0$. We shall always assume (6).

Proposition 1. *There exist* $c_1, c_2 > 0$ *(depending on* u*) such that*

$$\left[m(t)^{1-p} + c_1 x^2\right]^\alpha \le u(t,x) \le \left[m(t)^{\frac{1-p}{2}} + c_2|x|\right]^{2\alpha} \tag{9}$$

for all $T/2 \le t \le T$, $|x| \le 1$. *Moreover,*

$$c_3|x|^{2\alpha} \le u(T,x) \le c_4|x|^{2\alpha}, \qquad |x| \le 1,$$

for some $c_3, c_4 > 0$.

Theorem 1 is an immediate consequence of the following proposition by taking $y = 0$. It also provides precise information on the behavior of solutions near the dead-core point $x = 0$ as $t \to T$.

Proposition 2 ([10])**.** *There holds*

$$\lim_{t \to T^-} (T-t)^{-\alpha} u(t, y\sqrt{T-t}) = V_1(y) := k_p|y|^{2\alpha}, \tag{10}$$

uniformly in $|y| \le R$ *for each* $R > 0$, *where* $k_p = \left[\frac{(1-p)^2}{2(1+p)}\right]^\alpha$.

The proof of Proposition 2 relies on the use of Giga-Kohn self-similar variables:

$$T - t = e^{-s}, \quad y = x/\sqrt{T-t}, \quad u(t,x) = (T-t)^\alpha v(s,y).$$

Then v satisfies the equation

$$v_s = v_{yy} - \frac{y}{2}v_y + \alpha v - v^p \quad \text{in } D, \tag{11}$$

where $D := \{(s,y);\ -\log T < s < \infty,\ |y| < e^{s/2}\}$.

In the self-similar variables, Proposition 2 is equivalent to:

Proposition 3 ([10])**.** *There holds*

$$\lim_{s \to \infty} v(s,y) = V_1(y),$$

uniformly on $\{|y| < R\}$ *for each* $R > 0$.

Note that $V_1(y)$ is an unbounded stationary solution of (11), i.e., a solution of

$$V_{yy} - \frac{y}{2}V_y + \alpha V - V^p = 0, \quad y \in \mathbf{R}. \tag{12}$$

Proof of Proposition 3: We divide the proof into three steps as follows.

Step 1. Identify the stationary solutions of (11). Indeed, if $V \in C^2(\mathbf{R})$ by a solution of (12) such that

$$V = V(|y|), \quad \text{with} \quad V' \geq 0, \quad V > 0 \quad \text{for all } y > 0,$$

and such that V is polynomially bounded, then either $V = V_1$ or $V = V_2 := \kappa$. See also [2].

Step 2. Construct a suitable Lyapunov functional and apply Giga-Kohn's energy method. Let $\rho(y) = e^{-y^2/4}$, $R(s) = e^{s/2}$, and define

$$E(s) = \int_0^{R(s)} \Big(\frac{v_y^2}{2} + \frac{v^{p+1}}{p+1} - \frac{\alpha v^2}{2} \Big)(s, y)\rho(y)\, dy.$$

Using a priori bounds

$$v(s, y) \leq C(1 + |y|)^{\frac{2}{1-p}}, \quad |v_y(s, y)| \leq C(1 + |y|)^{\frac{1+p}{1-p}}$$

for all $-\log(T/2) =: s_0 < s < \infty$, $|y| < e^{s/2}$, we prove that all global solutions of (11) are attracted by the solutions of (12).

Step 3. Discard all the possible limits other than the stationary solution V_1. This follows from the lower bound estimate (9). □

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