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# ON THE EXPLICIT DYNAMICS OF A PLASMA-SHEATH INTERFACE

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ABSTRACT. We present recent results in [13] on the geometric level-set formulation of a plasma-sheath interface arising from the plasma physics. Based on the new formulation of a plasma-sheath, we formally derive the explicit dynamics of the interface from the Euler-Poisson equations using a new formulation

## 1. Introduction

The purpose of this paper is to present a summary of some recent results given in [13] of the dynamics of a plasma-sheath interface (in short "sheath interface") arising from the plasma sheath problem [1, 8, 16, 19, 20, 24, 25]. Consider a plasma consisting of cold ions and hot electrons confined to a domain $\Omega = \mathbb{R}^3 - \Omega_0$ which is exterior to a target $\Omega_0 \subset \mathbb{R}^3$. Both ions and electrons have constant temperature, the temperature of the ions being absolute zero Kelvin. The density of ions is denoted by $n$, the density of electrons is $e^{-\phi}$ (Boltzmann relation [20]), $-\phi$ is the potential field and $\mathbf{u}$ is the velocity of the ions. The dynamics of the plasma are governed by the Euler-Poisson system

$$\begin{cases} \partial_t n + \nabla \cdot (n\mathbf{u}) = 0, \quad (\mathbf{x}, t) \in \Omega \times (0, \infty), \\ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nabla \phi, \\ \varepsilon^2 \Delta \phi = n - e^{-\phi}, \end{cases} \tag{1.1}$$

subject to initial and boundary conditions:

$$\begin{aligned} &(n, \mathbf{u}, \phi)(\mathbf{x}, 0) = (n_0, \mathbf{u}_0, \phi_0)(\mathbf{x}), \quad \mathbf{x} \in \Omega, \\ &\nabla \phi \cdot \boldsymbol{\nu}_0 = \frac{g(\mathbf{x}, t)}{\varepsilon}, \quad (\mathbf{x}, t) \in \partial\Omega_0 \times [0, \infty). \end{aligned} \tag{1.2}$$

Here $\varepsilon$ is proportional to the Debye length $\lambda_D$ [20] and $\boldsymbol{\nu}_0$ is the exterior normal at the target boundary $\partial\Omega_0$. Typically away from the boundary $\partial\Omega_0$, the formal $\varepsilon \to 0$ limit in **(E-P)** can be used to yield the quasi-neutral relation $n = e^{-\phi}$. However near the boundary $\partial\Omega_0$, this quasi-neutrality breaks down (see Section 2) and a *sheath* boundary layer of width $\varepsilon$ forms.

In [18], Ha and Slemrod gave a description of sheath dynamics for the case of planar, cylindrically and spherically symmetric motion, generalizing earlier work of Daube and Riemann [24]. In [13], we generalized the result of [18] to find the dynamics of the sheath interface governed by the equations

$$\boxed{\frac{\delta\psi}{\delta t} = 0, \quad \frac{\delta n}{\delta t} = n\nabla\cdot\boldsymbol{\nu}, \quad (V+1) + \frac{\mathbf{h}\cdot\boldsymbol{\nu}}{n} = -\frac{1}{n}\nabla_s\cdot(V\nabla_s \ln n),}$$

where

(i) the level set $\mathcal{S}(t) = \{(\mathbf{x},t) : \psi(\mathbf{x},t) = 0\}$ is the sheath interface;
(ii) $\frac{\delta}{\delta t} = \partial_t + V\boldsymbol{\nu}\cdot\nabla$ is the normal time derivative following $\mathcal{S}(t)$ and $\nabla_s$ is the surface gradient on $\mathcal{S}(t)$;
(iii) $\boldsymbol{\nu}$ is the exterior unit normal to $\mathcal{S}(t)$. Since $\nabla\cdot\boldsymbol{\nu}$ is twice the mean curvature of $\mathcal{S}(t)$, motion is curvature driven;
(iv) $\mathbf{h}$ is the ion current and $n$ is the ion density on the sheath interface.

Usefulness of such models is seen in studying material processing [20] and in particular the plasma source ion implantation (PSII) technique invented by Conrad and his collaborators [9]. Other applications may be found in the related problems for the modelling of electron beam where again loss of quasi-neutrality is a crucial issue (see [2, 3, 4, 5, 10, 11, 12]).

## 2. Level-set formulation of the plasma-sheath interface

In this section, we recall the level-set formulation [13] of the sheath interface for general three-dimensional targets. This formulation was partly employed in [18] in the case of planar, cylindrical and spherical targets.

First we give a rather elementary description of the plasma sheath. Since the Debye length $\varepsilon$ is a small parameter in (1.1), the Poisson equation suggests that the quasi-neutral relation $n = e^{-\phi}$ should pervade in our problem. Substitution of this relation into (1.1) yields the quasi-neutral system:

$$\begin{cases} \partial_t n + \nabla\cdot(n\mathbf{u}) = 0, \quad (\mathbf{x},t)\in\Omega\times[0,\infty), \\ \partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla(\ln n) = \mathbf{0}, \end{cases} \tag{2.1}$$

with prescribed initial data for $n$ and $\mathbf{u}$ at $t = 0$ and boundary data $\nabla \ln n \cdot \boldsymbol{\nu}_0 = -g$ on $\partial\Omega_0$. In general, the initial-boundary value problem for (2.3) is not well-posed. For example, consider the symmetric cases of planar, cylindrical and spherical targets. In these cases, the Euler-Poisson system (1.1) becomes

$$\begin{cases} \partial_t\rho + \partial_r(\rho u) = 0, \quad r_0 \le r < \infty, \quad t > 0, \\ \partial_t u + \partial_r\left(\frac{u^2}{2}\right) = \partial_r\phi, \\ \varepsilon^2\partial_r(r^\nu\partial_r\phi) = \rho - \rho_e, \quad \rho = nr^\nu, \quad \rho_e = e^{-\phi}r^\nu, \end{cases} \tag{2.2}$$

where $\nu = 0, 1, 2$ correspond to the planar, cylindrical and spherical cases respectively.

With this one-dimensional symmetry, (2.1) possesses two distinct characteristic curves:

$$\frac{d\chi_1}{dt} = u - 1, \qquad \frac{d\chi_2}{dt} = u + 1,$$

which carry the prescribed data into the domain $(r_0,\infty)\times R_+$. Notice that when $u$ decreases below the critical value $u=-1$, both characteristics $\chi_1$ and $\chi_2$ run into the boundary $r=r_0$, thus making the initial-boundary value problem for (2.1) overdetermined and hence unsolvable in the class $C^1((r_0,\infty)\times(0,T))\cap C^0([r_0,\infty)\times[0,T))$, for some positive constant $T$. Hence near the "Bohm velocity" $u=-1$, quasi-neutrality breaks down and a sheath boundary layer forms. Since the Poisson equation reads

$$\varepsilon^2\partial_r(r^\nu\partial_r\phi)=r^\nu(n-e^{-\phi}),$$

the quasi-neutrality relation is violated when the left hand side becomes non-negligible. This has been quantified by Franklin and Ockendon for steady problems [14], where a matched asymptotic expansion yields $\partial_r\phi\approx\varepsilon^{-\beta},\quad 0<\beta<1$, so that the electric potential develops a large gradient near the sheath edge (see also [23]). Since $\phi$ has rapidly increased as the ions entered the "*sheath*" boundary layer, we formally set the electron density $\rho_e=0$ of (2.2) in the boundary layer to define the "*step sheath*" model which we now describe in more detail.

Specifically we return to the Euler-Poisson system. Since the sheath width is order of $\varepsilon$, we use fast variables $(\bar{\mathbf{x}},\bar{t})$:

$$\bar{\mathbf{x}}=\frac{\mathbf{x}}{\varepsilon},\qquad \bar{t}=\frac{t}{\varepsilon},$$

to get a rescaled system:

$$\text{(2.3)}\qquad \begin{cases}\partial_{\bar{t}}n+\nabla_{\bar{\mathbf{x}}}\cdot(n\mathbf{u})=0,\quad (\mathbf{x},t)\in\Omega\times(0,\infty),\\ \partial_{\bar{t}}\mathbf{u}+(\mathbf{u}\cdot\nabla_{\bar{\mathbf{x}}})\mathbf{u}=\nabla_{\bar{\mathbf{x}}}\phi,\\ \Delta_{\bar{\mathbf{x}}}\phi=n-e^{-\phi},\end{cases}$$

and rescaled initial and boundary data

$$\begin{cases}(n,\mathbf{u},\phi)(\bar{\mathbf{x}},0)=(n_0,\mathbf{u}_0,\phi_0)(\bar{\mathbf{x}}),\quad \bar{\mathbf{x}}\in\Omega,\\ \nabla_{\bar{\mathbf{x}}}\phi\cdot\boldsymbol{\nu}_0=g(\bar{\mathbf{x}},t),\quad (\bar{\mathbf{x}},\bar{t})\in\partial\Omega_0\times[0,\infty).\end{cases}$$

where now the gradient $\nabla_{\bar{x}}$ and the Laplacian $\Delta_{\bar{x}}$ are taken in terms of rescaled variables $\bar{\mathbf{x}}$.

As mentioned above in the sheath region, we formally set the electron density to be zero to get the rescaled sheath system **(S)**:

$$\text{(2.4)}\qquad \begin{cases}\partial_{\bar{t}}n+\nabla_{\bar{\mathbf{x}}}\cdot(n\mathbf{u})=0,\quad (\bar{\mathbf{x}},t)\in\Omega\times(0,\infty),\\ \partial_{\bar{t}}\mathbf{u}+(\mathbf{u}\cdot\nabla_{\bar{\mathbf{x}}})\mathbf{u}=\nabla_{\bar{\mathbf{x}}}\phi,\\ \Delta_{\bar{\mathbf{x}}}\phi=n.\end{cases}$$

In contrast, in the quasi-neutral region, we use the rescaled quasi-neutral system **(Q)**:

$$\text{(2.5)}\qquad \begin{cases}\partial_{\bar{t}}n+\nabla_{\bar{\mathbf{x}}}\cdot(n\mathbf{u})=0,\quad (\bar{\mathbf{x}},t)\in\Omega\times(0,\infty),\\ \partial_{\bar{t}}\mathbf{u}+(\mathbf{u}\cdot\nabla_{\bar{\mathbf{x}}})\mathbf{u}+\nabla_{\bar{\mathbf{x}}}(\ln n)=\mathbf{0}.\end{cases}$$

Of course it is readily noted that (2.5) is just the system of compressible inviscid isothermal gas dynamics.

The boundary surface $\partial\Omega_0$ is described by an implicit relation:

$$b(\mathbf{x}, \varepsilon) = 0, \quad \text{for a smooth function } b : R^3 \times R_+ \to R.$$

Furthermore we assume $b$ satisfies the scaling relation:

$$\text{If} \quad \bar{\mathbf{x}} = \frac{\mathbf{x}}{\varepsilon}, \text{ then } b(\varepsilon\bar{\mathbf{x}}, \varepsilon) = c(\varepsilon)\bar{b}(\bar{\mathbf{x}}), \text{ for some smooth functions } c, \text{ and } \bar{b}.$$

For these rescaled independent variables $\bar{\mathbf{x}}$, the boundary surface $\partial\Omega_0$ of the target $\Omega_0$ can be represented as

$$\bar{b}(\bar{\mathbf{x}}) = 0. \tag{2.6}$$

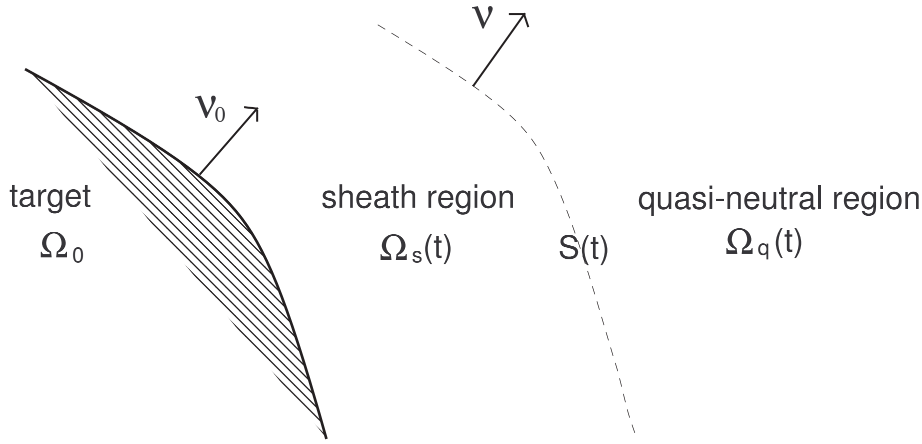


FIGURE 1. Schematic diagram of a physical domain at time $t$

In our model, for fixed time $t$, we decompose the domain $\Omega$ into the sheath region, the quasi-neutral region and their interface, i.e.,

$$\Omega = \Omega_s(t) \cup \mathcal{S}(t) \cup \Omega_q(t), \quad \Omega_s(t) : \text{ the sheath region,}$$
$$\Omega_q(t) : \text{ the quasi-neutral region,} \quad \mathcal{S}(t) : \text{ plasma-sheath interface.}$$

Now we return to the issue of the sheath edge. As noted above for the steady motion with planar, cylindrical and spherical symmetry, a matched asymptotic expansion [14] yields $\partial_r\phi = \varepsilon^{-\beta}, \ 0 < \beta < 1$. Hence in the formal quasi-neutral limit ($\varepsilon \to 0+$), we obtain the sheath edge relation

$$\partial_{\bar{r}}\phi = \varepsilon\partial_r\phi \approx \varepsilon^{1-\beta} \to 0, \qquad \text{as } \varepsilon \to 0+,$$

so that the normal component of $\nabla_{\mathbf{x}}\phi$ on the interface becomes zero, i.e.,

$$\nabla\phi \cdot \boldsymbol{\nu} = 0 \quad \text{on } \mathcal{S}(t). \tag{2.7}$$

We will incorporate this relation in defining the sheath interface below.

First drop the over bars in (2.4) and (2.5) for notational simplicity and set $n_e$ to be the electron density so that our governing equations become

$$\begin{aligned}&\partial_t n + \nabla_{\mathbf{x}} \cdot (n\mathbf{u}) = 0, \quad (\mathbf{x}, t) \in \Omega_s(t) \times (0, \infty),\\ &\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla_{\mathbf{x}})\mathbf{u} = \nabla\phi,\\ &\Delta_{\mathbf{x}}\phi = n, \ n_e = 0.\end{aligned} \tag{2.8}$$

in the sheath region and

$$\begin{aligned}&\partial_t n + \nabla_{\mathbf{x}} \cdot (n\mathbf{u}) = 0, \quad (\mathbf{x}, t) \in \Omega_q \times (0, \infty),\\ &\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla_{\mathbf{x}})\mathbf{u} + \nabla_{\mathbf{x}}(\ln n) = \mathbf{0}\\ &n = n_e = e^{-\phi} \text{ (quasi-neutrality and the Boltzmann relation)} ,\end{aligned} \tag{2.9}$$

in the quasi-neutral region.

Note that with overbars deleted the target boundary surface (2.6) can be rewritten as

$$b(\mathbf{x}) = 0.$$

We combine (2.7) with the "Bohm-relation" $\mathbf{u}\cdot\boldsymbol{\nu} = -1$ (where $\boldsymbol{\nu}$ is the unit exterior normal to the plasma-sheath interface) to give a definition of the sheath interface for (2.3). Specifically the definition of the plasma-sheath interface $\mathcal{S}(t)$ is defined by the level set of the normal component of the electric field and ion-velocity fields.

**Definition:** A plasma-sheath interface $\mathcal{S}(t)$ separating a quasi-neutral region and an ion-sheath region is the level set of the normal component of the ion velocity and electric fields,

$$\mathcal{S}(t) \equiv \{\mathbf{x} \in R^3 : (\mathbf{u} \cdot \boldsymbol{\nu})(\mathbf{x}, t) = -1, \quad (\nabla\phi \cdot \boldsymbol{\nu})(\mathbf{x}, t) = 0\}, \qquad t \geq 0,$$

where $\boldsymbol{\nu}$ is the exterior normal to the interface.

Notice our definition is motivated by the observation that in the symmetric case [18] only the normal component of fluid velocity $\mathbf{u}$ and electric field $\nabla\phi$ affect the sheath location.

## 3. Local existence of plasma-sheath interfaces

In this part, we briefly discuss a new geometric level-set formulation of a plasma-sheath interface and an explicit dynamics of sheath interfaces and present an local in time existence theorem on the 2D-interface.

In [13], we show that taking the normal time derivative $\delta_t = \partial_t + V\boldsymbol{\nu} \cdot \nabla$ along the sheath edge yields the dynamics of the plasma-sheath interface given by

$$\boxed{\frac{\delta\psi}{\delta t} = 0, \quad \frac{\delta n}{\delta t} = n\nabla \cdot \boldsymbol{\nu}, \quad (V + 1) + \frac{\mathbf{h} \cdot \boldsymbol{\nu}}{n} = -\frac{1}{n}\nabla_s \cdot (V\nabla_s \ln n),}$$

As a special case, we consider the case where the interface is given by a simple closed convex curve in the plane (see Figure 2). Suppose a portion $\Gamma$ of a curve which is represented by $x_2 = f(x_1, t)$ in $x_1$-$x_2$ plane. If we wish to study the evolution of a simple closed convex curve in the plane, it will be convenient to use polar coordinates:

$$x_1 = r\cos\beta, \qquad x_2 = r\sin\beta, \tag{3.10}$$

so that the evolving curve is represented by $r = r(\beta, t)$, and the portion $\Gamma$ becomes

$$r \sin\beta = f(r\cos\beta, t), \tag{3.11}$$

In the case when the sheath interface is a simple closed curve, the sheath edge is

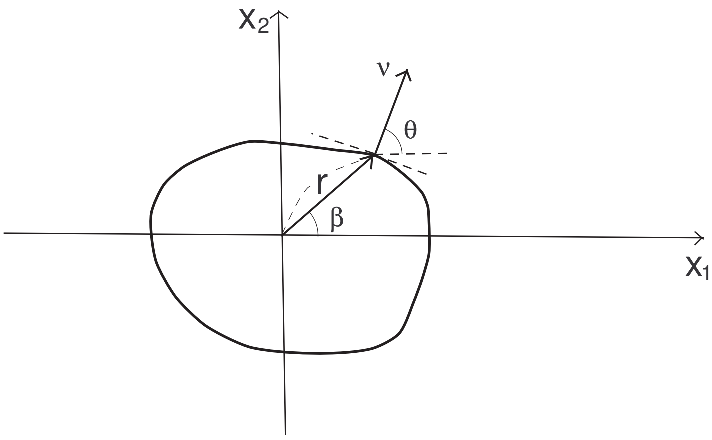


FIGURE 2. Schematic diagram of a sheath interface in the plane

governed by the system:

$$\begin{pmatrix} \partial_t\theta \\ \partial_t n_s \\ \partial_t r \end{pmatrix} - \frac{2\sin\beta}{r}\begin{pmatrix} V\cos\theta & 0 & 0 \\ 2n_s\sin\theta & V\cos\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \partial_\beta\theta \\ \partial_\beta n_s \\ \partial_\beta r \end{pmatrix} = \begin{pmatrix} -\frac{2\sin\theta\sin\beta\partial_\beta V}{r} \\ 0 \\ \frac{V}{\cos(\theta-\beta)} \end{pmatrix}$$
(3.12)

with the same constitutive equation for $V$:

$$(V+1) + \frac{\mathbf{h}\cdot(\cos\theta, \sin\theta)}{n} = -\frac{1}{n}\nabla_s\cdot(V\nabla_s \ln n).$$

Existence and uniqueness theorems for the system are discussed in [13] where we employed the small surface gradient approximation $\tilde{V}$ for $V$:

$$V = -1 - \frac{\mathbf{h}\cdot(\cos\theta, \sin\theta)}{n}.$$

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