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ON THE AXISYMMETRIC BOUSSINESQ EQUATIONS WITHOUT SWIRL COMPONENT

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Abstract. We obtain the Beale-Kato-Majda type blow-up criterion for the axisymmetric inviscid Boussinesq equation. We also proved the global existence of the strong solution of the axisymmetric viscous Boussinesq equations under some regularity condition.

1. Introduction

In this paper, we study the blow-up criterion for the solutions of the nondissipative axisymmetric Boussinesq equations without swirl component and the global well-posedness for the solutions of the dissipative axisymmetric Boussinesq equations without swirl component. Three dimensional Boussinesq (Oberbeck-Boussinesq) equations are written as follows.

$$(1)\qquad \begin{cases} \frac{\partial u}{\partial t} + (u\cdot\nabla)u + \nabla p - \nu\Delta u + \sigma\theta\vec{e}_3 = 0, \\ \nabla\cdot u = 0, \\ \frac{\partial\theta}{\partial t} + (u\cdot\nabla)\theta - \kappa\Delta\theta = 0, \\ u(0,x) = u_0(x),\ \theta(0,x) = \theta_0(x), \end{cases}$$

where u, p, θ, ν, κ, σ and $\sigma\theta\vec{e}_n$ denotes the fluid velocity vector field, the scalar pressure, the scalar temperature, the positive normalised fluid viscosity, the positive thermal diffusivity or Lewis number, the Rayleigh number, and the gravitational force, respectively.

The above equations was designed as a simplified model for the thermomechanical model of the isothermally incompressible viscous(inviscous) fluids. Since the heat convection is considered, the mathematical analysis of the Boussinesq equations is a little bit more complicate than that of the Navier-Stokes equations. For example, the similarity of the two dimensional Boussinesq equations with $\nu = \kappa = 0$ with the three dimensional Euler equations is indicated by many authors(see [7] and [6]). For the two-dimensional Boussinesq equations with $\kappa, \quad \nu > 0$, the global existence of the strong solution is rather well-known(see [2]). Recently, Chae[3] proved the global existence of the solutions for the case of the partial viscosity(one of the viscosities ν and κ are zero) by using Brexis-Wainger inequality. For the two dimensional inviscid flow, Chae and Nam[6] obtained the blow-up criterion imposing the condition on the temperature θ which is a Boussinesq version of the

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Beals-Kato-Majda criterion of the Euler equations. Eckhaus[8] studied the stability of the roll solutions for the three dimensional Boussinesq equation.
In this paper, we consider the axially symmetric three dimensional viscous and in viscous Boussinesq approximation without swirl component. Consider the cylindrical coordinate with the basis

$$e_r = (\frac{x_1}{r}, \frac{x_2}{r}, 0),\ e_\theta = (-\frac{x_2}{r}, \frac{x_1}{r}, 0),\ e_3 = (0,0,1),\ r = \sqrt{x_1^2 + x_2^2}.$$

We assume every vector fields and functions do not depend on the angle and swirl component of the velocity vector field vanishes i.e., $u = u^r e_r + u_3 e_3$. The equation are written as follows.

$$\begin{cases} \partial_t u^r + (\tilde{u}\cdot\tilde{\nabla})u^r + \partial_r p - \nu[\tilde{\Delta}u - \frac{u^r}{r^2}] = 0, \\ \partial_t u_3 + (\tilde{u}\cdot\tilde{\nabla})u_3 + \partial_3 p - \nu\tilde{\Delta}u + \sigma\theta = 0, \\ \frac{1}{r}\partial_r(ru^r) + r\partial_3 u_3 = 0, \\ \partial_t\theta + (\tilde{u}\cdot\tilde{\nabla})\theta - \kappa\tilde{\Delta}\theta = 0, \\ \tilde{u}(0,x) = \tilde{u}_0(x),\ \theta(0,x) = \theta_0(x) \end{cases} \tag{2}$$

where $\tilde{u^r} = (u^r, u_3)$, $\tilde{\nabla} = (\partial_r, \partial_3)$, and $\tilde{\Delta} = \frac{1}{r}\partial_r(r\partial_r) + \partial_3^2$. Taking the curl operator on the first and second equation of the (2), we obtain the following equation on the vorticity $\omega = \omega^\theta$.

$$\partial_t\omega + (\tilde{u}\cdot\tilde{\nabla})\omega - \nu[\tilde{\Delta}\omega - \frac{\omega}{r^2}] - \frac{u^r}{r}\omega + \sigma\partial_r\theta = 0. \tag{3}$$

Let Ω be any axially symmetric domain in $\mathbb{R}^3$. We denote by L^p, $1 \le p \le \infty$ the space of real functions defined on Ω with the p-th power absolutely integrable for the Lebesgue measure. The Sobolev space H^m is the space of functions in L^2 with derivatives of order less than or equal to m in L^2. We shall be concerned with the 3-dimensional vector functions with components in one of above spaces. We shall use the same notations for the vector functions if there is no ambiguity. Let $\mathcal{V}$ be the space $\mathcal{V} = \{u \in C_0^\infty(\Omega),\ \text{div}\ u = 0\}$. The closure of $\mathcal{V}$ in H^m is denoted by V^m.

2. Main Theorem

First we consider the domain as whole $\mathbb{R}^3$ and the inviscid case, i.e., $\nu = \kappa = 0$. For the axisymmetric Euler equations($\theta = 0$) with the swirl component, Chae and Kim[4] obtained the blow-up criterion on the swirl component of the vorticity. For the axisymmetric Boussinesq equations without swirl component, we obtained the following Beale-Kato-Majda type blow-up criterion.

Theorem 2.1. *Let the initial data $u_0 \in V^m$ and $\theta_0 \in H^m$ for some $m > 2$. Then we have*

$$\lim_{t\nearrow T}\sup(\|\tilde{u}(\cdot,t)\|_{V^m} + \|\theta(\cdot,t)\|_{H^m}) = \infty \text{ if and only if } \int_0^T \|\frac{u^r}{r}(\cdot,\tau)\|_{L^\infty} + \|\nabla\theta(\cdot,\tau)\|_{L^\infty} d\tau = \infty.$$

In the following, we consider the general axisymmetric domain $\Omega \subset \mathbb{R}^3$ and $\nu, \kappa > 0$ case. For the axisymmetric Navier-Stokes equations($\theta = 0$) with the swirl component, the author[5] obtained the variousregularity criterion(see also [12]).

And Kim[9] showed the global existence in the weighted space. For the axisymmetric Navier-Stokes equations($\theta = 0$) without swirl component, the global unique-existence of the solutions was announced by Ladyzhenskaya[10] and simultaneously by Uchovskii and Yudovich[14]. The main idea is that the term $[\nabla\omega]u$ is formally orthogonal to $\frac{\omega}{r^2}$. Recently, the above formal argument is justified rigorously by Leonardi et al.[11]. First, we can easily show the existence of the weak solution of the axisymmetric Boussinesq equation.

Proposition 2.2. *Suppose that the initial data $\tilde{u}_0 \in L^2$, $\theta_0 \in L^2$. There exist weak solutions $\tilde{u}, \theta$ for the axisymmetric Boussinesq equations satisfying $\tilde{u} \in L^\infty(0,T;L^2) \cap L^2(0,T;V^1)$ and $\theta \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$.*

Proof) We provide a priori estimates on the solutions.
Multiplying $\tilde{u}$ and θ on the both sides of equations (2) and integrating over Ω, respectively, we have

$$\begin{aligned}\tfrac{1}{2}\tfrac{d}{dt}(\|\tilde{u}\|^2_{L^2} + \|\theta\|^2_{L^2}) + \nu\|\tilde{\nabla}\tilde{u}\|^2_{L^2} + \nu\|\tfrac{u^r}{r}\|^2_{L^2} + \kappa\|\tilde{\nabla}\theta\|^2_{L^2} \\ \le \tfrac{|\sigma|}{2}(\|\theta\|^2_{L^2} + \|\tilde{u}\|^2_{L^2}).\end{aligned}$$

Using Gronwall's inequality, we have

$$\begin{aligned}\sup(\|\tilde{u}\|^2_{L^2} + \|\theta\|^2_{L^2}) + \nu\int_0^T \|\tilde{\nabla}\tilde{u}\|^2_{L^2} + \nu\|\tfrac{u^r}{r}\|^2_{L^2}dt + \kappa\int_0^T \|\tilde{\nabla}\theta\|^2_{L^2}dt \\ \le (\|\theta_0\|^2_{L^2} + \|\tilde{u}_0\|^2_{L^2})\exp(|\sigma|T).\end{aligned}$$

The remainder of the proof is rather standard, for example, we can justify the above a priori estimates and existence of the solutions by using the Galerkin approximations. We omit the details. □

For the unique-existence of the axisymmetric Boussinesq equation, we use the equation for $\frac{\omega}{r}$. We can obtain the apriori estimates for the ω and $\tilde{\nabla}\theta$ and justify all the process by the method presented in [11].

Theorem 2.3. *Suppose that $\tilde{u}_0 \in V^1$, $\frac{\omega_0}{r} \in L^2$, and $\theta_0 \in H^1$ and $\int_0^T \|\frac{\theta}{r}\|_{L^2}dt$ is finite. Then there exists a unique solution u, θ for the axisymmetric Boussinesq equations satisfying $u \in L^\infty(0,T;V^1) \cap L^2(0,T;V^2)$ and $\theta \in L^\infty(0,T;H^1) \cap L^2(0,T;H^2)$.*

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