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UNIQUENESS OF POSITIVE RADIAL SOLUTIONS TO SEMILINEAR ELLIPTIC PROBLEMS IN ANNULAR DOMAINS.

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ABSTRACT. In this note, we investigate the multiplicity of positive radial solutions of $\Delta u + f(u) = 0$ using maximum principle and shooting argument.

1. Introduction

We consider the uniqueness of positive solution of the problems

$$(BP_1) \quad \begin{cases} u'' + \frac{n-1}{r}u' + f(u) = 0 & \text{for } 0 < a < r < b, \\ u(a) = u(b) = 0, \end{cases}$$

$$(BP_2) \quad \begin{cases} u'' + \frac{n-1}{r}u' + f(u) = 0 & \text{for } 0 \le a < r < b, \\ u'(a) = u(b) = 0, \end{cases}$$

where $f : [0, \infty) \to R$ is a smooth function of class C^1 with appropriate properties according to the type of the problems.

For $f(u) = -u + u^p$, $1 < p < \frac{n+2}{n-2}$, the uniqueness of positive solution of (BP_2) was studied by Coffman [3], Kwong [4] and extended by Chen and Lin [1] to more general f. The uniqueness of positive solution of (BP_1) with the same $f(u) = -u + u^p$ was proved by Coffman [4] in the case where $n = 3$, $1 < p \le 3$ and by Yadava [7] for the case $n \ge 3$, $p \ge \frac{n+2}{n-2}$. With $f(u) = u^p + u^q$, the uniqueness of positive solution of (BP_2) was studied by Zhang [8] for the case $1 = q < p \le \frac{n+2}{n-2}$ and the uniqueness of positive solution of (BP_1) was shown by Yadava [7] in the case where $1 < p < q < \frac{n+2}{n-2}$ and $\frac{p-1}{q+1} \le \frac{2}{n}$. It was also proved by Yadava [7] that if $\frac{n+2}{n-2} < p \le q$ and $\frac{p-1}{q+1} \ge \frac{2}{n}$, then (BP_1) possesses ay most one solution. For sub/supercritical case, a non-uniqueness result was obtained by Ni and Nussubaum. In [6], with other interesting uniqueness results, they have shown that if $1 < q < \frac{n+2}{n-2} < p$, there exists a and b such that the problem (BP_1) admits at least three positive solutions. In this note we obtain the uniqueness results of (BP_1) and (BP_2) with more general f including the examples mentioned above.

2. Theorems

Throughout this note the nonlinear term f is assumed to be a smooth function satisfying either

(f1) $f(u) \ge 0$, for all $u \ge 0$,

(f2) $uf'(u) \ge f(u)$, for all $u \ge 0$

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or

(F1) $f(0) = 0$ and there exists $\varepsilon_0 > 0$ such that $f'(u) \leq 0$ for $0 \leq u \leq \varepsilon_0$,

(F2) There exists a positive constant u_0 such that $f(\bar{u}) = 0$, $f'(u_0) > 0$ and f has no other positive zero,

(F3) For $u \geq u_0$, $f'(u)(u - u_0) > f(u)$.

Theorem A. *Suppose f satisfies* (f1), (f2) *and that there exists $\beta > 0$ such that, for every $u > 0$,*

$$\Phi_\beta(u) = \beta(uf'(u) - f(u)) - 2f(u) \leq 0, \tag{A-1}$$

$$2nF(u) - (n-2)uf(u) + u\Phi_\beta(u) \geq 0. \tag{A-2}$$

Then the problems (BP_1) and (BP_2) possesses at most one solution.

Corollary A. *Let $f(u) = u^q + u^p$ and suppose $1 \leq q < p \leq \frac{n+2}{n-2}$ and $n \geq 3$. Given a and b with $0 < a < b < \infty$, we have at most one positive solution of $(BP_{1,2}$ if either $q = 1$ or $q > 1$ and $\frac{p-1}{q+1} \leq \frac{2}{n}$.*

Theorem B. *Suppose that there exists $\beta > 0$ such that, for every $u > 0$,*

$$\Phi_\beta(u) = \beta(uf'(u) - f(u)) - 2f(u) \geq 0, \tag{B-1}$$

$$2nF(u) - (n-2)uf(u) + u\Phi_\beta(u) \leq 0. \tag{B-2}$$

Then the problem (P) possesses at most one solution.

Corollary B1. *Suppose $\frac{n+2}{n-2} < p < q$ and $n \geq 3$ and let $f(u) = u^q + u^p$. Given a and b with $0 < a < b < \infty$, (BP) admits a unique solution provided $\frac{p-1}{q+1} \geq \frac{2}{n}$.*

Corollary B2. *Suppose $\frac{n+2}{n-2} \leq p$ and $n \geq 3$ and let $f(u) = -u + u^p$. Given a and b with $0 < a < b < \infty$, (BP) admits at most one solution.*

Theorem C1. *Let $n \geq 3$ and let f satisfy* (F1), (F2) *and* (F3). *If Φ_β has at most one zero in (u_0, ∞) for every $\beta \in (0, n-2]$, then the problem (BP_2) possesses at most one solution.*

Theorem C2. *Let $n = 2$ and let f satisfy* (F1), (F2) *and* (F3). *If Φ_β has at most one zero in (u_0, ∞) for every $\beta \in (0, \infty)$, then the problem (BP_2) possesses at most one solution.*

2. Proofs

Since the main idea and the arguments for the proof of Theorem A,B,C are the same, we only provide a sketch the proof of Theorem A in this article for the sake of briefness.

As usual, we use the shooting argument to investigate multiplicity of solutions of (BP_1) and (BP_2).

To make use of the usual shooting method we consider the corresponding initial value problems

$$(IP_1)_\alpha \quad \begin{cases} u'' + \frac{n-1}{r}u' + f(u) = 0 & \text{for } 0 < a < r < b, \\ u(a) = 0,\ u'(a) = \alpha, \end{cases}$$

$$(IP_2)_\alpha \quad \begin{cases} u'' + \frac{n-1}{r}u' + f(u) = 0 & \text{for } 0 \le a < r < b, \\ u(a) = \alpha,\ u'(a) = 0. \end{cases}$$

We denote the solution of $(IP_{1,2})$ by $u_\alpha = u(\alpha, \cdot)$ and the first zero of $u(\alpha, \cdot)$ by $b(\alpha)$ if it exists. As is well known, the uniqueness of positive solution of (BP_1) depends on the number of zeros of the solution of the corresponding linearized equation

$$(LP)_\alpha \phi'' + \frac{n-1}{r}\phi' + f'(u)\phi = 0 \quad \text{for } 0 < a < r < b,$$

where u is the solution of (BP_1). In fact, $\phi_\alpha = \frac{\partial u}{\partial \alpha}(\alpha, \cdot)$ is the solution of (LP) and since

$$0 = \frac{d}{d\alpha}u(\alpha, b(\alpha)) = \phi_\alpha(\alpha, b(\alpha)) + u'(\alpha, b(\alpha))b'(\alpha),$$

if $\phi_\alpha(\alpha, b(\alpha))$ has one sign for all α, so does $b'(\alpha)$. Therefore $b = b(\alpha)$ is monotone in α and this implies the uniqueness. As was observed by McLeod and Serrin [5] and Kwong [4], $v_\beta = ru' + \beta u$ satisfies

$$v_\beta'' + \frac{n-1}{r}v_\beta' + f'(u)v_\beta = \beta(uf'(u) - f(u)) - 2f(u).$$

Hence the function v_β with the information on the sign of

$$\Phi_\beta(u) = \beta(uf'(u) - f(u)) - 2f(u)$$

plays an important role as a comparison function to the solution of the problem (LP).

Lemma A1. *Let u a solution of (BP_1). If ϕ satisfies (LP), then ϕ has at least one zero in (a, b).*

Proof. By (F_1), u satisfies

$$u'' + \frac{n-1}{r}u' + f'(u)u = uf'(u) - f(u) \geq 0.$$

If ϕ has no zero in (a, b), we may assume $\phi > 0$ in (a, b). But then, there exists a constant $c > 0$ and $r_0 \in (a, b)$ such that $c\phi \geq u$ in (a, b) and $\phi(r_0) = u(r_0)$. This contradicts the maximum principle.

Using the same argument, we obtain another Lemma.

Lemma A2. *Let ϕ be the solutions of (LP) such that $\phi(b) = u(b) = 0$ and $\phi'(b) = u'(b)$. If r_1 is the largest zero in (a, b), then $u > \phi$ in (r_1, b).*

Proof. Omit.

Lemma A3. *Let ϕ be the solutions of (LP) such that $\phi(b) = u(b) = 0$ and $\phi'(b) = u'(b)$. If r_1 is the largest zero in (a, b), then $v_\beta(r_1) > 0$.*

Proof. Let ϕ be the solution of (LP) such that $\phi(b) = 0$ and $\phi'(b) = u'(b)$ and let

$$z_\beta(r) = r^{n-1}(\phi v'_\beta - \phi' v_\beta).$$

A straightforward computation shows

$$z'_\beta(r) = r^{n-1}\{(\phi v''_\beta - \phi'' v_\beta) + \frac{(n-1)}{r}(\phi v'_\beta - \phi' v_\beta)\} = r^{n-1}\phi(r)\Phi_\beta(u(r)).$$

Then, using Lemma A2 on the first inequality and the Pohozaev Identity on the third equality, we have

$$\begin{aligned}
z_\beta(r_1) - z_\beta(b) &= b^{n-1}\phi'(b)v(b) - r_1^{n-1}\phi'(r_1)v_\beta(r_1) \\
&= -\int_{r_1}^{b} r^{n-1}\phi(r)\Phi_\beta(u(r))dr \\
&\leq -\int_{r_1}^{b} r^{n-1}u(r)\Phi_\beta(u(r))dr \\
&\leq \int_{r_1}^{b} r^{n-1}\{2nF(u) - (n-2)uf(u)\}dr \\
&\leq \int_{a}^{b} r^{n-1}\{2nF(u) - (n-2)uf(u)\}dr \\
&= b^{n-1}u'(b)v_{n-2}(b) - a^{n-1}u'(a)v_{n-2}(a) \\
&= b^{n-1}u'(b)v(b) - a^{n}u'(a)^2
\end{aligned}$$

Hence we obtain

$$z_\beta(r_1) \leq z_\beta(b) + b^{n-1}u'(b)v(b) - a^n u'(a)^2 = -a^n u'(a)^2 < 0$$

and therefore $v_\beta(r_1) = -z_\beta(r_1)(r_1^{n-1}\phi'(r_1))^{-1} > 0$.

Lemma A4. *Let ϕ be the solutions of (LP) such that $\phi(b) = u(b) = 0$ and $\phi'(b) = u'(b)$. If r_1 is the unique zero in (a, b), then $\phi(a) < 0$.*

Proof. Since $z_\beta(r_1) < 0$ and

$$z'_\beta(r) = r^{n-1}\phi(r)\Phi_\beta(u(r)) \geq 0, \ a < r < r_1,$$

$0 < -a^{n-1}\phi'(a)v_\beta(a) = z_\beta(a) \leq z_\beta(r_1) < 0$ if $\phi(a) = 0$, a contradiction.

Now let u_α be the solution of the initial value problem $(IP)_\alpha$ and let $\alpha_1 = \inf\{\alpha > 0 \ : \ u_\alpha$ is a solution of $(BP_1)\}$. Using maximum principle, one can show that $u_\alpha(b) > 0$ for all $\alpha \in (0, \alpha_1)$ and then $\phi_{\alpha_1} = \frac{\partial}{\partial\alpha}u(\alpha_1, \cdot)$ has exactly one zero in (a, b). By Lemma A4, $\phi_{\alpha_1}(b) < 0$. Since $\phi_\alpha(b(\alpha))$ depends on α continuously, Lemma A4 implies that $\phi_\alpha(b(\alpha)) < 0$ for all $\alpha > 0$. This completes the proof of Theorem A.

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