UNIQUENESS OF POSITIVE RADIAL SOLUTIONS TO SEMILINEAR ELLIPTIC PROBLEMS IN ANNULAR DOMAINS.

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Abstract. In this note, we investigate the multiplicity of positive radial solutions of $\Delta u + f(u) = 0$ using maximum principle and shooting argument.

1. Introduction

We consider the uniqueness of positive solution of the problems

$$(BP_1) \begin{cases} u'' + \frac{n-1}{r} u' + f(u) = 0 & \text{for } 0 < a < r < b, \\ u(a) = u(b) = 0, \end{cases}$$

$$(BP_2) \begin{cases} u'' + \frac{n-1}{r} u' + f(u) = 0 & \text{for } 0 \leq a < r < b, \\ u'(a) = u(b) = 0, \end{cases}$$

where $f : [0, \infty) \to \mathbb{R}$ is a smooth function of class $C^1$ with appropriate properties according to the type of the problems.

For $f(u) = -u + u^p$, $1 < p < \frac{n+2}{n-2}$, the uniqueness of positive solution of $(BP_2)$ was studied by Coffman [3], Kwong [4] and extended by Chen and Lin [1] to more general $f$. The uniqueness of positive solution of $(BP_1)$ with the same $f(u) = -u + u^p$ was proved by Coffman [4] in the case where $n = 3$, $1 < p \leq 3$ and by Yadava [7] for the case $n \geq 3$, $p \geq \frac{n+2}{n-2}$. With $f(u) = u^p + u^q$, the uniqueness of positive solution of $(BP_2)$ was studied by Zhang [8] for the case $1 = q < p \leq \frac{n+2}{n-2}$ and the uniqueness of positive solution of $(BP_1)$ was shown by Yadava [7] in the case where $1 < p < q < \frac{n+2}{n-2}$ and $\frac{p-1}{q+1} \geq \frac{2}{n}$. It was also proved by Yadava [7] that if $\frac{n+2}{n-2} < p < q < \frac{n+2}{n-2}$ and $\frac{p-1}{q+1} \geq \frac{2}{n}$, then $(BP_1)$ possesses at most one solution. For sub/supercritical case, a non-uniqueness result was obtained by Ni and Nussbaum. In [6], with other interesting uniqueness results, they have shown that if $1 < q < \frac{n+2}{n-2} < p$, there exists $a$ and $b$ such that the problem $(BP_1)$ admits at least three positive solutions. In this note we obtain the uniqueness results of $(BP_1)$ and $(BP_2)$ with more general $f$ including the examples mentioned above.

2. Theorems

Throughout this note the nonlinear term $f$ is assumed to be a smooth function satisfying either

(f1) $f(u) \geq 0$, for all $u \geq 0$,

(f2) $uf'(u) \geq f(u)$, for all $u \geq 0$

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or

(F1) \( f(0) = 0 \) and there exists \( \varepsilon_0 > 0 \) such that \( f'(u) \leq 0 \) for \( 0 \leq u \leq \varepsilon_0 \),

(F2) There exists a positive constant \( u_0 \) such that \( f(\bar{u}) = 0, f'(u_0) > 0 \) and \( f \) has no other positive zero,

(F3) For \( u \geq u_0, f'(u)(u - u_0) > f(u) \).

**Theorem A.** Suppose \( f \) satisfies (F1), (F2) and that there exists \( \beta > 0 \) such that, for every \( u > 0 \),

(A-1) \( \Phi_\beta(u) = \beta(uf'(u) - f(u)) - 2f(u) \leq 0 \),

(A-2) \( 2nF'(u) - (n - 2)uf(u) + u\Phi_\beta(u) \geq 0 \).

Then the problems (BP\textsubscript{1}) and (BP\textsubscript{2}) possesses at most one solution.

**Corollary A.** Let \( f(u) = u^q + u^p \) and suppose \( 1 \leq q < p \leq \frac{n+2}{n-2} \) and \( n \geq 3 \). Given \( a \) and \( b \) with \( 0 < a < b < \infty \), we have at most one positive solution of (BP\textsubscript{1,2}) if either \( q = 1 \) or \( q > 1 \) and \( \frac{q-1}{q+1} \leq \frac{2}{n} \).

**Theorem B.** Suppose that there exists \( \beta > 0 \) such that, for every \( u > 0 \),

(B-1) \( \Phi_\beta(u) = \beta(uf'(u) - f(u)) - 2f(u) \geq 0 \),

(B-2) \( 2nF'(u) - (n - 2)uf(u) + u\Phi_\beta(u) \leq 0 \).

Then the problem (P) possesses at most one solution.

**Corollary B1.** Suppose \( \frac{n+2}{n-2} < p < q \) and \( n \geq 3 \) and let \( f(u) = u^q + u^p \). Given \( a \) and \( b \) with \( 0 < a < b < \infty \), (BP) admits a unique solution provided \( \frac{q-1}{q+1} \geq \frac{2}{n} \).

**Corollary B2.** Suppose \( \frac{n+2}{n-2} \leq p \) and \( n \geq 3 \) and let \( f(u) = -u + u^p \). Given \( a \) and \( b \) with \( 0 < a < b < \infty \), (BP) admits at most one solution.

**Theorem C1.** Let \( n \geq 3 \) and let \( f \) satisfy (F1), (F2) and (F3). If \( \Phi_\beta \) has at most one zero in \((u_0, \infty)\) for every \( \beta \in (0, n - 2] \), then the problem (BP\textsubscript{2}) possesses at most one solution.
Theorem C2. Let \( n = 2 \) and let \( f \) satisfy (F1), (F2) and (F3). If \( \Phi_\beta \) has at most one zero in \((u_0, \infty)\) for every \( \beta \in (0, \infty) \), then the problem \((BP_2)\) possesses at most one solution.

2. Proofs

Since the main idea and the arguments for the proof of Theorem A,B,C are the same, we only provide a sketch the proof of Theorem A in this article for the sake of briefness.

As usual, we use the shooting argument to investigate multiplicity of solutions of \((BP_1)\) and \((BP_2)\).

To make use of the usual shooting method we consider the corresponding initial value problems

\[
(IP_1)_\alpha \quad \left\{ \begin{array}{l}
 u'' + \frac{n-1}{r}u' + f(u) = 0 \quad \text{for } 0 < a < r < b, \\
 u(a) = 0, \ u'(a) = \alpha,
\end{array} \right.
\]

\[
(IP_2)_\alpha \quad \left\{ \begin{array}{l}
 u'' + \frac{n-1}{r}u' + f(u) = 0 \quad \text{for } 0 \leq a < r < b, \\
 u(a) = \alpha, \ u'(a) = 0.
\end{array} \right.
\]

We denote the solution of \((IP_{1,2})\) by \( u_\alpha = u(\alpha, \cdot) \) and the first zero of \( u(\alpha, \cdot) \) by \( b(\alpha) \) if it exists. As is well known, the uniqueness of positive solution of \((BP_1)\) depends on the number of zeros of the solution of the corresponding linearized equation

\[
(LP)_\alpha \phi'' + \frac{n-1}{r} \phi' + f'(u) \phi \equiv 0 \quad \text{for } 0 < a < r < b,
\]

where \( u \) is the solution of \((BP_1)\). In fact, \( \phi_\alpha = \frac{\partial u}{\partial \alpha}(\alpha, \cdot) \) is the solution of \((LP)\) and since

\[
0 = \frac{d}{d\alpha} u(\alpha, b(\alpha)) = \phi_\alpha(\alpha, b(\alpha)) + u'(\alpha, b(\alpha)) b'(\alpha),
\]

if \( \phi_\alpha(\alpha, b(\alpha)) \) has one sign for all \( \alpha \), so does \( b'(\alpha) \). Therefore \( b = b(\alpha) \) is monotone in \( \alpha \) and this implies the uniqueness. As was observed by McLeod and Serrin [5] and Kwong [4], \( v_\beta = ru' + \beta \alpha \) satisfies

\[
v''_\beta + \frac{n-1}{r} v'_\beta + f'(u)v_\beta = \beta(uf'(u) - f(u)) - 2f(u).
\]

Hence the function \( v_\beta \) with the information on the sign of

\[
\Phi_\beta(u) = \beta(uf'(u) - f(u)) - 2f(u)
\]

plays an important role as a comparison function to the solution of the problem \((LP)\).
Lemma A1. Let $u$ a solution of $(BP_1)$. If $\phi$ satisfies $(LP)$, then $\phi$ has at least one zero in $(a, b)$.

Proof. By $(F_1)$, $u$ satisfies
\[
u'' + \frac{n - 1}{r} u' + f'(u)u = uf'(u) - f(u) \geq 0.
\]

If $\phi$ has no zero in $(a, b)$, we may assume $\phi > 0$ in $(a, b)$. But then, there exists a constant $c > 0$ and $r_0 \in (a, b)$ such that $c\phi \geq u$ in $(a, b)$ and $\phi(r_0) = u(r_0)$. This contradicts the maximum principle.

Using the same argument, we obtain another Lemma.

Lemma A2. Let $\phi$ be the solutions of $(LP)$ such that $\phi(b) = u(b) = 0$ and $\phi'(b) = u'(b)$. If $r_1$ is the largest zero in $(a, b)$, then $u > \phi$ in $(r_1, b)$.

Proof. Omit.

Lemma A3. Let $\phi$ be the solutions of $(LP)$ such that $\phi(b) = u(b) = 0$ and $\phi'(b) = u'(b)$. If $r_1$ is the largest zero in $(a, b)$, then $\nu_\beta(r_1) > 0$.

Proof. Let $\phi$ be the solution of $(LP)$ such that $\phi(b) = 0$ and $\phi'(b) = u'(b)$ and let
\[
z_\beta(r) = r^{n-1}(\phi v_\beta' - \phi' v_\beta).
\]

A straightforward computation shows
\[
z_\beta'(r) = r^{n-1}\{(\phi v_\beta'' - \phi'' v_\beta) + \frac{(n - 1)}{r}(\phi v_\beta' - \phi' v_\beta)\} = r^{n-1}\phi(r)\Phi_\beta(u(r)).
\]

Then, using Lemma A2 on the first inequality and the Pohozaev Identity on the third equality, we have
\[
z_\beta(r_1) - z_\beta(b) = b^{n-1} \phi'(b) v(b) - r_1^{n-1} \phi'(r_1) v_\beta(r_1)
\]
\[
= - \int_{r_1}^b r^{n-1} \phi(r) \Phi_\beta(u(r))dr
\]
\[
\leq - \int_{r_1}^b r^{n-1} u(r) \Phi_\beta(u(r))dr
\]
\[
\leq \int_a^b r^{n-1}\{2nF(u) - (n - 2)uf(u)\}dr
\]
\[
= b^{n-1} u'(b)v_{n-2}(b) - a^{n-1} u'(a)v_{n-2}(a)
\]
\[
= b^{n-1} u'(b)v(b) - a^n u'(a)^2
\]

Hence we obtain
\[
z_\beta(r_1) \leq z_\beta(b) + b^{n-1} u'(b)v(b) - a^n u'(a)^2 = -a^n u'(a)^2 < 0
\]

and therefore $v_\beta(r_1) = -z_\beta(r_1)(r_1^{n-1} \phi'(r_1))^{-1} > 0$. 

Lemma A4. Let $\phi$ be the solutions of $(LP)$ such that $\phi(b) = u(b) = 0$ and $\phi'(b) = u'(b)$. If $r_1$ is the unique zero in $(a, b)$, then $\phi(a) < 0$.

Proof. Since $z_\beta(r_1) < 0$ and $z_\beta'(r) = r^{n-1} \phi(r) \Phi(u(r)) \geq 0$, $a < r < r_1$,

$0 < -a^{n-1} \phi(a) v_\beta(a) = z_\beta(a) \leq z_\beta(r_1) < 0$ if $\phi(a) = 0$, a contradiction.

Now let $u_\alpha$ be the solution of the initial value problem $(IP)_\alpha$ and let $\alpha_1 = \inf\{\alpha > 0 : u_\alpha$ is a solution of $(BP_1)\}$. Using maximum principle, one can show that $u_\alpha(b) > 0$ for all $\alpha \in (0, \alpha_1)$ and then $\phi_\alpha_1 = \frac{2}{\alpha_1} u(\alpha_1, \cdot)$ has exactly one zero in $(a, b)$. By Lemma A4, $\phi_\alpha_1(b) < 0$. Since $\phi_\alpha(b(\alpha))$ depends on $\alpha$ continuously, Lemma A4 implies that $\phi_\alpha(b(\alpha)) < 0$ for all $\alpha > 0$. This completes the proof of Theorem A.

References


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