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FINE STRUCTURES ARISING IN REACTION-DIFFUSION EQUATIONS

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ABSTRACT. We study the functionals containing a small parameter ε and a long-range interaction. Such functionals arise from the stationary problem of the reaction-diffusion system of FitzHugh-Nagumo type and from models for phase separation in diblock copolymers. We consider stationary solutions obtained as global minimizers of the functional.

1. BACKGROUND

Reaction-diffusion systems have been used as the mathematical models of pattern formation in physics, chemistry, mathematical biology and so on. We consider a system of two unknown functions u and v which denote the concentrations of diffusive substances with nonlinear interaction. When the ratio of the diffusion constants is extremely small, very complicated nonuniform steady states appear even in the uniform environment. In order to deal with this case, we study the limit to let the ratio of the diffusion constants tend to 0, as a mathematical approach. Since this is a singular perturbation problem, we need to find out what the limiting state is. In many cases with bistable nonlinearity, the domain is divided into two regions (u is close to 1 in one side and u is close to 0 in the other, for example) and the remaining part becomes the thin layer. We call this "internal layered pattern". In the layer region, the value of u changes very rapidly. The width of the layer tends to 0 in the limit, and the discontinuity surface inside the domain, which is called "sharp interface" generally, arises. So the basic questions are where this surface appears and what kind of shape it has. We usually obtain the curvature dependent type equation for this problem. There are many studies about these questions for similar equations. However in some parameter scaling, since the well-behaved patterns whose limiting state becomes a smooth surface do not appear and the profile of solutions is much more complicated, we had difficulty in analysis by methods as before. Namely, the pattern oscillates rapidly and becomes finer and finer. As a result, the shape is not convergent. This is completely different. Since the treatment of this micro patterns with fine structure is difficult, we do not know many things.

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We deal with this case by taking the reaction-diffusion system of FitzHugh-Nagumo type as a model equation.

$$(1)\qquad \begin{aligned} u_t &= \varepsilon^2 \Delta u - F'(u) - v && \text{in } \Omega \times \mathbb{R}^+, \\ \tau v_t &= \Delta v + u - \gamma v - m && \text{in } \Omega \times \mathbb{R}^+, \\ \frac{\partial u}{\partial n} &= \frac{\partial v}{\partial n} = 0 && \text{on } \partial\Omega \times \mathbb{R}^+, \end{aligned}$$

where $\tau, \varepsilon, \gamma > 0$ are constantsC $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with smooth boundary, $\partial/\partial n$ is outward normal derivative, m is a constant, F is a function such that

(A) $m \in (0,1)$;

(F1) $F \in C^3(\mathbb{R}),\ F(0) = F(1) = 0 < F(s)\ \forall s \in \mathbb{R}\backslash\{0,1\},\ F''(0)F''(1) \neq 0.$

A typical example of F is $F(s) = s^2(1-s)^2$.

The system of two coupled parabolic equations (1) is a typical activator-inhibitor system, which consists of two substances called activator and inhibitor respectively. These equations are the simplified model of Hodgkin-Huxley nerve-impulse conduction equations [10], proposed by FitzHugh [8] and later by Nagumo et al. [14]. This system is also used as models for other chemical and biological systems [9, 28].

Remark. *In our equations, we choose the coordinate system* (u, v) *so that* (A) *and* (F1) *are satisfied. This does not change the boundary condition.*

Let

$$\mathbf{E}(u, \varepsilon) = \int_\Omega \left\{ \frac{\varepsilon^2}{2} |\nabla u|^2 + F(u) + \frac{1}{2}(\gamma v^2 + |\nabla v|^2) \right\} dx, \quad (u, v) \in \mathbf{K}$$

where

$$\mathbf{K} = \{(u, v) \in (H^1(\Omega))^2 \mid -\Delta v + \gamma v = u - m \quad \text{in } \Omega, \quad \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial\Omega\}.$$

Then Euler-Lagrange equation of the energy functional $\mathbf{E}$ corresponds to the stationary problem of (1).

Remark. *Also in the case* $\gamma = 0$*, this functional arises in the model of phase separation of diblock copolymers* [1, 4, 17].

2. Recent developments

In [18], by the variational approach, we succeeded in finding internal layered patterns with complicated structure which increases the intensity of oscillation. Indeed, we constructed, in any space dimension, non-constant stable steady state solutions for small $\varepsilon > 0$ and showed that the total variation denoting the complexity of layered patterns diverges to infinity as $\varepsilon \to 0$ by studying Young measure generated in the limit.

In [19], we deal with a singular limiting problem with a slightly different parameter scalings (ε multiplies the term v in the first equation). In this case, global minimizers have a compactness property in some sense, and they can have the limit. So we can characterize the limit pattern and we find the parameter range where the limit pattern has very large total variation. The advantage here is that we can consider the interface equation corresponding to Euler-Lagrange equation of the limit functional so that we can treat the convergence to interfacial patterns and the complexity of the limit patterns separately.

In [2], we consider, on an interval of arbitrary length, global minimizers of the energy functional **E**. It is shown that every global minimizer is periodic with a period of order $\varepsilon^{1/3}$. Moreover we identify the number of global minimizers and provide asymptotic expansions for the periods and the global minimizers. We not only generalize earlier works but also introduce some new ideas which can make the proof simpler.

In [3], we consider fine structures of the solutions which oscillate more and more rapidly in the singular limit in two dimensional space. In this problem, it has been known since early times that the fine periodic structures with infinite dots often appear in both the experiments and the numerical simulations. However the mathematical analysis was done only for the case which is essentially one dimensional and there were no rigorous analysis of fine dot patterns. We derive the "rescaled reduced energy" from **E** and compute the energy of periodic dot patterns, and show that the hexagonal structure gives the least energy. To put it concretely, by making use of Néron's function which is important in the theory of elliptic curves, the modular invariance of the modular function, and the maximum principle for harmonic functions, we show some monotonicity property of the energy of periodic patterns when deforming the shape of the cell (the periodic unit), and it is shown analytically that the minimum is attained only when the cell is the regular hexagon. Moreover, by using the analytical expression obtained for hexagonal cells, we compare the energy of dot patterns and one of stripe patterns. As the result, it is shown that dot patterns have less energy than stripe patterns in some parameter range, so we can give a theoretical evidence to infinite dots appearing often.

Remark. *The readers can find some other results in* [5, 6, 7, 11, 12, 13, 15, 20, 21, 22, 23, 27, 29] *for FitzHugh-Nagumo type system and* [16, 24, 25, 26] *for diblock copolymer equation.*

References

[1] M. Bahiana & Y. Oono, Cell dynamical system approach to block copolymers, Phys. Rev., 41 (1990), 6763–6771.

[2] X. Chen & Y. Oshita, Periodicity and uniqueness of global minimizers of an energy functional containing a long-range interaction, to appear in SIAM J. Math. Anal.

[3] X. Chen & Y. Oshita, An application of the Modular Function in Interfacial Dynamics, preprint.

[4] R. Choksi & X. F. Ren, On the derivation of a density functional theory for microscopic separation of diblock copolymers, J. Statistical Physics, 113 (2003), 151–176.

[5] E. N. Dancer & S. Yan, Multipeak solutions for the Neumann problem of an elliptic system of FitzHugh-Nagumo type. Proc. London Math. Soc. (3) 90 (2005), 209–244.

[6] E. N. Dancer & S. Yan, A minimization problem associated with elliptic systems of FitzHugh-Nagumo type. Ann. Inst. H. Poincaré Anal. Non Linéaire 21 (2004), 237–253.

[7] E. N. Dancer & S. Yan, Peak solutions for an elliptic system of FitzHugh-Nagumo type. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 2 (2003), 679–709.

[8] R. FitzHugh, Impulses and physiological states in theoretical models of nerve membrane, Biophy. J. 1 (1961), 445–466.

[9] S. P. Hastings, Some mathematical problems from neurobiology. Amer. Math. Monthly 82 (1975), 881–895.

[10] A. Hodgkin & A. Huxley, A quantitative description of membrane current and its application to conduction and excitation in nerve, J. Physiol. 117 (1952), 500–544.

[11] G. A. Klaasen & E. Mitidieri, Standing wave solutions for a system derived from the FitzHugh–Nagumo equation for nerve conduction, SIAM J. Math. Anal. 17 (1986), 74–83.

[12] G. A. Klaasen & W. C. Troy, Stationary wave solutions of a system of reaction–diffusion equations derived from the FitzHugh–Nagumo equations, SIAM J. Appl. Math. 44 (1984), 96–110.

[13] H. Matsuzawa, Asymptotic profiles of variational solutions for a FitzHugh-Nagumo-type elliptic system. Differential Integral Equations 16 (2003), 897–926.

[14] J. S. Nagumo, S. Arimoto & Y. Yoshizawa, An active pulse transmission line simulating nerve axon, Proc. Inst. Radio. Engineers 50 (1962), 2061–2070.

[15] Y. Nishiura & H. Suzuki, Nonexistence of higher dimensional stable Turing patterns in the singular limit. SIAM J. Math. Anal. 29 (1998), 1087–1105.

[16] Y. Nishiura & H. Suzuki, Higher dimensional SLEP equation and applications to morphological stability in polymer problems. SIAM J. Math. Anal. 36 (2004), 916–966.

[17] T. Ohta & K. Kawasaki, Equilibrium morphology of block copolymer melts, Macromolecules, 19 (1986), 2621–2632.

[18] Y. Oshita, On stable nonconstant stationary solutions and mesoscopic patterns for FitzHugh Nagumo equation in higher dimensionC J. Differential Equations, 188 (2003), 110-134.

[19] Y. Oshita, Phase separations and interfaces arising in reaction-diffusion systemsC SIAM. J. Math. Anal. 36 (2004), 479–497.

[20] C. J. Reinecke & G. Sweers, Solutions with internal jump for an autonomous elliptic system of FitzHugh-Nagumo type. Math. Nachr. 251 (2003), 64–87.

[21] C. J. Reinecke & G. Sweers, Existence and uniqueness of solutions on bounded domains to a FitzHugh-Nagumo type elliptic system, Pacif. J. Math. 197 (2001), 183–211.

[22] C. J. Reinecke & G. Sweers, A boundary layer solution to a semilinear elliptic system of FitzHugh-Nagumo type, C. R. Acad. Sci. Paris Sér. I Math. 329 (1999), 27–32.

[23] C. Reinecke & G. Sweers, A positive solution on $\mathbf{R}^N$ to a system of elliptic equations of FitzHugh-Nagumo type. J. Differential Equations 153 (1999), 292–312.

[24] X. Ren & J. Wei, Concentrically layered energy equilibria of the di-block copolymer problem, European J. Appl. Math., 13 (2002), 479–496.

[25] X. Ren & J. Wei, On energy minimizers of the diblock copolymer problem, Interfaces and Free Boundaries, 5 (2003), 193–238.

[26] X. Ren & J. Wei, On the spectra of three-dimensional lamellar solutions of the Diblock copolymer problem, SIAM J. Math. Anal., 35 (2003), 1–32.

[27] X. Ren & J. Wei, Nucleation in the FitzHugh-Nagumo system: interface-spike solutions. J. Differential Equations 209 (2005), 266–301.

[28] G. Sweers & W. C. Troy, On the bifurcation curve for an elliptic system of FitzHugh-Nagumo type. Physica D 177 (2003), 1–22.

[29] J. Wei & M. Winter, Clustered spots in the FitzHugh-Nagumo system, to appear in Journal of Differential Equations.

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