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ON THE REPULSIVE VLASOV-POISSON SYSTEM NEAR VACUUM

MYEONGJU CHAE AND SEUNG-YEAL HA

ABSTRACT. In this note, we present recent results in [3] on collision potential and L^1-stability for the Vlasov-Poisson system near vacuum and address two unresolved issues.

1. Introduction

In this paper, we consider a collisionless plasma consisting of single species under the effect of an electromagnetic field. The issue can be understood via the Vlasov-Maxwell system. When the speed of light is taken to be infinity and a magnetic field is ignored, the Vlasov-Poisson system with a self-consistent electrostatic field is addressed. Suppose there are single species of particles with mass m, charge q and density $f = f(x, v, t)$ in phase space. Here $(x, v) \in \mathbb{R}^N \times \mathbb{R}^N$ denotes position and velocity respectively. The self-consistent electric field is denoted by $E = -\nabla_x \phi$ and the total charge density ρ is

$$\rho(x, t) := q \int_{\mathbb{R}^N} f(x, v, t) dv.$$

In this case, the Vlasov-Poisson system reads as

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \dfrac{q}{m} \nabla_x \phi \cdot \nabla_v f = 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}_+, \\ -\Delta_x \phi = N(N-2)\omega_N \rho, \end{cases} \tag{1.1}$$

with the prescribed initial datum

$$f(x, v, 0) = f_0(x, v) \quad \text{in } \mathbb{R}^N \times \mathbb{R}^N, \tag{1.2}$$

where ω_N is the volume of the unit ball in $\mathbb{R}^N$.

The Vlasov-Poisson system has many applications in the modelling of an electron gun, plasma sheath and galaxies as a large ensemble of stars in plasma physics and astrophysics respectively. The global existence of a smooth solution in one space variable has been proved by Iordanskii [12], two space variables by Ukai and Okabe [17]. On the other hand, for three-dimensional case, Horst [9] established the global existence for spherical and cylindrically symmetric data respectively. For general but small data, Bardos and Degond [2] obtained global existence and finally Pfaffelmoser [14] proved the global existence of a smooth solution with large data and simpler proofs were provided by Schaeffer [16] and Lions and Perthame [13].

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For other issues such as weak solutions, relativistic effects, stablility and dispersion estimates, we refer to [1, 18].

In this paper, we first present a collision potential for the repulsive Vlasov-Poisson system when the dimension of physical space $N \neq 2$. This functional measures the future collisions between charged particles, although the Vlasov-Poisson system does not register them. As in the Boltzmann equation [7, 8], these functionals might be useful to establish L^1-scattering type result for the Vlasov-Poisson-Boltzmann equation near vacuum [6]. Secondly, we study the L^1-stability of smooth C^2-solutions to (1.1) using Gronwall type estimates when the dimension of physical space is sufficiently large, and initial data is smooth and decays fast enough in phase space. Throughout the paper, we denote C by a universal positive constant independent of time t.

The rest of this note is organized in the following manner. In Section 2, we present main results on the collision potential and L^1-stability. In Section 3, we briefly discuss some open problems related to the materials addressed in Section 2.

2. **Main results**

In this section, we present main results in [3] without detailed proofs. We first briefly explain main stability mechanisms leading our main results.

The stability estimates are subject to the following estimates on the time-decay of the electric field and velocity variation of the density function f.

- (E1). The electric field $E(x,t)$ decays and integrable in t:

$$\int_0^\infty ||E(t)||_{L^\infty(\mathbb{R}^N_x)} dt < \infty.$$

- (E2). For fixed $(x,t) \in \mathbb{R}^N \times \mathbb{R}_+$,

$$\Big|\Big| \int_{\mathbb{R}^N} |\nabla_v f(t)| dv \Big|\Big|_{L^\infty(\mathbb{R}^N_x)} \le \frac{C}{(1+t)^{N-1}} \quad \text{and} \quad \Big|\Big| \int_{\mathbb{R}^N} |\nabla_v f(t)| dv \Big|\Big|_{L^1(\mathbb{R}^N_x)} \le C(1+t).$$

Based on the time-integrable decay of the electric field E, we first construct a collision potential $\mathcal{D}(f)$ which is non-increasing along smooth solutions to (1.1) with $N \geq 3$, and satisfies a Lyapunov estimate

$$\mathcal{D}(f(t)) + \int_0^t \Lambda(f(s)) ds = \mathcal{D}(f_0), \qquad t \geq 0,$$

where $\Lambda(f(s)) \geq 0$ is an collision production functional

$$\Lambda(f(t)) := \iiint_{\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N} f(x,v,t) f(x,v_*,t) dv_* dv dx.$$

The above functional measures the possible crossings of the projected particle trajectories in physical space, and hence it can be regarded as a kinetic counterpart of Glimm's interaction potential [5] in hyperbolic conservation laws in one-space dimension. Due to the strong dispersive effect of the free transport part, the crossings of projected trajectories in physical space will tend to be non-increasing in time t. In fact, in the absence of external forces, such collision potentials have been constructed for the full Boltzmann equation in [3, 7, 8] based on the free transport equation $\partial_t f + v \cdot \nabla_x f = 0$ with small initial data. Unlike the Boltzmann equation, where the collision potential is constructed based on free transport part, the external force term $\nabla_x \phi \cdot \nabla_v f$ produces error terms which cannot be controlled by the collision production functional $\Lambda(f(t))$. We overcome this difficulty by devising a new functional capturing the nonlinear feature of the system, and by essentially using time-integrable decay of the electric field; thus small initial data is assumed.

Secondly, we establish the uniform L^1-stability for (1.1) with $N \geq 4$:

$$\sup_{0 \leq t < \infty} ||f(t) - \bar{f}(t)||_{L^1} \leq G||f_0 - \bar{f}_0||_{L^1}, \tag{2.1}$$

where G is a positive constant independent of t, and a simplified notation for L^1-norm is used ;

$$||f(t) - \bar{f}(t)||_{L^1} := ||f(\cdot,\cdot,t) - \bar{f}(\cdot,\cdot,t)||_{L^1(\mathbb{R}^N \times \mathbb{R}^N)}.$$

Unlike the Boltzmann equation [7, 8], the nonlinear functional approach incorporating the Lyapunov functional $\mathcal{D}(f)$ cannot be applied to L^1-stability estimates for (1.1); By direct calculation, $|f - \bar{f}|$ satisfies

$$\partial_t|f - \bar{f}| + v \cdot \nabla_x|f - \bar{f}| + E(f) \cdot \nabla_v|f - \bar{f}| \leq |E(\bar{f}) - E(f)||\nabla_v \bar{f}|. \tag{2.2}$$

This yields

$$\frac{d}{dt}||f(t) - \bar{f}(t)||_{L^1} \leq \iint_{\mathbb{R}^N \times \mathbb{R}^N} |E(\bar{f}) - E(f)||\nabla_v \bar{f}| dv dx. \tag{2.3}$$

In contrast the time-derivative of $\mathcal{D}(|f - \bar{f}|)$ based on the left hand sides of (1.1) and (2.2) yields good decay terms:

$$-\iiint_{\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N} |f - \bar{f}|(x, v, t) f(x, v_*, t) dv_* dv dx,$$

however, these terms cannot control the right hand side of (2.3). In fact these good terms can be used to control the error terms due to the $Q(f, f) - Q(\bar{f}, \bar{f})$ in the L^1-stability estimates of the Vlasov-Poisson-Boltzmann system (see [7, 8] for the related issue of the Boltzmann equation), hence instead of using the nonlinear functional approach for (2.1), we employ Gronwall type estimates incorporationg (E2) in Theorem B:

$$||f(t) - \bar{f}(t)||_{L^1} \leq ||f_0 - \bar{f}_0||_{L^1} + C\int_0^t (1 + s)^{-(N-2)}||f(s) - \bar{f}(s)||_{L^1} ds. \tag{2.4}$$

Since

$$\int_0^t (1 + s)^{-(N-2)} < C \quad \text{for} \quad N \geq 4,$$

we can establish the uniform L^1-stability (2.1) for physical space dimension $N \geq 4$ using the Gronwall estimate.

The main hypotheses (H) employed in this paper are as follows.

- (H1). Initial data f_0 are twice continuously differentiable:

$$f_0 \in C^2(\mathbb{R}^N \times \mathbb{R}^N).$$

- (H2). Initial data are small and decay at infinity in phase space:

$$\sum_{0 \le i,j \le 2} \sup_{x,v} (1+|x|^2)^{\frac{\mu_1}{2}} (1+|v|^2)^{\frac{\mu_2}{2}} |\nabla_x^i \nabla_v^j f_0(x,v)| \le \varepsilon_0,$$
$$\text{where} \quad \mu_1 > N+2, \quad \mu_2 > N+1 \quad \text{and} \quad 0 < \varepsilon_0 \ll 1.$$

Our first main result shows that the system (1.1) with $N \ge 3$ admits a collision potential.

Theorem A: Suppose the main hypotheses (H) with $N \ge 3$ hold, and let f be a smooth solution to (1.1) corresponding to initial datum f_0. Then there exists a collision potential $\mathcal{D}(f)$ satisfying a Lyapunov estimate:

$$\mathcal{D}(f(t)) + \int_0^t \Lambda(f(s))ds = \mathcal{D}(f_0), \qquad t \ge 0.$$

Remark A 1. Following the arguments in [2], the global existence of smooth C^2-solutions satisfying the main estimates (E1) - (E2) can be shown.
2. For $N = 1$, we can construct a similar Lyapunov functional under the monotonicity assumption of the electric field (see [3]).

Our second main theorem is concerned with the uniform L^1-stability.

Theorem B: Suppose the main hypotheses (H) with $N \ge 4$ hold, and let f and $\bar{f}$ be smooth solutions to (1.1) corresponding to initial data f_0 and $\bar{f}_0$ respectively. Then smooth solutions are uniformly L^1-stable with respect to initial data:

$$\sup_{0 \le t < \infty} ||f(t) - \bar{f}(t)||_{L^1} \le G||f_0 - \bar{f}_0||_{L^1},$$

where G is a positive constant independent of time t.

3. **Open problems**

In this part, we briefly discuss unresolved questions related to the main results in Section 2.

(1) Question A: When $N = 2$, construct a collision potential decreasing in time like $N \ge 3$ case, or find a functional implying that the Vlasov-Poisson system with initially finite collisions evolves into the system with infinite collisions .

For $N = 2$, the electric field E decays like $\frac{1}{t}$ in uniform norm in x, hence (E1) in Section 1 does not hold, moreover it seems to us that the uniform separability of characteristics is not known.

$$\frac{1}{C_1}|t-s||v-w| \le |X(s;t,x,v) - X(s;t,x,w)| \le C_1|t-s||v-w|, \qquad 0 \le s \le t.$$

Here C_1 is a positive constant independent of t and s.

(2) Question B: Prove or disprove L^1-stability in low dimensions $N \leq 3$.

For lower dimensional case, the estimate (2.4) does not yield uniform L^1-stability estimate. For example $N = 2, 3$, we have

$$||f(t) - \bar{f}(t)||_{L^1} \leq C(t)||f_0 - \bar{f}_0||_{L^1} \qquad \text{for } t \geq 0.$$

Here $C(t)$ is a unbounded function depending on t. Therefore, the direct estimates based on only the Gronwall' type estimate does not yield the uniform stability (2.1), however when $N = 3$, we expect the time-decreasing collision potential can play a role to provide uniform stability as in [7, 8].

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School of Mathematics, KIAS, Korea
E-mail address: mchae@kias.re.kr

Department of Mathematical Sciences, Seoul National University, Seoul 151-747, Korea
E-mail address: syha@math.snu.ac.kr