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LINEAR ELASTICITY IN 3-DIMENSIONAL LIPSCHITZ DOMAINS

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ABSTRACT. We study the Green's function of the linear elasticity in a three dimensional bounded Lipschitz domain Ω. Our main results assert that if G is Green's function of the linear elasticity, then $|G(X,Y)| \leq c\frac{1}{|X-Y|}$ and $|G(X,Y)| \leq c\frac{min(dist(X,\partial\Omega), dist(Y,\partial\Omega))^\eta}{|X-Y|^{1+\eta}}$ for some $\eta > 0$ and for all $X, Y \in \overline{\Omega}$. For its application, we deduce $L^p(\Omega)$ estimate of solutions of inhomogeneous elasticity equations and show the solvability of elasticity equations with boundary data $B^1_\alpha(\partial\Omega)$.

1. INTRODUCTION

The main purpose of this work is to study the Green's function G of elasticity equations

$$\mu\Delta u + (\lambda + \mu)\nabla div\, u = 0 \quad \text{in} \quad \Omega \tag{1.1}$$

where $\mu > 0$ and $\lambda > \frac{-2\mu}{3}$ are constants (Lamé moduli) and Ω is bounded Lipschitz domain in $\mathbf{R}^3$.

Let $\Gamma(X) = (\Gamma_{ij}(X))_{1\leq i,j\leq 3}$ be the matrix of fundamental solutions

$$\Gamma_{i,j}(X) = \frac{3\mu + \lambda}{4\pi\mu(2\mu+\lambda)}\frac{\delta_{ij}}{|X|} + \frac{\mu - \lambda}{4\pi\mu(2\mu+\lambda)}\frac{X_i X_j}{|X|^3}.$$

Then there is a matrix Green's function $G(X,Y) = (G_{ij}(X,Y))_{1\leq i,j\leq 3}$ where

$$G(X,Y) = \Gamma(X-Y) - v^X(Y)$$

and, for $X \in \Omega$, v^X is the matrix-valued solution to the Dirichlet problem (1.1) with boundary data

$$v^X(Q) = \Gamma(X-Q) \quad \text{on} \quad \partial\Omega.$$

(See [1] and [3].)

In the case of Laplace equation, the pointwise estimates for the Green's function L is well-known. When Ω is bounded Lipschitz domain in $\mathbf{R}^n$ with $n \geq 3$, there exist constants $c > 0$, $\eta > 0$ such that

$$|L(X,Y)| \leq c\frac{1}{|X-Y|^{n-2}} \quad \text{and} \quad |L(X,Y)| \leq c\frac{min(dist(X,\partial\Omega), dist(Y,\partial\Omega))^\eta}{|X-Y|^{n-2+\eta}}$$

for all $X, Y \in \overline{\Omega}$.

When Y is located away from $\partial\Omega$, an interior estimate provide critical growth estimates. In fact, for any $n > 2$ and Lipschitz domain Ω in $\mathbf{R}^n$, we have

$$|G_{ij}(X,Y)| \leq c_d|X-Y|^{2-n}, \quad |D_k G_{ij}(X,Y)| \leq c_d|X-Y|^{1-n}$$

for all $X \in \overline{\Omega}$ and for all $Y \in \Omega$ with $dist(Y, \partial\Omega) \geq d > 0$, $c_d \to \infty$ as $d \to 0$ where $D_k u = \frac{\partial u}{\partial X_k}$. Moreover, if Ω is of class $C^{1,\lambda}$ in $\mathbf{R}^n$, $\lambda \in (0,1)$, then the Green's function G of (1.1) satisfies that

$$|G_{ij}(X,Y)| \leq c|X-Y|^{2-n}, \quad |D_k G_{ij}(X,Y)| \leq c|X-Y|^{1-n} \tag{1.2}$$

for all $X, Y \in \bar{\Omega}$. Such estimates hold even in the Stokes equations. We refer [8] for 3-dimension and an extension of (1.2) to higher dimensional Stokes equations can be found in a result of Solonnikov(1970)[9].

Here, we will show that the estimates of Green's function G of elasticity equations is similar with the estimates of Green's function L of Laplace equation if Ω is a bounded Lipschitz domain in $\mathbf{R}^3$.(See Theorem 2.2.) Our estimates hold only in 3-dimension since an ϵ-type integrabilty gain is possible in 3-dimension only.

For its application, we study the L^q-estimates of the solution of the inhomogeneous elasticity equations

$$\begin{cases} \mu\Delta u + (\lambda+\mu)\nabla(div\, u) = f & \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{1.3}$$

We also show that solvability of the homogeneous and inhomogeneous boundary value problems.

We let Ω be a bounded Lipschitz domain in $\mathbf{R}^3$. We define $B(P,R)$ by an open ball in $\mathbf{R}^3$ with center P and radius R and $D(P,R) = \Omega \cap B(P,R)$. We denote $X, Y, Z \in \Omega$ are points in Ω, P, Q, are points on $\partial\Omega$ and c is a positive real number dependent only on the size and Lipschitz character of Ω. We also define Sobolev spaces $W^{k,p}(\Omega)$ by $W^{k,p}(\Omega) = \{f : ||f||_{W^{k,p}(\Omega)} = \left(\int_\Omega |f|^p + |\nabla^k f|^p dx\right)^{1/p} < \infty\}$.

2. Main results

First, we state a localized pointwise estimate near boundary.

Theorem 2.1. *We suppose that $u \in W^{1,2}(D(P,R))$ is a solution of*

$$\begin{cases} \mu\Delta u + (\lambda+\mu)\nabla(div\, u) = 0 & in \quad D(P,R) \\ u = 0 \quad on \quad B(P,R)\cap\partial\Omega. \end{cases} \tag{2.4}$$

Then there exists $R_0 > 0$, depending only Lipschitz property of Ω, such that for any $q > 0$, $0 < r < R$,

$$\sup_{D(P,r)} |u| \leq c_q \left(\frac{1}{(R-r)^3} \int_{D(P,R)} |u|^q dX \right)^{1/q} \tag{2.5}$$

where c_q is only dependent on q, Ω and independent on r, R, P, u.

Green's functions have zero boundary data and hence, with the estimate on the balls $B(X, dist(X,Y))$ like Theorem 2.1, we announce the following theorem congruous to Laplace equation;

Theorem 2.2. *Let $G(X,Y)$ be Green's function of the linear elasticity equations (1.1). Then, we have*

$$|G(X,Y)| \le c\frac{1}{|X-Y|}, \quad |G(X,Y)| \le c\frac{min(dist(X,\partial\Omega), dist(X,\partial\Omega))^{\eta}}{|X-Y|^{1+\eta}}$$

for some $\eta > 0$ and for all $X, Y \in \bar{\Omega}$.

As a natural application, we obtain $L^p - L^q$ estimate of the nonhomogeneous equations.

Theorem 2.3. *Let $f \in L^q(\Omega), q > 1$ and u be a solution of the equation (1.3). If $q > \frac{3}{2}$, then*

$$||u||_{L^\infty(\Omega)} \le c||f||_{L^q(\Omega)}$$

and if $q < \frac{3}{2}$, then

$$||u||_{L^p(\Omega)} \le c||f||_{L^q(\Omega)}$$

where $\frac{1}{p} = \frac{1}{q} - \frac{2}{3}$.

For $1 \le q < \infty$ and $0 < \alpha < 1$, we define $B^q_\alpha(\Omega)$ to be the collection of functions u on Ω with the norm

$$||u||_{B^q_\alpha(\Omega)} \equiv ||u||_{L^q(\Omega)} + \left(\int\int_{\Omega\times\Omega} \frac{|u(X)-u(Y)|^q}{|X-Y|^{3+\alpha q}} dXdY\right)^{1/q} < \infty.$$

For $1 < \alpha < 2$ we define

$$B^q_\alpha(\Omega) \equiv ||u||_{L^q(\Omega)} + ||Du||_{B^q_{\alpha-1}(\Omega)} < \infty.$$

We may define $B^q_\alpha(\partial\Omega)$ for $0 < \alpha < 1$ and $1 \le q < \infty$ in a similar manner with the integral over $\partial\Omega \times \partial\Omega$.

Theorem 2.4. *There exists $\eta > 0$ depending only on the Lipschitz property of Ω such that, for all α, $1-\eta < \alpha < 1$, and all $g \in B^1_\alpha(\partial\Omega)$, there is a unique solution u to the homogeneous Dirichlet problem*

$$\begin{cases} \mu\Delta u + (\lambda+\mu)\nabla(div\, u) = 0 \quad in \quad \Omega, \\ u = g \quad on \quad \partial\Omega, \end{cases} \tag{2.6}$$

satisfying

$$\int_\Omega \delta^{1-\alpha}|\nabla^2 u| + |\nabla u| + |u| \le c||g||_{B^1_\alpha(\partial\Omega)}.$$

In particular, $\nabla u \in B^1_\alpha(\Omega)$, namely, $u \in B^1_{1+\alpha}(\Omega)$.

Using Theorem 2.4 and the results in [1], [3], [4] and interpolation, we obtain the a solvability in critical Besov spaces. A corresponding result to Laplace equation had been obtained by Jerison and Kenig[7].

Corollary 2.5. *Let Ω be a bounded Lipschitz domain in $\mathbf{R}^3$. For a given $0 < \eta \le 1$, we define p_0 and p_0' by $\frac{1}{p_0} = \frac{1+\eta}{2}$ and $\frac{1}{p_0'} = \frac{1-\eta}{2}$. We also let s and p be numbers satisfying one of the following:*
(a) $p_0 < p < p_0'$ and $0 < s < 1$.
(b) $1 < p \le p_0$ and $\frac{2}{p} - 1 - \eta < s < 1$.

(c) $p_0' \le p < \infty$ *and* $0 < s < \frac{2}{p} + \eta$.

Then there exists $\eta \in (0,1)$ *depending only on the Lipschitz constant of* Ω *such that for every* $g \in B^p_s(\partial\Omega)$ *there exists a unique solution* u *of (2.6) such that* $u = g$ *on* $\partial\Omega$ *and* $u \in L^p_{s+1/p}(\Omega)$. *Also, the solution* u *belongs to* $B^p_{s+1/p}(\Omega)$.

Corollary 2.6. *Let* Ω *be a bounded Lipschitz domain in* $\mathbf{R}^3$. *There exists* $\eta, 0 < \eta \le 1$, *depending only on the Lipshitz property of* Ω *such that for every* $f \in L^p_{\alpha-2}$ *there is a unique solution* $u \in L^p_\alpha(\Omega)$ *to the (1.3) provided one of the following holds:*
(a) $p_0 < p < p_0'$ *and* $\frac{1}{p} < \alpha < 1 + \frac{1}{p}$
(b) $1 < p \le p_0$ *and* $\frac{3}{p} - 1 - \epsilon < \alpha < 1 + \frac{1}{p}$
(c) $p_0' \le p < \infty$ *and* $\frac{1}{p} < \alpha < \frac{3}{p} + \eta$
where $\frac{1}{p_0} = \frac{1}{2} + \frac{\eta}{2}$ *and* $\frac{1}{p_0'} = \frac{1}{2} - \frac{\eta}{2}$. *Moreover, we have the estimate*

$$||u||_{L^p_\alpha(\Omega)} \le c||f||_{L^p_{\alpha-2}(\Omega)}$$

for all $f \in L^p_{\alpha-2}(\Omega)$.

Next, we consider the estimate of derivative of Green's function of G. In fact, in the case of corned domain, the derivative blows up as approaching to the corner. Hence, we can not expect an estimate of the type $|X-Y|^{-2}$ near boundary.

Theorem 2.7. *If* $|X - Y| \ge \frac{1}{2} min(dist(X), dist(Y))$, *then*

$$|\nabla_X G(X,Y)| \le c \frac{min(dist(X)^{\eta-1}, dist(Y)^{\eta-1})}{|X-Y|^{1+\eta}}$$

and if $|X - Y| \le \frac{1}{2} min(dist(X), dist(Y))$, *then*

$$|\nabla_X G(X,Y)| \le c \frac{1}{|X-Y|^2}.$$

Using Theorem 2.7, we obtain the estimate of derivative of G.

Theorem 2.8. *For all* $X \in \Omega$, $\nabla_X G(X, \cdot) \in L^p(\Omega)$ *for all* $1 \le p \le 1 + \eta$ *for some* $\eta > 0$. *Furthermore, there exists positive* $c > 0$ *independent on* X *such that*

$$||\nabla_X G(X, \cdot)||_{L^p(\Omega)} < c\, max(dist(X)^{3-2p}, dist(X)^{1-2p}).$$

3. Proof of Theorem 2.1

We regard $div(u)$ as a forcing term to Laplace equation and then an elliptic estimate holds near boundary. Hence, we have a maximum estimate by $L^q, q > 3$ norm of $div(u)$.

Lemma 3.1. *Let* $P \in \Omega$, $D(P,R)$ *and* u *satisfy the assumption in Theorem 2.1 and* $q > 3$. *Then* u *satisfies that*

$$\sup_{D(P,\frac{1}{2}R)} |u| \le c\{ \left(\frac{1}{R^3} \int_{D(P,R)} |u|^2 dX \right)^{1/2} + R^{1-\frac{3}{q}} ||div\, u||_{L^q(D(P,R))} \}$$

where c *is positive constant independent on* u, P.

To control $||div\, u||_{L^q(D_R)}$, we introduce a Proposition which is basically trace estimate.

Proposition 3.2. *Let Ω be a bounded Lipschitz domain in $\mathbf{R}^3$. There exists $\epsilon > 0$ so that if $2 \le q \le 3+\epsilon$, and $g \in W^{1-1/q,q}(\partial\Omega)$ then the solution u of*

$$\begin{cases} \mu\Delta u + (\lambda+\mu)(\nabla div\, u) = 0 \quad in \quad \Omega, \\ u = g \quad on \quad \partial\Omega \end{cases}$$

satisfies

$$(\int_\Omega |Du(X)|^q dX)^{1/q} \le c||g||_{W^{1-1/q,q}(\partial\Omega)}.$$

Proof.(See Theorem 1.3 in [1] and Theorem 3 in [4]). □

Lemma 3.3. *Let u satisfy the assumption in Theorem 2.1. Then, we have that*

$$(\int_{D_R} |Du|^q)^{\frac{1}{q}} \le cR^{\frac{3}{q}-1}(\frac{1}{R^3}\int_{D_{3R}} |u|^2)^{\frac{1}{2}}$$

for all $P \in \partial\Omega, R > 0$ where c is independent on P, R.

Proof.

We take $q > 3$, then by Proposition 3.2 and Sobolev inequality, it follows that, for all $1 < \tau < \frac{3}{2}$,

$$||Du||_{L^q(D_R)} \le ||Du||_{L^q(D_{\tau R})} \le c||u||_{W^{1-1/q,q}(\partial D_{\tau R})} \le c||u||_{W^{1,\frac{2}{3}q}(\partial D_{\tau R})}.$$

Hence, in integral expression, we have

$$(3.7)\quad (\int_{D_R} |Du|^q)^{\frac{2}{3}} \le c\left(\int_{\partial D_{\tau R}} |u|^{\frac{2}{3}q} + |\nabla_T u|^{\frac{2}{3}q}\right) \le c\left(\int_{\Omega\cap\partial D_{\tau R}} |u|^{\frac{2}{3}q} + |\nabla_T u|^{\frac{2}{3}q}\right).$$

For the last inequality, we used the fact $u = 0$ on $\partial D_\tau \cap \partial\Omega$. Integrating (3.7) over $\tau \in [1, \frac{3}{2}]$, we obtain that

$$(\int_{D_R} |Du|^q)^{\frac{1}{q}} \le c\left(\frac{1}{R}\int_{D_{\frac{3}{2}R}} |u|^{\frac{2}{3}q} + |\nabla u|^{\frac{2}{3}q}\right)^{\frac{3}{2q}}.$$

We take $q > 3$ so that the reverse Hölder inequality holds, that is,

$$(3.8)\qquad (\frac{1}{R^3}\int_{D_{\frac{3}{2}R}} |\nabla u|^{\frac{2}{3}q})^{\frac{3}{2q}} \le c(\frac{1}{R^3}\int_{D_{2R}} |\nabla u|^2)^{\frac{1}{2}}.$$

Using (3.8) and Caccioppoli inequality, we obtain

$$\begin{aligned} (\frac{1}{R}\int_{D_{\frac{3}{2}R}} |Du|^{\frac{2}{3}q})^{\frac{3}{2q}} &\le R^{3/q}(\frac{1}{R^3}\int_{D_{\frac{3}{2}R}} |Du|^{\frac{2}{3}q})^{\frac{3}{2q}} \\ &\le R^{3/q}(\frac{1}{R^3}\int_{D_{2R}} |Du|^2)^{\frac{1}{2}} \\ &\le R^{3/q-1}(\frac{1}{R^3}\int_{D_{3R}} |u|^2)^{\frac{1}{2}}. \end{aligned}$$

To estimate $(\frac{1}{R}\int_{D_{\frac{3}{2}R}}|u|^{\frac{2}{3}q})^{\frac{3}{2q}}$, we use Sobolev inequality and Caccioppoli inequality so that

$$\begin{aligned}(\tfrac{1}{R}\int_{D_{\frac{3}{2}R}}|u|^{\frac{2}{3}q})^{\frac{3}{2q}} &\leq cR^{-\frac{3}{2q}}\left(R^{\frac{-6q}{9+2q}}\int_{D_{2R}}|u|^{\frac{6q}{9+2q}}+\int_{D_{2R}}|Du|^{\frac{6q}{9+2q}}\right)^{\frac{9+2q}{6q}}\\ &\leq cR^{-\frac{3}{2q}-1+\frac{9+2q}{2q}}(\tfrac{1}{R^3}\int_{D_{2R}}|u|^2)^{1/2}+R^{-\frac{3}{2q}+\frac{9+2q}{2q}}(\tfrac{1}{R^3}\int_{D_{2R}}|Du|^2)^{1/2}\\ &\leq cR^{\frac{3}{q}}(\tfrac{1}{R^3}\int_{D_{3R}}|u|^2)^{1/2}\end{aligned}$$

and this completes the proof of Lemma 3.3.□

By Lemma 3.1 and Lemma 3.3,

$$sup_{D_{\frac{1}{2}R}}|u| \leq c(\frac{1}{R^3}\int_{D_R}|u|^2)^{\frac{1}{2}}.$$

Applying the above result to $B(Y,\theta R)$ for any $Y \in D_{\theta R}$, we obtain the following inequality

$$||u||_{L^\infty(D_{\theta R})} \leq c\frac{1}{[(1-\theta)R]^{\frac{3}{2}}}||u||_{L^2(D_R)}$$

and this is the proof of (2.5) when $p=2$. It remains the case $p \neq 2$

In the case $p>2$, (2.5) is trivial from Hölder inequality. Now we prove the case for $p \in (0,2)$. For $p \in (0,2)$ we have

$$\int_{D_R}(u)^2 \leq ||u||^{2-p}_{L^\infty(B_R)}\int_{B_R}(u)^p$$

and hence by Hölder inequality

$$\begin{aligned}||u||_{L^\infty(D_{\theta R})} &\leq c\frac{1}{[(1-\theta)R]^{\frac{3}{2}}}||u||^{1-\frac{p}{2}}_{L^\infty(D_R)}\left(\int_{D_R}(u)^p dx\right)^{\frac{1}{2}}\\ &\leq \frac{1}{2}||u||_{L^\infty(D_R)}+c\frac{1}{[(1-\theta)R]^{\frac{3}{p}}}\left(\int_{D_R}(u)^p dx\right)^{\frac{1}{p}}.\end{aligned}$$

Set $f(t)=||u||_{L^\infty(D_t)}$ for $t \in (0,1]$. Then for any $0<r<R$

$$f(r) \leq \frac{1}{2}f(R)+c\frac{1}{(R-r)^{\frac{3}{p}}}||u||_{L^p(D_R)}.$$

Applying the following lemma, we obtain (2.5) and finish the proof of Theorem 2.1.□

Lemma 3.4. *Let $f(t) \geq 0$ be bounded in $[\tau_0,\tau_1]$ with $\tau_0 \geq 0$. Suppose for $\tau_0 \leq t < s \leq \tau_1$ we have*

$$f(t) \leq \theta f(s)+\frac{A}{(s-t)^\alpha}+B$$

for some $\theta \in [0,1)$. Then for any $\tau_0 \leq t < s \leq \tau_1$ there holds

$$f(t) \leq c(\alpha,\theta)\left(\frac{A}{(s-t)^\alpha}+B\right).$$

Proof. See Lemma 3.1 in [[6], page 161].

4. Proof of Theorem 2.2

We fix $X_0, Y_0 \in \Omega$ and let $r = \frac{|X_0 - Y_0|}{4}$. Suppose that $f \in C_0^\infty(\Omega \cap B(Y_0, r))$. Let $u(X) = \int_\Omega G(X,Z)f(Z)dZ$. Then u satisfies that

$$\begin{cases} \mu\Delta u + (\lambda+\mu)\nabla(div\, u) = f \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

By Sobolev inequality,

$$||u||^2_{L^6(\Omega)} \le c||\nabla u||^2_{L^2(\Omega)} = c\int_\Omega -(\operatorname{div} u)^2 + < f, u >$$

$$\le c\int_\Omega | < f, u > | \le c||u||^2_{L^6(\Omega)}||f||_{L^{\frac{6}{5}}(\Omega)}.$$

Hence

$$||u||^2_{L^6(\Omega)} \le c||f||_{L^{\frac{6}{5}}(\Omega)}.$$

If $B(X_0, \frac{1}{2}r) \subset \Omega$, then the energy estimate implies

$$|u(X_0)| \le c(\frac{1}{r^3}\int_{B(X_0,\frac{1}{2}r)} |u|^2 dX)^{\frac{1}{2}}.$$

If $B(X_0, \frac{1}{2}r) \cap \partial\Omega \ne \emptyset$, then the maximum estimate Theorem 2.1 brings

$$|u(X_0)| \le c\left(\frac{1}{r^3}\int_{\Omega\cap B(X_0,r)} |u(Y)|^2 dY\right)^{1/2}$$

$$\le c\frac{1}{r^{\frac{1}{2}}}||u||_{L^6(\Omega)} \le c\frac{1}{r^{\frac{1}{2}}}||f||_{L^{6/5}(\Omega)}.$$

With the characterization of L^6 norm by its dual space $L^{6/5}$ we have

$$\left(\int_{\Omega\cap B(Y_0,r)} |G(X_0,Y)|^6 dY\right)^{1/6} \le \frac{c}{r^{\frac{1}{2}}}.$$

Again, from the maximum estimate Theorem 2.1, we conclude that

$$\begin{aligned} |G(X_0,Y_0)| \quad &\le c\left(\tfrac{c}{r^3}\textstyle\int_{\Omega\cap B(Y_0,r)} |G(X_0,Y)|^2 dY\right)^{1/2} \\ &\le c\left(\tfrac{c}{r^3}\textstyle\int_{\Omega\cap B(Y_0,r)} |G(X_0,Y)|^6 dY\right)^{1/6} \\ &\le c\tfrac{c}{r}. \end{aligned}$$

Therefore the proof of first part of Theorem 2.2 is completed.

To prove the second part of Theorem 2.2, we need a local Hölder estimate which is stated in the following lemma. But, the Hölder exponent is limited as a small number only.

Lemma 4.1. *Let (u, π) satisfy (2.4). Then u is in Holder space. Furthermore,*

$$|u(X) - u(Y)| \le c\left(\frac{|X-Y|}{R}\right)^\eta \left(\frac{1}{R^3}\int_{D_R} |u|^2 dX\right)^{\frac{1}{2}}$$

for all $X, Y \in \overline{D}_{\frac{1}{2}R}$ for some $\eta > 0$.

Proof.

By Morrey inequality,

$$|u(X) - u(Y)| \leq c|X - Y|^{\eta}||Du||_{L^q(D_R)}$$

for all $X, Y \in \overline{D}_{\frac{1}{2}R}$ where $q > 3$ is in Proposition 3.2 and $\eta = 1 - \frac{3}{q}$. By Lemma 3.3, L^q norm of Du is dominated by L^2 norm of u.□

Let $|X - Y| = 4R$. If $R \leq \frac{1}{16}dist(Y, \partial\Omega)$, then $\min(dist(X, \partial\Omega), dist(Y, \partial\Omega))$ is comparable to $|X - Y|$ and the second estimate follows from the first estimate of $G(X, Y)$. If $R > \frac{1}{16}dist(Y, \partial\Omega)$, then $u(Z) = G(X, Z)$ satisfies (2.4) in $D(Y, R) = \Omega \cap B(Y, R)$. Hence by the Hölder estimate Lemma 4.1, we obtain that

$$|G(X, Z_1) - G(X, Z_2)| \leq c\left(\frac{|Z_1 - Z_2|}{R}\right)^{\eta}\left(\frac{1}{R^3}\int_{D_R}|G(X, Z)|^2 dZ\right)^{\frac{1}{2}}$$

for all $Z_1, Z_2 \in \overline{D(Y, \frac{1}{2}R)}$. We take $Z_1 = Y$ and $Z_2 \in \partial\Omega$ such that $|Y - Z_2| = dist(Y, \partial\Omega)$. Hence

$$|G(X, Y)| \leq c(\frac{dist(Y, \partial\Omega)}{R})^{\eta}\left(\frac{1}{R^3}\int_{D_R}|G(X, Z)|^2 dZ\right)^{\frac{1}{2}}.$$

Since $|G(X, Z)| \leq c\frac{1}{|X - Y|}$ for all $Z \in D(Y, R)$, it follows that

$$\left(\frac{1}{R^3}\int_{D_R}|G(X, Z)|^2 dZ\right)^{\frac{1}{2}} \leq c(\frac{1}{R^3}\frac{1}{R^2}R^3)^{\frac{1}{2}} = c\frac{1}{R}.$$

Hence we have that

$$|G(X, Y)| \leq c(\frac{dist(Y, \partial\Omega)}{R})^{\eta}\left(\frac{1}{R^3}\int_{D_R}|G(X, Z)|^2 dZ\right)^{\frac{1}{2}} \leq c\frac{dist(Y, \partial\Omega)^{\eta}}{|X - Y|^{1+\eta}}.$$

Therefore we complete the proof of Theorem 2.2

5. Proof of Theorem 2.3

The solution u of (1.3) is represented as

$$u(X) = \int_{\Omega} G(X, Y)f(Y)dY$$

for all $X \in \Omega$. By Theorem 2.2, we have that for all $X \in \Omega$

$$|u(X)| \leq \int_{\Omega}|G(X, Y)||f(Y)|dY \leq c\int_{\Omega}\frac{1}{|X - Y|}|f(Y)|dY.$$

When $q > \frac{3}{2}$, we use Hölder inequality to get

$$|u(X)| \leq c||\frac{1}{|X - \cdot|}||_{L^{\frac{q}{q-1}}(\Omega)}||f||_{L^q(\Omega)}$$

for all $X \in \Omega$. Since $\frac{q}{q-1} < 3$, X is chosen arbitrarily and Ω is bounded, we obtain

$$||u||_{L^{\infty}(\Omega)} \leq ||f||_{L^q(\Omega)}.$$

When $q < \frac{3}{2}$, we remind the Riesz potential estimate[10]. In conclusion, we have that, for $\frac{1}{p} = \frac{1}{q} - \frac{2}{3}$,

$$||u||_{L^p(\Omega)} \leq ||f||_{L^q(\Omega)}.$$

Therefore we complete the proof of Theorem 2.3 □

6. Proof of Theorem 2.4

To prove Theorem 2.4, we use the atomic characterization of $B^1_\alpha(\partial\Omega)$.(See [5]) An $(\alpha,1)$-atom is a function a on $\partial\Omega$ satisfying

$$|a| \le r^{\alpha-2}, \quad |\nabla_T a| \le r^{\alpha-3}, \quad \text{and} \quad \text{supp } a \subset \Delta(P,r)$$

for some "surface ball" $\Delta(P,r) = \partial\Omega \cap B(P,r)$ of size r. Every $g \in B^1_\alpha(\partial\Omega)$ has the form $g = \sum s_k a_k$ where each function a_k is an atom and $\sum |s_k| \le c||g||_{B^1_\alpha(\partial\Omega)}$. For the proof we need only check the case of the single atom $g = a$. Since the domain Ω is bounded, it suffices to consider only $r < r_0$ for some fixed constant r_0, $0 < r_0 < 1$.

Since Ω is locally Lipschitz graph domain, there exists $r_1 > 0$ such that, for all $P \in \partial\Omega$, we have a local coordinate system and a Lipschitz function $\phi : \mathbf{R}^{n-1} \to R$ with $|\nabla\phi| \le M$ such that

$$B(P,r_1)\cap\Omega \subset \{(x,y) \in B(P,r_1) \mid y > \phi(x)\} \quad \text{and} \quad \partial\Omega\cap B(P,r_1) \subset \{(x,\phi(x)) \mid |x| < r_1\}.$$

For notational simplicity we suppose that the surface ball Δ on which a is supported has center at $P = 0$, the origin. Also, we denote

$$U(\rho) = \{(x,y) \in \mathbf{R}^{n-1} \times \mathbf{R} \mid |x| < \rho \quad \text{and} \quad |y| < 2M\rho\}$$

and $\Delta_r = \Delta(0,r)$. We can suppose that in a neighborhood $U = U(r_1)$ of 0, with $r_1 > 100r_0$, the boundary $\partial\Omega$ is given by the graph of the Lipschitz function.

The solution u of (2.6) with the boundary data $(\alpha,1)$-atom a is represented as

$$u(X) = \int_{\partial\Omega} \frac{\partial G}{\partial\rho_Q}(X,Q)a(Q)dQ$$

where

$$\frac{\partial G}{\partial\rho_Q}(X,Q) = \lambda(div\, u)\mathbf{n} + \mu(\nabla u + \nabla u^T)\mathbf{n} + \frac{2\mu^2}{3\mu+\lambda}((div\, u)\mathbf{n} - (\nabla\, u)^T\mathbf{n})$$

for $Q \in \partial\Omega$.(See [1] and [3])

Lemma 6.1. *If $X \in \Omega$ and $|X| \ge Mr$, then*

$$\int_{\Delta_r} |\frac{\partial G}{\partial\rho_Q}(X,Q)|dQ \le c(\frac{r}{|X|})^{1+\eta}$$

where η is defined in Lemma 4.1.

Proof.

For $\tau \in [1, \frac{3}{2}]$, we have(See Lemma 1.5, Lemma 1.14 and Lemma 1.23 in [3])

$$\begin{aligned}
&(\textstyle\int_{\Delta_r} |\frac{\partial G}{\partial\rho_Q}(X,Q)|dQ)^2 \\
&\quad \le r^2 \textstyle\int_{\partial U(\tau r)} |\frac{\partial G}{\partial\rho_Q}(X,Q)|^2 dQ \\
&\quad \le cr^2 \textstyle\int_{\partial U(\tau r)} |\nabla G(X,Q)|^2 dQ \\
&\quad \le cr^2 \textstyle\int_{\partial U(\tau r)} |\nabla_T G(X,Q)|^2 dQ + cr^2 \int_{U(\frac{3}{2}r)} |\nabla G(X,Y)|^2 + G(X,Y)|^2 dY
\end{aligned}$$

where ∇_T denotes the tangential derivative on $\partial U(\tau r)$.

Integrating with respect to $\tau \in [1,2]$, we have an estimate by the volume integral such that

$$\begin{aligned}(\int_{\Delta_r} |\tfrac{\partial G}{\partial \rho_Q}(X,Q)|dQ)^2 \\ &\le cr \int_{U(\frac{3}{2}r)} |\nabla G(X,Y)|^2 dY + cr^2 \int_{U(\frac{3}{2}r)} |\nabla G(X,Y)|^2 + G(X,Y)|^2 dY \\ &\le cr^{-1} \int_{U(2r)} |G(X,Q)|^2 dQ\end{aligned}$$

where we used the fact that $G(X,\cdot) = 0$ on $\partial\Omega$ in the first inequality and Caccioppoli's inequality in the second inequality.

Consequently, by Theorem 2.2, we have

$$\begin{aligned}(\int_{\Delta_r} |\tfrac{\partial G}{\partial \rho_Q}(X,Q)|dQ)^2 &\le cr^{-1} \int_0^{2r} \tfrac{t^{2\eta}}{|X|^{2(1+\eta)}} r^2 dt \\ &= c(\tfrac{r}{|X|})^{2(1+\eta)}\end{aligned}$$

and the Lemma 6.1 is proved.□

Lemma 6.2. *Let ρ satisfy $4r \le \rho \le \frac{r_1}{16}$. Then there exists ρ' such that $2\rho < \rho' < 4\rho$ and*

$$\int_{\Omega \cap \partial U(\rho')} |\nabla u|^2 d\sigma \le c \max_{\Omega \cap U(8\rho) \setminus U(\rho)} |u|^2 \le cr^{2\alpha - 4}(\frac{r}{\rho})^{2+2\eta}.$$

Proof.

Since u vanishes on $(U(8\rho) \setminus U(\rho)) \cap \partial\Omega$ it follows from Caccioppoli's inequality that

$$\int_{U(4\rho)\cap\Omega\setminus U(2\rho)} |\nabla u|^2 \le c\rho^{-2} \int_{U(8\rho)\cap\Omega\setminus U(\rho)} |u|^2 \le c\rho \max_{U(8\rho)\cap\Omega\setminus U(\rho)} |u|^2.$$

By Fubini's theorem there is at least one number ρ', $2\rho < \rho' < 4\rho$, for which

$$\int_{\Omega\cap\partial U(\rho')} |\nabla u|^2 \le c\rho^{-1} \int_{U(4\rho)\cap\Omega\setminus U(2\rho)} |\nabla u|^2.$$

Combining the previous inequalities, the first inequality of Lemma 6.2 follows. Finally, Lemma 6.1 says that if $X \in U(8\rho) \cap \Omega \setminus U(\rho)$, then

$$|u(X)| \le \int_{\Delta_r} |\frac{\partial G}{\partial \rho_Q}(X,Q)||a(Q)|dQ \le c(\frac{r}{\rho})^{1+\eta} \max |a| \le cr^{\eta-2}(\frac{r}{\rho})^{1+\eta}$$

and with the first inequality, this implies the second inequality of Lemma 6.2.□

Let N be the largest integer for which $2^{N+3}r < r_1$. Applying Lemma 6.2 with $\rho = 2^{k-1}r$, we find that for every integer k such that $3 \le k \le N$, there exists ρ_k such that $2^k r < \rho_k < 2^{k+1}r$ and

$$\int_{\Omega\cap\partial U(\rho_k)} |\nabla u|^2 d\sigma \le cr^{2(\alpha-2)}(2^{-k})^{2(\eta+1)}. \tag{6.9}$$

Define

$R_k = \Omega \cap (U(\rho_{k+3}) \setminus U(\rho_k))$ for $k \ge 3$ and $R_2 = U(\rho_5) \cap \Omega$

$\delta_k(X) = dist(X, \partial R_k)$ for $X \in R_k$; $\delta_k(X) = 0$ for $X \in \mathbf{R}^n \setminus R_k$.

Let ∇_T denote the tangential component of the gradient on ∂R_k. For $k \geq 3$, u and $\nabla_T u$ vanish on $(\partial R_k) \cap \partial\Omega$. Also, $|\nabla_T u| \leq |\nabla u|$, so that (6.9) implies

$$\int_{\partial R_k} |\nabla_T u|^2 d\sigma \leq c r^{2(\alpha-2)} (2^{-k})^{2(\eta+1)}. \tag{6.10}$$

For $k = 2$,

$$\int_{\partial R_2 \cap \partial\Omega} |\nabla_T u|^2 d\sigma = \int_{\Delta_r} |\nabla_T a|^2 d\sigma \leq c r^{2\alpha-4}.$$

This is the same as the bound given by (6.10) for the other portion of the boundary, $\Omega \cap \partial R_2$. Therefore, (6.10) is valid for $k = 2$ as well. Theorem 1.3 in [1] and Theorem 3 in [4] and the interior estimates imply

$$\int_{R_k} \delta_k |\nabla^2 u|^2 \leq c \int_{\partial R_k} |\nabla_T u|^2 d\sigma.$$

Note that this inequality is dilation invariant, so that R_k can be rescaled to unit size. It follows from Schwarz's inequality that

$$\begin{aligned} \int_{R_k} \delta_k^{1-\alpha} |\nabla^2 u| &\leq \left(\int_{R_k} \delta_k |\nabla^2 u|^2\right)^{1/2} \left(\int_{R_k} \delta_k^{1-2\alpha}\right)^{1/2} \\ &\leq c (2^k r)^{\frac{4-2\alpha}{2}} \left(\int_{\partial R_k} |\nabla_T u|^2 d\sigma\right)^{1/2}. \end{aligned} \tag{6.11}$$

Finally, (6.11) and (6.10) yield

$$\sum_{k=2}^{N} \int_{R_k} \delta_k^{1-\alpha} |\nabla^2 u| \leq c \sum_{k=2}^{N} (2^k)^{-\eta+1-\alpha} \leq c'$$

because $-\eta + 1 - \alpha < 0$. The overlap of the sets R_k imply that $c \sum \delta_k \leq \delta$, so we have

$$\int_\Omega \delta^{1-\alpha} |\nabla^2 u| \leq c$$

whenever the solution u of elasticity equations has boundary value equal to an atom. The bound in Lemma 6.2 implies

$$\int_{R_k} |u| \leq c r^{\alpha-2} (2^{-k})^{\eta+1} |R_k|.$$

Hence,

$$\int_\Omega |u| \leq \sum_{k=2}^{N} \int_{R_k} |u| \leq c r^{\alpha+1} \sum_{k=2}^{N} (2^k)^{2-\eta} = c r^{\alpha-1+\eta} < c,$$

because 2^N is comparable to $\frac{1}{r}$, $2 - \eta > 0$, and $\alpha - 1 + \eta > 0$. For $k \geq 3$ we can estimate $|\nabla u|$ as in Lemma 6.2 by Caccioppoli's inequality

$$\begin{aligned} \left(\int_{R_k} |\nabla u|^2\right)^{1/2} &\leq c (2^k r)^{-1} r^{\alpha-2} (2^{-k})^{\eta+1} |R_k|^{1/2} \\ &= c r^{\alpha-\frac{3}{2}} 2^{k(-\frac{1}{2}-\eta)}. \end{aligned}$$

In the case $k = 2$, by Theorem 3.7 in [3], we obtain that

$$\int_{R_2} |\nabla u|^2 \leq c r \int_{\partial\Omega} ((\nabla u)^*)^2 \leq c r \int_{\partial\Omega} |\nabla_T a|^2 \leq c r^{2\alpha-3}.$$

This is the same estimate as in (6.12) for larger k. Thus by Schwarz's inequality we have

$$\int_\Omega |\nabla u| \le c\sum_{k=2}^{N}\int_{R_k}|\nabla u| \le c\sum_{k=2}^{N}(\int_{R_k}|\nabla u|^2)^{1/2}(2^k r)^{3/2}$$
$$\le cr^\alpha \sum_{k=2}^{N}(2^k)^{1-\eta} \le cr^{\alpha-1+\eta} < c.$$

The final two inequalities are true because 2^N is comparable to $1/r$, $1-\eta>0$, and $\alpha-1+\eta>0$. This concludes the proof of existence and regularity in Theorem 2.4

To prove uniqueness, it suffices to consider a function $u \in B^1_{1+\alpha}(\Omega)$ such that $\mu\Delta u + (\lambda+\mu)\nabla(div\, u) = 0$ in Ω and $u=0$ on $\partial\Omega$. Choose smooth domains Ω_j such that $\overline{\Omega}_j \subset \Omega, \phi_j \in C_0^\infty(\Omega_j)$, and $\partial\Omega_j$ is uniformly Lipschitz and tends to $\partial\Omega$ as $j\to\infty$,

$$||u||_{B^1_\alpha(\partial\Omega_j)} \to 0.$$

The regularity estimate just proved says that

$$||u||_{B^1_{1+\alpha}(\Omega_j)} \le c||u||_{B^1_\alpha(\partial\Omega_j)}.$$

Since the right-hand side tends to zero, $u\equiv 0$.□

7. Proof Theorem 2.7

Let $X, Y\in\Omega$ and $r=|X-Y|$ and $|X-Y|\ge \frac{1}{2}min(dist(X), dist(Y))$. Then $X\notin B(Y,\frac{1}{2}dist(Y)), Y\notin B(X,\frac{1}{2}dist(X))$ and by the energy estimate and Caccioppoli inequality, we have

$$|\nabla_X G(X,Y)| \le c\, min\left((\frac{1}{dist(X)^5}\int_{B(X,\frac{1}{2}dist(X))}|G(Z,Y)|dZ)^{\frac{1}{2}}, \right.$$
$$\left. (\frac{1}{dist(Y)^5}\int_{B(,\frac{1}{2}dist(Y))}|G(X,Z)|dZ)^{\frac{1}{2}} \right).$$

By Theorem 2.2, the first term of right hand side can be estimated as

$$\begin{aligned}(\tfrac{1}{dist(X)^5}\textstyle\int_{B(X,\frac{1}{2}dist(X))}|G(Z,Y)|dZ)^{\frac{1}{2}} &\le c(\tfrac{1}{dist(X)^5}\textstyle\int_{B(X,\frac{1}{2}dist(X))}\tfrac{dist(Z)^{2\eta}}{|Z-Y|^{2(1+\eta)}}dZ)^{\frac{1}{2}}\\ &\le c(\tfrac{1}{dist(X)^5}\tfrac{dist(X)^{2\eta}}{|X-Y|^{2(1+\eta)}}dist(X)^3)^{\frac{1}{2}}\\ &= c\tfrac{dist(X)^{\eta-1}}{|X-Y|^{1+\eta}}.\end{aligned}$$

Similarly, we obtain that

$$|G(X,Y)| \le c\frac{dist(Y)^{\eta-1}}{|X-Y|^{1+\eta}}.$$

Hence we proved the first part of Theorem 2.7. To prove the second part of Theorem 2.7, we let $|X-Y|\le\frac{1}{2}min(dist(X), dist(Y))$. The ball $B(X,\frac{1}{2}|X-Y|)$ lies in Ω

and by the energy estimate and Caccioppoli inequality and Theorem 2.2, we have

$$\begin{aligned}|\nabla_X G(X,Y)| &\le c(\tfrac{1}{|X-Y|^5}\int_{B(X,\frac{1}{2}|X-Y|)}|G(Z,Y)|^2 dZ)^{\frac{1}{2}}\\ &\le c(\tfrac{1}{|X-Y|^5}\tfrac{1}{|X-Y|^2}|X-Y|^3)^{\frac{1}{2}}\\ &= c\tfrac{1}{|X-Y|^2}.\end{aligned}$$

□

8. Proof of Theorem 2.8

Let $X \in \Omega$. We define $d(\Omega)$ by the diameter of Ω and $\partial\Omega_t = \{Z \in \Omega\,;\, dist(Z) = t\}$. Then

$$\begin{aligned}||\nabla_X G(X,\cdot)||^p_{L^p(\Omega)} &= \int_{B(X,\frac{1}{2}dist(X))}|\nabla_X G(X,Z)|^p dZ + \int_{\Omega\setminus B(X,\frac{1}{2}dist(X))}|\nabla_X G(X,Z)|^p dZ\\ &= I_1 + I_2.\end{aligned}$$

To calculate I_1, we use Theorem 2.7 and hence

$$\begin{aligned}I_1 &\le c\int_{B(X,\frac{1}{2}dist(X))}\tfrac{1}{|X-Z|^{2p}}dZ\\ &= c\int_0^{\frac{1}{2}dist(X)}\tfrac{1}{t^{2p}}t^2 dt\\ &= c\,dist(X)^{3-2p}.\end{aligned}$$

We divide I_2 as two part I_{21}, I_{22}, where I_{21} is integral over $\{Z \in \Omega\setminus B(X,\frac{1}{2}dist(X))\,;\, dist(Z) \le dist(X)\}$ and I_{22} is integral over $\{Z \in \Omega \setminus B(X,\frac{1}{2}dist(X))\,;\, dist(Z) \ge dist(X)\}$. Then, using Theorem 2.7,

$$\begin{aligned}I_{21} &\le c\int_0^{dist(X)}\tfrac{dist(X)^{p(\eta-1)}}{dist(X)^{p(1+\eta)}}dZ\\ &= c\,dist(X)^{1-2p},\end{aligned}$$

and

$$\begin{aligned}I_{22} &\le c\int_{dist(X)}^{d(\Omega)}\tfrac{t^{p(\eta-1)}}{dist(X)^{p(1+\eta)}}dZ\\ &\le c\,dist(X)^{1-2p}.\end{aligned}$$

Therefore, combining all the estimates, we complete the proof of Theorem 2.8.□

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