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EXISTENCE RESULTS FOR SECOND ORDER SINGULAR BOUNDARY VALUE PROBLEMS VIA BIFURCATION METHOD

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Abstract. We establish an existence result of unbounded continuum of solutions from the trivial branch for a second order singular boundary value problem. As an application, we prove several existence and multiplicity results of positive solutions for certain singular boundary value problems.

1. Introduction

In the study of nonlinear phenomena, for example, in fluid theory and boundary layer theory, many mathematical and physical models give rise to the following type of boundary value problems

$$\begin{aligned} &\triangle u + f(|x|, u) = 0, \quad \text{in } \Omega \\ &u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $n > 2$. If Ω is an exterior domain like $\Omega = \{x \in R^n \mid |x| > r_0\}$, we have the following Dirichlet boundary value problem via suitable transformations and substitution $u(t) = z(r_0(1-t)^{\frac{1}{2-n}})$:

$$\begin{aligned} &u''(t) + g(t, u(t)) = 0, \quad t \in (0,1) \\ &u(0) = 0 = u(1), \end{aligned} \tag{2}$$

with

$$g(t,u) = \frac{{r_0}^2}{(n-2)^2}(1-t)^{\frac{2n-2}{2-n}} f(r_0(1-t)^{\frac{1}{2-n}}, u).$$

We notice that the problem (2) is singular at 1. When the nonlinear part of the equation is independent of t, we have the following type of singular boundary value problems

$$\begin{aligned} &u''(t) + \lambda q(t) f(u(t)) = 0, \quad t \in (0,1), \\ &u(0) = 0 = u(1), \end{aligned} \tag{P_λ}$$

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where λ a positive real parameter, $f \in C(\mathbf{R}_+, \mathbf{R}_+)$, $\mathbf{R}_+ = [0, \infty)$ and $q \in C((0,1),(0,\infty))$ may be singular at $t = 0$ and/or 1. Existence and multiplicity of positive solutions for (P_λ) have been studied by several authors. For example, if q satisfies $\int_0^1 s(1-s)q(s) < \infty$ and f satisfies either $f_0 \triangleq \lim_{u\to 0^+} \frac{f(u)}{u} = 0$ and $f_\infty \triangleq \lim_{u\to\infty} \frac{f(u)}{u} = \infty$ or $f_0 = \infty$ and $f_\infty = 0$. Then (P_λ) has at least one positive solution for all $\lambda > 0$. For more details, one may refer to Zhang[17], Lee[9], Zhao[18] and Cheng and Zhang[4]. On the other hand, if f satisfies $f_0 = \infty$ and $f_\infty = \infty$, then under additional nondecreasing condition on f, Lee[10] proved that there exists $\lambda_0 > 0$, such that (P_λ) has at least two , one or no positive solutions according to $0 < \lambda < \lambda_0$, $\lambda = \lambda_0$ or $\lambda > \lambda_0$, respectively. For more details and recent development, one may refer to Choi[5], Wong[14], Dalmasso[6], Liu and Li[12], Ha and Lee[8], Agarwal, Wong and Lian[1], Erbe and Mathsen[7], Yang[16] and Xu and Ma[15]. We see that problem (P_λ) may have variety of bifurcation phenomena mainly depending on the shape of f. Our main concern is to study bifurcation phenomena of positive solutions for (P_λ) specially when f satisfies $0 < f_0 < \infty$.

We first introduce the global bifurcation theorem due to Rabinowitz[13]. Consider

$$u = \lambda L u + H(\lambda, u), \tag{1.1}$$

where $L : E \to E$ is a bounded linear operator, $H : \mathbf{R} \times E \to E$ continuous and E a real Banach space with norm $\|\cdot\|$. Let $\mathcal{S}$ denote the closure of set of nontrivial solutions of (1.1). Assume L and H are compact on E and $\mathbf{R} \times E$ respectively. Furthermore, assume $H(\lambda, 0) = 0$ for all $\lambda \in \mathbf{R}$ and $H(\lambda, u) = o(\|u\|)$ as $\|u\| \to 0$. Then we have a global bifurcation theorem from the trivial branch as follows.

Theorem 1.1. ([13]) *If μ is a characteristic value of L of odd multiplicity, then there exists a subcontinuum C in $\mathcal{S}$ bifurcating from $\{(\lambda, 0) \mid \lambda \in \mathbf{R}\}$ at $(\mu, 0)$and either*

- (i) *C is unbounded in $\mathbf{R} \times E$ or*
- (ii) *$C \cap (\mathbf{R} \setminus \{\mu\}) \times E \neq \emptyset$.*

Now consider problem (G_λ)

$$\begin{aligned} &u''(t) + \lambda r(t)u(t) + \lambda g(t, u(t)) = 0, \quad t \in (0,1), \\ &u(0) = 0 = u(1), \end{aligned} \tag{G_λ}$$

and the following assumptions.

- (H_1) $r \in \mathcal{A}$ and there exist $\beta \in \mathcal{A}$ and $\phi \in C(\mathbf{R}, \mathbf{R}_+)$ such that $|g(t,u)| \leq \beta(t)\phi(t)$ for all $(t,u) \in (0,1) \times \mathbf{R}$.
- (H_2) $\phi(u) = o(u)$ as $u \to 0$.
- (H_3) $g(t, -u) = -g(t,u)$.

We see mainly by condition (H_1) that problem(G_λ) can be equivalently written as the following integral equation

$$u(t) = \lambda \int_0^1 G(t,s)[r(s)u(s) + g(s, u(s))]ds, \tag{1.2}$$

where $G(t,s)$ is Green's function explicitly written as

$$G(t,s) = \begin{cases} s(1-t), & 0 \le s \le t, \\ t(1-s), & t \le s \le 1. \end{cases}$$

Define $L : C_0[0,1] \to C_0[0,1]$ and $H : \mathbf{R} \times C_0[0,1] \to C_0[0,1]$ by taking

$$Lu(t) = \int_0^1 G(t,s)r(s)u(s)ds,$$
$$H(\lambda,u)(t) = \lambda \int_0^1 G(t,s)g(s,u(s))ds$$

respectively. Then it is not hard to check that L is a bounded linear operator and H continuous satisfying $H(\lambda,u) = o(\|u\|_\infty)$ as $\|u\|_\infty \to 0$. It is also known that L and H are completely continuous in $C_0[0,1]$ and $\mathbf{R} \times C_0[0,1]$ respectively. Recently, the existence of characteristic values of L and its properties are studied by Asakawa[2] as follows.

Proposition 1.1. *Let $r \in \mathcal{A}$. Then*

(1) *The set of all characteristic values of L is a countable set $\{\mu_n \mid n \in N\}$ satisfying*
$0 < \mu_1 < \cdots < \mu_n < \mu_{n+1} < \cdots \to \infty$.
(2) *For each n, $Ker(I - \mu_n L)$ is a subspace of $C^1[0,1]$and its dimension is 1.*
(3) *Let u_n be a corresponding characteristic function to μ_n, then the number of interior zeros of u_n is $n-1$.*

Assume $(H1) \sim (H3)$, then by Proposition 1.1 and Theorem 1.1, we conclude that there exists a subcontinuum C_k of solutions of (G_λ) bifurcating from $(\mu_k, 0)$ and either it is unbounded in $\mathbf{R} \times C_0[0,1]$ or it meets $(\mu_j, 0)$, for some $j \ne k$. In the following section, we will prove that only the first alternative is possible. We end up with this section giving some notations and useful lemmas. Let N_k^+ denote the set of $u \in C_0[0,1]$ such that u has exactly $k-1$ simple interior zeros, $u > 0$ near 0 and all zeros of u in [0,1] are simple. Let ${N_k}^- = -N_k$ and $N_k = {N_k}^+ \cup {N_k}^-$. We notice that $N_k \cap N_j = \emptyset$ if $k \ne j$. Also notice that ${N_k}^\pm$ and N_k are neither open nor closed in $C_0[0,1]$. We give well-known Gronwall-Bellman inequality for later use.

Lemma 1.1. ([3]) *Let $\epsilon > 0$ and let $m \in L^1(0,T)$ be such that $m \ge 0$ a,e in $(0,T)$. Suppose $u \in C[0,T]$ and*

$$u(t) \le \epsilon + \int_0^t m(s)u(s)ds,$$

for all $t \in [0,T]$. Then $u(t) \le \epsilon e^{\int_0^t m(s)ds}$ for all $t \in [0,T]$.

We also give a modification of Gronwall-Bellman inequality for readers convenience. Proof can be done by obvious modification of Lemma 1.1.

Lemma 1.2. *Let $\epsilon > 0$ and let $m \in L^1(0,T)$ be such that $m \geq 0$ a,e in $(0,T)$. Suppose $u \in C[0,T]$ and*

$$u(t) \leq \epsilon + \int_t^T m(s)u(s)ds,$$

for all $t \in [0,T]$. Then $u(t) \leq \epsilon e^{\int_t^T m(s)ds}$ for all $t \in [0,T]$.

2. Unbounded subcontinuum

In this section, we prove that subcontinuum C_k known to exist in Section 1 is unbounded. Let us consider problem (G_λ).

$$(G_\lambda) \qquad \begin{aligned} &u''(t) + \lambda[r(t)u(t) + g(t,u(t))] = 0, \quad t \in (0,1),\\ &u(0) = 0 = u(1). \end{aligned}$$

Denote $\mathcal{A} = \{h \in C((0,1),(0,\infty)) \mid \int_0^1 s(1-s)h(s)ds < \infty\}$.

Lemma 2.1. *If u is a solution of (G_λ) and u has a double zero, then $u \equiv 0$.*

Let μ_k denote the k-th characteristic value of linear operator L given in Section 1.

Lemma 2.2. *Let C_k be a subcontinuum of solutions of (G_λ) bifurcating from $(\mu_k, 0)$. Then $C_k \cap \mathbf{R} \times \{0\} \subset \cup_{j=1}^\infty \{(\mu_j, 0)\}$.*

Lemma 2.3. *Every nontrivial solution of (G_λ) has at most finite interior zeros.*

Lemma 2.4. *Let u_n and u be nontrivial solutions of (G_{λ_n}) and (G_λ) respectively. If $u_n \to u$ and $\lambda_n \to \lambda \neq 0$, then there exist $\delta_1, \delta_2 > 0$ such that*

$$\cup_{n=1}^\infty \{t \in (0,1) \mid u_n(t) = 0\} \subset [\delta_1, \delta_2] \subset (0,1).$$

Lemma 2.5. *Let u_n and u be nontrivial solutions of (G_{λ_n}) and (G_λ) respectively. Assume that there exists $k > 0$ such that $u_n \in N_k$ for all n, $u_n \to u$ and $\lambda_n \to \lambda \neq 0$. Then either $u \in N_k$ or $u \equiv 0$.*

Lemma 2.6. *For each $j > 0$, there exists a neighborhood $\mathcal{O}_j$ of $(\mu_j, 0)$ such that $(\lambda, u) \in \mathcal{O}_j \cap \mathcal{S}$ and $u \not\equiv 0$ implies $u \in N_j$.*

Now we give the main theorem in this section.

Theorem 2.1. *The subcontinuum C_k known to exist in Section 1 is unbounded.*

Proof. If we show $C_k \subset (\mathbf{R} \times N_k) \cup \{(\mu_k, 0)\}$, then C_k is unbounded by Lemma 2.6, Theorem 1.1 and by the fact $N_j \cap N_k = \emptyset$ for $j \neq k$. Suppose $C_k \not\subset (\mathbf{R} \times N_k) \cup \{(\mu_k, 0)\}$. Then there exists $(\lambda, u) \in C_k \cap (\mathbf{R} \times \partial N_k)$ such that $(\lambda, u) \neq (\mu_k, 0)$, $u \notin N_k$ and $(\lambda_n, u_n) \to (\lambda, u)$ with $(\lambda_n, u_n) \in C_k \cap (\mathbf{R} \times N_k)$. By Lemma 2.2 and Lemma

2,6, $u \not\equiv 0$. Since $\partial N_k \subset \cup_{i=1}^{k} N_i \cup \{u \in C_0[0,1] |\ u$ has a double zero$\}$, $u \notin N_k$ and u does not have a double zero by Lemma 2.1, there exists j with $0 < j < k$ such that $u \in N_j$. Consequently $u_n \in N_k$ and $u \in N_j$ $(j \neq k)$ are nontrivial solutions of (G_{λ_n}) and (G_λ) respectively and $u_n \to u$, $\lambda_n \to \lambda \neq 0$. Thus by Lemma 2.5, we get a contradiction. ■

3. Application

In this section, we prove various existence results of positive solutions for the following problem

$$(P_\lambda) \qquad \begin{aligned} &u''(t) + \lambda q(t) f(u) = 0, \quad t \in (0,1),\\ &u(0) = 0 = u(1), \end{aligned}$$

where λ is a positive real parameter, $f \in C(\mathbf{R}_+, \mathbf{R}_+)$ and $q \in C((0,1),(0,\infty))$. The assumptions we are interested in this section are as follows.

(C_1) $\int_0^1 s(s-1)q(s)ds < \infty$.
(C_2) $0 < f_0 < \infty$.
(C_3) $f_\infty = \infty$.

Define $h : \mathbf{R} \to \mathbf{R}$ by

$$h(u) = \begin{cases} f(u), \ u \geq 0, \\ -f(-u), \ u < 0, \end{cases}$$

and consider the following problem

$$(H_\lambda) \qquad \begin{aligned} &u''(t) + \lambda f_0 u(t) + \lambda q(t)[h(u(t)) - f_0 u(t)] = 0, \quad t \in (0,\ 1),\\ &u(0) = 0 = u(1). \end{aligned}$$

We know that a positive solution of problem (H_λ) is a positive solution of problem (P_λ). Furthermore, problem (H_λ) is equivalently written as the following operator equation

$$u = \lambda L u + H(\lambda, u),$$

where

$$Lu(t) = \int_0^1 G(t,s) f_0 q(s) u(s) ds,$$
$$H(\lambda,u)(t) = \lambda \int_0^1 G(t,s) q(s)[h(u(s)) - f_0 u(s)]ds.$$

We notice that the characteristic values of linear operator L is the eigenvalues of problem

$$(3.1) \qquad \begin{aligned} &u''(t) + \lambda f_0 q(t) u = 0, \quad t \in (0,1),\\ &u(0) = 0 = u(1). \end{aligned}$$

Assume (C_1) and (C_2), then problem (H_λ) satisfies conditions $(H_1) \sim (H_3)$ with $r(t) = f_0 q(t)$, $\beta(t) = q(t)$ and $\phi(u) = h(u) - f_0 u$. Thus by Theorem 2.1, (H_λ) has an unbounded subcontinuum C_k bifurcating from $(\mu_k, 0)$, where μ_k is the k-th eigenvalue of problem (3.1). Since we are interested in positive solutions of (H_λ), we focus on the shape of branch C_1. In what is to follows, we assume $f(u) > 0$, for all $u > 0$.

Lemma 3.1. *Assume* (C_1), (C_2) *and* (C_3). *If* f *satisfies* $f(u) \geq f_0 u$ *for all* $u > 0$ *and let* u *be a positive solution of* (H_λ). *Then* $\lambda < \mu_1$.

Lemma 3.2. *Assume* (C_1), (C_2) *and* (C_3). *Let* u *be a positive solution of* (H_λ). *If* $f(u) - f_0 u$ *is sign changing, then there exists* $\lambda_0 > \mu_1$ *such that* $\lambda \leq \lambda_0$.

Remark 3.1. If $f(u) - f_0 u$ is sign changing, $f(u) - f_0 u < 0$, for all $u \in (0, \tilde{u})$ and $f(\tilde{u}) - f_0 \tilde{u} = 0$, for some $\tilde{u} > 0$, then (P_λ) has a solution with $\|u\|_\infty < \tilde{u}$ for $\lambda > \mu_1$.

Lemma 3.3. *Assume* (C_1), (C_2) *and* (C_3). *Let* J *be a compact interval in* $(0, \infty)$. *Then there exists* $b_J > 0$ *such that for all* $\lambda \in J$ *and all possible positive solutions* u *of* (H_λ), *one has* $\|u\|_\infty < b_J$.

Lemma 3.4. *Assume* $(C_1), (C_2)$ *and* (C_3). *If* $(\lambda_n, u_n) \in C_1$ *and* $\|u_n\|_\infty \to \infty$. *Then* $\lambda_n \to 0$.

Suitable modifications of Lemma 3.1 through Lemma 3.4 provide the following theorem.

Theorem 3.1. *Assume* $(C_1), (C_2)$ *and* (C_3). *If* f *satisfies* $f(u) > f_0 u$ *for all* $u > 0$. *Then* (P_λ) *has at least one positive solution for* $0 < \lambda < \mu_1$ *and no positive solution for* $\lambda \geq \mu_1$. *If* f *satisfies that there exists* $\tilde{u} > 0$ *such that* $f(\tilde{u}) = f_0 \tilde{u}$ *and* $f(u) > f_0 u$ *for all* $u \in (0, \tilde{u})$. *Then there exists* $\lambda^* \geq \mu_1$ *such that* (P_λ) *has at least one positive solution for* $0 < \lambda < \mu_1$ *and no positive solution for* $\lambda > \lambda^*$. *If* f *satisfies that there exists* $\tilde{u} > 0$ *such that* $f(\tilde{u}) = f_0 \tilde{u}$ *and* $f(u) < f_0 u$ *for all* $u \in (0, \tilde{u})$. *Then there exists* $\lambda^* > \mu_1$ *such that* (P_λ) *has at least two, one or no positive solutions according to* $\lambda \in (\mu_1, \lambda^*)$, $\lambda \in (0, \mu_1] \cup \{\lambda^*\}$ *or* $\lambda \in (\lambda^*, \infty)$ *respectively.*

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