

FIRST-ORDER SYSTEM LEAST-SQUARES METHODS FOR PARTIAL DIFFERENTIAL EQUATIONS

BYEONG CHUN SHIN

1. FOSLS WITH FINITE ELEMENT APPROXIMATIONS

Let Ω be the square $(-1, 1)^d$ with $d = 2$ or 3 . We consider the second-order elliptic boundary value problem:

$$(1.1) \quad \begin{cases} -\nabla \cdot A \nabla p + \mathbf{b} \cdot \nabla p + c_0 p = f, & \text{in } \Omega, \\ p = 0, & \text{on } \Gamma_D, \\ \mathbf{n} \cdot A \nabla p = 0, & \text{on } \Gamma_N, \end{cases}$$

where $\partial\Omega = \Gamma_D \cup \Gamma_N$ denotes the boundary of Ω , A is a 2×2 symmetric matrix of bounded functions, f is a given continuous function, \mathbf{b} is a bounded vector function and c_0 is a given bounded function, and \mathbf{n} is the outward unit vector normal to the boundary. We assume that the matrix A is uniformly elliptic such as

$$(1.2) \quad 0 < \lambda \boldsymbol{\xi}^t \boldsymbol{\xi} \leq \boldsymbol{\xi}^t A(x, y) \boldsymbol{\xi} \leq \Lambda \boldsymbol{\xi}^t \boldsymbol{\xi} < \infty$$

for all $\boldsymbol{\xi} \in \mathbb{R}^2$ and almost all $(x, y) \in \bar{\Omega}$.

Setting the flux variable $\mathbf{u} = \nabla p$, we have the first-order system of linear equations equivalent to (1.1):

$$(1.3) \quad \begin{cases} -\nabla \cdot \mathbf{u} + \mathbf{b} \cdot \mathbf{u} + c_0 p = f, & \text{in } \Omega, \\ \mathbf{u} - A \nabla p = \mathbf{0}, & \text{in } \Omega, \\ \nabla \times A^{-1} \mathbf{u} = 0, & \text{in } \Omega, \\ p = 0, & \text{on } \Gamma_D, \\ \mathbf{n} \cdot \mathbf{u} = 0, & \text{on } \Gamma_N \\ \mathbf{n} \times A^{-1} \mathbf{u} = 0, & \text{on } \Gamma_D. \end{cases}$$

In recent years there has been lots of interest in the use of first-order system least-squares method (FOSLS) for numerical approximations of elliptic partial differential equations, Stokes equations, elasticity and Navier-Stokes equations. In this paper we will provide a brief review of FOSLS around elliptic problems and Stokes equations.

FOSLS is to find a minimization solution which minimizes least-squares functional defined by summing appropriate norms of residual equations :

$$G(\mathbf{v}, q; f) = \|f + \nabla \cdot \mathbf{v} - \mathbf{b} \cdot \mathbf{v} - c_0 q\|_{\alpha}^2 + \|\mathbf{v} - A \nabla q\|_{\beta}^2 + \|\nabla \times A^{-1} \mathbf{v}\|_{\gamma}^2$$

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With appropriate Hilbert spaces V and W , the variational problem is to find (\mathbf{u}, p) in $V \times W$ satisfying

$$a(\mathbf{u}, p; \mathbf{v}, q) = f(\mathbf{v}, q), \quad \forall (\mathbf{v}, q) \in V \times W$$

where the positive definite symmetric bilinear form $a(\cdot, \cdot)$ is given by

$$\begin{aligned} a(\mathbf{u}, p; \mathbf{v}, q) &= (\nabla \cdot \mathbf{u} - \mathbf{b} \cdot \mathbf{u} - c_0 p, \nabla \cdot \mathbf{v} - \mathbf{b} \cdot \mathbf{v} - c_0 q)_\alpha \\ &\quad + (\mathbf{u} - A\nabla p, \mathbf{v} - A\nabla q)_\beta + (\nabla \times A^{-1}\mathbf{u}, \nabla \times A^{-1}\mathbf{v})_\gamma \end{aligned}$$

and linear form $f(\cdot)$ is given by

$$f(\mathbf{v}, q) = -(f, \nabla \cdot \mathbf{v} - \mathbf{b} \cdot \mathbf{v} - c_0 q)_\alpha.$$

Note that

$$G(\mathbf{v}, q; 0) = a(\mathbf{v}, q; \mathbf{v}, q).$$

Main focus is to design the norms $\|\cdot\|_\alpha$, $\|\cdot\|_\beta$ and $\|\cdot\|_\gamma$ that the homogeneous least-squares functional $G(\mathbf{v}, q; 0)$ is equivalent to the norm $\|\mathbf{v}\|_V^2 + \|q\|_W^2$ over $V \times W$.

1.1. L^2 **FOSLS**. $H_{div} - H^1$ equivalence : In [2], Cai, Lazarov, Manteuffel and McCormick developed a $H(\text{div}) - H^1$ -norm equivalent least squares functional for scalar second order elliptic partial differential equations :

$$G_{div}(\mathbf{v}, q; f) = \|f + \nabla \cdot \mathbf{v} - \mathbf{b} \cdot \mathbf{v} - c_0 q\|_{L^2}^2 + \|\mathbf{v} - A\nabla q\|_{L^2}^2.$$

Then there exists a positive constant C such that

$$\frac{1}{C}(\|\mathbf{v}\|_{H_{div}}^2 + \|q\|_{H^1}^2) \leq G_{div}(\mathbf{v}, q; 0) \leq C(\|\mathbf{v}\|_{H_{div}}^2 + \|q\|_{H^1}^2)$$

over $H_{div}(\Omega) \times H_D^1(\Omega)$ where

$$H_{div}(\Omega) := \{\mathbf{v} \in L^2(\Omega)^d : \nabla \cdot \mathbf{v} \in L^2(\Omega) \quad \text{and} \quad \mathbf{n} \cdot \mathbf{v} = 0 \text{ on } \Gamma_N\}$$

and

$$H_D^1(\Omega) := \{q \in H^1(\Omega) : q = 0 \text{ on } \Gamma_D\}$$

$H^1 - H^1$ equivalence : In [3], Cai, Manteuffel and McCormick have improved L^2 -norm least-squares approaches for scalar second order elliptic partial differential equations by adding the auxiliary equation given in the third equation of (1.3) to the previous functional :

$$G_1(\mathbf{v}, q; f) = \|f + \nabla \cdot \mathbf{v} - \mathbf{b} \cdot \mathbf{v} - c_0 q\|_{L^2}^2 + \|\mathbf{v} - A\nabla q\|_{L^2}^2 + \|\nabla \times A^{-1}\mathbf{v}\|_{L^2}^2.$$

Then, under appropriate assumptions (see [3] for more details) there exists a positive constant C such that

$$\frac{1}{C}(\|\mathbf{v}\|_{H^1}^2 + \|q\|_{H^1}^2) \leq G_1(\mathbf{v}, q; 0) \leq C(\|\mathbf{v}\|_{H^1}^2 + \|q\|_{H^1}^2)$$

over $H_{div}(\Omega) \cap H_{curl}(\Omega) \times H_D^1(\Omega)$ where

$$H_{curl}(\Omega) := \{\mathbf{v} \in L^2(\Omega)^d : \nabla \times A^{-1}\mathbf{v} \in L^2(\Omega) \quad \text{and} \quad \mathbf{n} \times A^{-1}\mathbf{v} = 0 \text{ on } \Gamma_D\}.$$

The limitation of above L^2 -norm FOSLS is the requirement of sufficient smoothness of the underlying problem which guarantees the equivalence of norms between $H_{div}(\Omega) \cap H_{curl}(\Omega)$ and $H^1(\Omega)^d$, where $d = 2$ or 3 , so that it can be approximated

by standard continuous finite element space. But those two spaces are not equivalent in general when the domain Ω is not smooth or not convex, or the coefficient is not continuous.

The error estimate is as follows. If the exact solution $(\mathbf{u}, p) \in H^2(\Omega)^{d+1}$ and the approximation $(\mathbf{u}_h, p_h) \in (\mathcal{P}_1^h)^{d+1}$ where \mathcal{P}_1^h is the space of continuous piecewise linear functions, then

$$\|\mathbf{u} - \mathbf{u}_h\|_{H^1} + \|p - p_h\|_{H^1} \leq Ch (\|\mathbf{u}\|_{H^2} + \|p\|_{H^2}).$$

1.2. H^{-1} **FOSLS. $L^2 - H^1$ equivalence** : To overcome such a limitation of L^2 FOSLS, in [1] Bramble, Lazarov and Pasciak developed the negative norm FOSLS summing the H^{-1} -norm of the first residual equation and L^2 -norm of the second residual equation in (1.3) :

$$G_{-1}(\mathbf{v}, q; f) = \|f + \nabla \cdot \mathbf{v} - \mathbf{b} \cdot \mathbf{v} - c_0 q\|_{H^{-1}}^2 + \|\mathbf{v} - A\nabla q\|_{L^2}^2.$$

Then, there exists a positive constant C such that

$$\frac{1}{C} (\|\mathbf{v}\|_{L^2}^2 + \|q\|_{H^1}^2) \leq G_1(\mathbf{v}, q; 0) \leq C (\|\mathbf{v}\|_{L^2}^2 + \|q\|_{H^1}^2)$$

over $L^2(\Omega)^d \times H_D^1(\Omega)$.

To make the method computationally feasible, they proposed discrete negative norm least-squares. Let $T = (I - \Delta)^{-1}$ be the solution operator under appropriate boundary conditions. That is, define $T : H^{-1}(\Omega) \rightarrow H_D^1(\Omega)$ by $p = Tg \in H_D^1(\Omega)$ for $g \in H^{-1}(\Omega)$ where p is the unique solution satisfying

$$(\nabla p, \nabla q)_{L^2} + (p, q)_{L^2} = \langle g, q \rangle \quad \forall q \in H_D^1(\Omega).$$

Then it is easily shown that

$$(p, q)_{H^{-1}} = \langle q, Tp \rangle \quad \text{and} \quad \|q\|_{H^{-1}}^2 = \langle q, Tq \rangle.$$

Let A_h be the discrete solution operator corresponding to T in a finite element approximation and let B_h be a multigrid V-cycle preconditioner which is spectrally equivalent to A_h .

Using the operator

$$T_h = \alpha h^2 I + B_h$$

they defined the discrete negative norm and scalar product :

$$(p, q)_{-1,h} = \langle q, T_h p \rangle \quad \text{and} \quad \|q\|_{-1,h}^2 = \langle q, T_h q \rangle.$$

Using the discrete negative norm $\|\cdot\|_{-1,h}$, they defined the discrete negative norm least-squares functional :

$$G_{-1,h}(\mathbf{v}, q; f) = \|f + \nabla \cdot \mathbf{v} - \mathbf{b} \cdot \mathbf{v} - c_0 q\|_{-1,h}^2 + \|\mathbf{v} - A\nabla q\|_{L^2}^2.$$

The error estimate is as follows. If the exact solution $(\mathbf{u}, p) \in H^1(\Omega)^d \times H^2(\Omega)$ and the approximation $(\mathbf{u}_h, p_h) \in (\mathcal{P}_1^h)^{d+1}$ where \mathcal{P}_1^h is the space of continuous piecewise linear functions, then

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^2} + \|p - p_h\|_{H^1} \leq Ch (\|\mathbf{u}\|_{H^1} + \|p\|_{H^2}).$$

One may find the computational results for Stokes equations and linear elasticity in [8] and [9], respectively.

1.3. Discrete FOSLS. $L^2 - H^1$ equivalence : An alternative approach to the $L^2 - H^1$ equivalence is the discrete least-squares method given by Z. Cai and B. C. Shin in [5] and [10].

The discrete FOSLS uses direct approximation of $H_{div}(\Omega) \cap H_{curl}(\Omega)$ -type space based on the Helmholtz decomposition :

For any $\mathbf{u} \in H_{div}(\Omega) \cap H_{curl}(\Omega)$,

$$\mathbf{u} = A\nabla s + \nabla \times t,$$

where $s \in H_D^1(\Omega)$ is the unique solution of

$$\begin{cases} \nabla \cdot (A\nabla s) = \nabla \cdot \mathbf{u} & \text{in } \Omega, \\ s = 0 & \text{on } \Gamma_D, \\ \mathbf{n} \cdot (A\nabla s) = 0 & \text{on } \Gamma_N \end{cases}$$

and $t \in H_N^1(\Omega)$ is the unique solution of

$$\begin{cases} \nabla \times \nabla \times t = \nabla \times \mathbf{u} & \text{in } \Omega, \\ \boldsymbol{\tau} \cdot (A^{-1}\nabla \times t) = 0 & \text{on } \Gamma_D, \\ t = 0 & \text{on } \Gamma_N. \end{cases}$$

It is then natural to approximate the scalar functions $s \in H_D^1(\Omega)$ and $t \in H_N^1(\Omega)$ by standard continuous piecewise polynomials.

Using the discrete divergence operator $\nabla_h \cdot : L^2(\Omega)^d \rightarrow [\mathcal{P}_{1,D}^h]^d$ and curl operator $\nabla_h \times : L^2(\Omega) \rightarrow \mathcal{P}_{1,N}^h$ where the function $q \in \mathcal{P}_{1,B}^h$ vanishes on the boundary Γ_B , $B = D$ or N , we define

$$G_h(\mathbf{v}, q; f) = \|f + \nabla_h \cdot \mathbf{v} - Q_h(\mathbf{b} \cdot \mathbf{v}) - cq\|_{L^2}^2 + \|\mathbf{v} - A\nabla q\|_{L^2}^2 + \|\nabla_h \times A^{-1}\mathbf{v}\|_{L^2}^2.$$

Then, there exists a positive constant C such that

$$\frac{1}{C} |||(\mathbf{v}, q)|||_h^2 \leq G_h(\mathbf{v}, q; 0) \leq C |||(\mathbf{v}, q)|||_h^2$$

over $(A\nabla\mathcal{P}_{1,D}^h \oplus \nabla \times \mathcal{P}_{1,N}^h)^d \times \mathcal{P}_{1,D}^h$ where

$$|||(\mathbf{v}, q)|||_h = \left(\|\mathbf{v}\|^2 + \|\nabla_h \cdot A\mathbf{v}\|^2 + \|\nabla_h \times \mathbf{v}\|^2 + \|q\|_1^2 \right)^{\frac{1}{2}}.$$

Then we have the following error estimate (see [5] for more details).

If (\mathbf{u}, p) is in $H^1(\Omega)^2 \times H^2(\Omega)$, then

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^2} + \|p - p_h\|_{H^1} \leq C h (\|\mathbf{u}\|_{H^1} + \|p\|_{H^2}).$$

1.4. FOSLL*. In [4] Cai, Manteuffel, McCormick and Ruge developed a new approach using adjoint equation, so-called FOSLL*.

Introducing a new vector variable $\mathbf{u} = A^{\frac{1}{2}}\nabla p$, we have an extended first-order system of equations equivalent to(1.1):

$$\mathcal{L}_e(\mathbf{u}, p, \phi) := \begin{cases} A^{-\frac{1}{2}}\mathbf{u} - \nabla p - \nabla \times \phi = \mathbf{0}, & \text{in } \Omega, \\ -\nabla \cdot A^{\frac{1}{2}}\mathbf{u} + \mathbf{b} \cdot A^{-\frac{1}{2}}\mathbf{u} + cp = f, & \text{in } \Omega, \\ \nabla \times A^{-\frac{1}{2}}\mathbf{u} = 0, & \text{in } \Omega \end{cases}$$

where the slack variable ϕ is given so that the system becomes a square system.

Using the adjoint operator \mathcal{L}_e^* they defined a least-squares functional :

$$G^*(\mathbf{v}, s, \psi; \mathbf{0}, f, 0) := \|\mathcal{L}_e^*(\mathbf{v}, s, \psi) - (\mathbf{u}, p, \phi)\|_{L^2}^2.$$

Then the corresponding variational problem is to find the adjoint variables $(\mathbf{w}, r, \chi) \in \mathcal{V}^*$ satisfying

$$a^*(\mathbf{w}, r, \chi; \mathbf{v}, s, \psi) = f^*(\mathbf{w}, r, \chi) \quad \forall (\mathbf{w}, r, \chi) \in \mathcal{V}^*$$

where \mathcal{V}^* is an appropriate space (see [4]),

$$a^*(\mathbf{w}, r, \chi; \mathbf{v}, s, \psi) := (\mathcal{L}_e^*(\mathbf{w}, r, \chi), \mathcal{L}_e^*(\mathbf{v}, s, \psi))_{L^2}$$

and

$$f^*(\mathbf{w}, r, \chi) := (\mathbf{u}, p, \phi, \mathcal{L}_e^*(\mathbf{v}, s, \psi))_{L^2} = ((\mathbf{0}, f, 0), (\mathbf{v}, s, \psi))_{L^2}.$$

Then we have the following adjoint equation and normal equation :

$$\mathcal{L}_e^*(\mathbf{w}, r, \chi) = (\mathbf{u}, p, \phi)^t \quad \text{and} \quad \mathcal{L}_e \mathcal{L}_e^*(\mathbf{w}, r, \chi) = (\mathbf{0}, f, 0)^t.$$

It has been known that the adjoint least-squares finite element approach has one of the merits which possesses the full efficiency of the L^2 norm least-squares approach by allowing less regularities of the solutions for a given boundary value problem.

2. FOSLS WITH SPECTRAL COLLOCATION METHODS

In [6], Kim, Lee and Shin recently developed the fusion method combining the concept of L^2 FOSLS and spectral collocation method, so-called least-squares spectral collocation method, to solve second order elliptic boundary value problems with constant coefficients. They derived the spectral error estimates which allow us to get the spectral convergence for both Legendre and Chebyshev approximations.

Let

$$V := \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\},$$

and

$$\mathbf{W} := \{\mathbf{v} \in H^1(\Omega)^2 : \mathbf{n} \cdot \mathbf{v} = 0 \text{ on } \Gamma_N, \quad \boldsymbol{\tau} \cdot \mathbf{v} = 0 \text{ on } \Gamma_D\},$$

where \mathbf{n} and $\boldsymbol{\tau}$ are unit normal and tangent vector, respectively. Let $\mathbf{W}_N = \mathcal{Q}_N^2 \cap \mathbf{W}$ and $V_N = \mathcal{Q}_N \cap V$ where \mathcal{Q}_N denotes the space of all polynomials of degree less than or equal to N with respect to each single variable x and y .

For any continuous functions u and v on $\bar{\Omega}$, the discrete scalar product and norm are defined by

$$\langle u, v \rangle_{w, N} = \sum_{i, j=0}^N \mathbf{w}_{ij} u(x_{ij}) v(x_{ij}) \quad \text{and} \quad \|v\|_{w, N} = \langle v, v \rangle_{w, N}^{\frac{1}{2}}$$

where x_{ij} and \mathbf{w}_{ij} are Legendre or Chebyshev Gauss Lobatto points and the corresponding quadrature weights, respectively.

Define the discrete least-squares functional using the discrete spectral norm as

$$G_N(\mathbf{v}, q; f) = \|f + \nabla \cdot \mathbf{v} - \mathbf{b} \cdot \mathbf{v} - c_0 q\|_{w, N}^2 + \|\mathbf{v} - \nabla q\|_{w, N}^2 + \|\nabla \times \mathbf{v}\|_{w, N}^2$$

for $(\mathbf{v}, q) \in \mathbf{W}_N \times V_N$.

Then, for any $(\mathbf{v}, q) \in \mathbf{W}_N \times V_N$, there exists a constant C such that

$$\frac{1}{C} (\|\mathbf{v}\|_W^2 + \|q\|_{H^1}^2) \leq G_N(\mathbf{v}, q; 0) \leq C (\|\mathbf{v}\|_W^2 + \|q\|_{H^1}^2)$$

where

$$\|\mathbf{v}\|_W := \left(\|\mathbf{v}\|_{L_w^2}^2 + \|\nabla \cdot \mathbf{v}\|_{L_w^2}^2 + \|\nabla \times \mathbf{v}\|_{L_w^2}^2 \right)^{\frac{1}{2}}.$$

Also we have the spectral convergence.

Assume that the exact solution $(\mathbf{u}, p) \in H_w^s(\Omega)^3$ for some $s \geq 1$ and $f \in H_w^\ell(\Omega)$ for some integer $\ell \geq 2$. Let $(\mathbf{u}_N, p_N) \in \mathbf{W}_N \times V_N$ be the approximate solution. Then there exists a constant C such that

$$\|\mathbf{u} - \mathbf{u}_N\|_W + \|p - p_N\|_{H^1} \leq C [N^{1-s} (\|\mathbf{u}\|_{H_w^s} + \|p\|_{H_w^s}) + N^{-\ell} \|f\|_{H_w^\ell}].$$

In [7] one may also find the theory and numerical results of least-squares spectral collocation method for Stokes problem.

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DEPARTMENT OF MATHEMATICS, CHONNAM NATIONAL UNIVERSITY, KWANGJU 500-757, KOREA
E-mail address: bcshin@jnu.ac.kr