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DECAY BOUNDS FOR MAGNETOHYDRODYNAMIC GEOPHYSICAL FLOW

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Abstract. This paper establishes exponential decay bounds for steady magnetohydrodynamic geophysical pipe flow when homogeneous lateral surface boundary conditions are applied. In the spirit of earlier work of Horgan and Wheeler (SIAM J. Appl. Math. 35 (1978) 789) and Ames and Payne (SIAM J. Math. Anal. 20 (1989) 97), the decay to fully developed flow as a function of distance from the entry section is investigated. Here, it is not assumed that the flow is fully developed at the exit section. Energy inequalities are derived that lead to estimates for the "energy" associated with the velocity and magnetic field represented by the difference between the entrance flow and the fully developed flow in a portion of the pipe near the exit section. This paper also indicates how to bound the total energy.

1. Introduction

A number of spatial decay studies for problems involving various systems of differential equations have appeared in the literatures, e.g., for a survey of Saint-Venant type spatial decay results see Hogran and Knowles [6] and Horgan [4, 5], and for a decay estimates for the Navier-Stokes equations see Ames and Payne [1], and Horgan and Wheeler [7], and for other type spatial decay results see Quintailla [9] and Ames and Straughan [3]. More recently, Song [10] established exponential decay bounds for a problem in steady magnetohydrodynamical pipe flow. In this paper, we shall consider a problem arising from geophysics with a view to deriving spatial decay inequalities of the same type exhibited in [10].

We shall assume the a stationary flow occupies the interior of a cylindrical pipe R of length l with boundary ∂R. The pipe has arbitrary cross section denoted by D and the boundary ∂D and we assume that the generators of the pipe are parallel to the x_3 axis. The entry and exit cross section will be denoted by D_0 and D_l. The system of equations we analyze can be written in the following form (see Zhao [11]):

$$\nu\Delta\hat{u}_i = \hat{u}_j\hat{u}_{i,j} - \hat{B}_j\hat{B}_{i,j} + \pi_{,i} \quad \text{in } R, \tag{1.1}$$

$$\hat{\nu}\Delta\hat{B}_i = \hat{u}_j\hat{B}_{i,j} - \hat{B}_j\hat{u}_{i,j} + \hat{\pi}_{,i} \quad \text{in } R, \tag{1.2}$$

$$\hat{u}_{j,j} = 0, \quad \text{in } R. \tag{1.3}$$

$$\hat{B}_{j,j} = 0 \quad \text{in } R. \tag{1.4}$$

Equations (1.1)–(1.4) are subject to the boundary conditions:

$$\hat{u}_i = 0, \quad \hat{B}_i = 0 \quad \text{on } \partial R \backslash (D_0 \cup D_l), \tag{1.5}$$

$$\hat{u}_i = f_i(x_1, x_2) \quad \text{on } D_0, \quad \hat{u}_i = g_i(x_1, x_2) \quad \text{on } D_l, \tag{1.6}$$

$$\hat{B}_i = h_i(x_1, x_2) \quad \text{on } D_0, \quad \hat{B}_i = a_i(x_1, x_2) \quad \text{on } D_l. \tag{1.7}$$

Here $\hat{u}_i$ and $\hat{B}_i$ are the unknown velocity and magnetic field components, π and $\hat{\pi}$ denotes the pressures, ν is the kinematic viscosity and the positive constant $\hat{\nu}$ related to magnetic permeability and electrical resistivity has the same dimension as ν. In (1.1)–(1.4) the symbol Δ denotes the Laplace operator, a comma is used to denote partial differentiation, and the summation convention of summing in any term over a repeated index, Latin subscripts ranging from 1 to 3 and the Greek subscript α from 1 to 2 is adopted. In addition, the prescribed entrance and exit flow profiles are assumed to be compatible with the lateral surface conditions (1.5). Our goal here is to study the steady state system (1.1)–(1.7) in order to establish spatial exponential decay results for an energy integral $E(z)$. To this end we derive an integro-differential inequality of the form

$$\frac{dE}{dz} + \alpha \int_z^l E(\xi) d\xi \leq \beta E(z) + \gamma \tag{1.8}$$

for positive constants α, β, and γ. Our derivation of the inequality and the subsequent attempt to bound the total energy $E(0)$ closely follows that of Song [10] with the exception that we need to introduce a coupling parameter in $E(z)$ in order to bound several nonlinear terms that arise since our system unlike Song's [10], is not symmetric.

2. Formulation of problem

In this section we formulate the boundary value problem that provides the framework for our investigation of deriving (1.8). For convenience, we introduce the notation:

$$R_z = \{(x_1, x_2, x_3) | (x_1, x_2) \in D, \ l > x_3 > z \geq 0\},$$
$$D_z = \{(x_1, x_2, x_3) | (x_1, x_2) \in D, \ x_3 = z\}.$$

In order to establish a exponential spatial decay for the flow, we introduce several auxiliary functions. Since we cannot expect the fully developed flow at the exit end of a finite length pipe, following the arguments of Ames and Payne [1], we find it convenient to compare the entrance flow to the fully developed flow indirectly through the introduction of a linearized Stokes flow that enters the pipe with a velocity field equivalent to the fully developed one. To this end, we introduce auxiliary Stokes problems which are

$$\nu\Delta v_i = q_{,i} \quad \text{in } R, \tag{2.1}$$
$$v_{j,j} = 0 \quad \text{in } R, \tag{2.2}$$
$$v_i = 0 \quad \text{on } \partial R\backslash(D_0 \cup D_l), \tag{2.3}$$
$$v_i = \hat{v}\delta_{3i} \quad \text{on } D_0, \tag{2.4}$$
$$v_i = \hat{u}_i \quad \text{on } D_l, \tag{2.5}$$

$$\nu\Delta c_i = \hat{q}_{,i} \quad \text{in } R, \tag{2.6}$$
$$c_{j,j} = 0 \quad \text{in } R, \tag{2.7}$$
$$c_i = 0 \quad \text{on } \partial R\backslash(D_0 \cup D_l), \tag{2.8}$$
$$c_i = \hat{c}\delta_{3i} \quad \text{on } D_0, \tag{2.9}$$
$$c_i = \hat{B}_i \quad \text{on } D_l, \tag{2.10}$$

where q and $\hat{q}$ are functions that are not prescribed *a* priori but are determined up to constants by the fact that all of the equations (2.1)–(2.10) are to be satisfied, and $(0, 0, \hat{v}(x_\alpha))$ and $(0, 0, \hat{c}(x_\alpha))$ represent the fully developed velocity fields corresponding to the net inflows, respectively

$$\int_{D_0} f_3 dA \equiv Q, \quad \int_{D_0} h_3 dA \equiv \hat{Q}. \tag{2.11}$$

Thus $\hat{v}(x_\alpha)$ and $\hat{c}(x_\alpha)$ can be characterized as the solutions of the boundary value problems

$$\nu\hat{v}_{,\alpha\alpha} = r_{,3} \quad \text{in } D_z, \tag{2.12}$$
$$\hat{v} = 0 \quad \text{on } \partial D_z, \tag{2.13}$$
$$\int_{D_z} \hat{v} dA = Q, \tag{2.14}$$

$$\nu\hat{c}_{,\alpha\alpha} = \hat{r}_{,3} \quad \text{in } D_z, \tag{2.15}$$
$$\hat{c} = 0 \quad \text{on } \partial D_z, \tag{2.16}$$
$$\int_{D_z} \hat{c}\, dA = \hat{Q}. \tag{2.17}$$

The gradients of the pressures r and $\hat{r}$ in (2.12) and (2.15) are such that $r_{,i} = -P\delta_{3i}$ and $\hat{r}_{,i} = -\hat{P}\delta_{3i}$ where P and $\hat{P}$ are positive constants. We define

$$u_i = \hat{u}_i - v_i,\ b_i = \hat{B}_i - c_i,\ p = \pi - q,\ \hat{p} = \hat{\pi} - \hat{q}. \tag{2.18}$$

Then the boundary value problem for u_i and c_i is

$$\begin{aligned}
&\nu\Delta u_i = (u_j+v_j)(u_i+v_i)_{,j} - (b_j+c_j)(b_i+c_i)_{,j} + p_{,i} \quad \text{in } R,\\
&\hat{\nu}\Delta b_i = (u_j+v_j)(b_i+c_i)_{,j} - (b_j+c_j)(u_i+v_i)_{,j} + \hat{p}_{,i} \quad \text{in } R,\\
&u_{i,i}=0, \quad b_{i,i}=0 \quad \text{in } R,\\
&u_i = 0, \quad b_i = 0 \quad \text{on } \partial R\backslash D_0,\\
&u_i = f_i - \hat{v}\delta_{3i}, \quad b_i = h_i - \hat{c}\delta_{3i} \quad \text{on } D_0.
\end{aligned} \tag{2.19}$$

In view of (2.11),(2.14), and (2.17) we have

$$\int_{D_z} u_3 dA = 0, \int_{D_z} b_3 dA = 0 \quad \text{for } 0\le z\le l. \tag{2.20}$$

3. Decay estimates

To derive an inequality which will imply exponential decay, we first consider the energy integral

$$E(z) = \int_{R_z} (u_{i,j}u_{i,j} + k\, b_{i,j}b_{i,j})dx, \tag{3.1}$$

where the coupling parameter $k = \hat{\nu}/\nu$. We integrate by parts and use boundary conditions to obtain

$$\begin{aligned}
E(z) = &-\int_{D_z} (u_i u_{i,3} + k\, b_i b_{i,3})dA\\
&-\frac{1}{\nu}\int_{R_z} u_i\big[(u_j+v_j)(u_{i,j}+v_{i,j}) - (b_j+c_j)(b_{i,j}+c_{i,j}) + p_{,i}\big]dx\\
&-\frac{k}{\hat{\nu}}\int_{R_z} b_i\big[(u_j+v_j)(b_{i,j}+c_{i,j}) - (b_j+c_j)(u_{i,j}+v_{i,j}) + \hat{p}_{,i}\big]dx.
\end{aligned} \tag{3.2}$$

From the definition of $E(z)$, it follows that

$$\frac{dE}{dz} = -\int_{D_z} (u_{i,j}u_{i,j} + k\, b_{i,j}b_{i,j})dA. \tag{3.3}$$

To establish the desired integro-differential inequality (1.8), using (3.2) we first form

$$\frac{dE}{dz} + \alpha\int_z^l E(\xi)d\xi = -I_1 + I_2, \tag{3.4}$$

where α is a positive parameter,

$$I_1 = \int_{D_z} (u_{i,j}u_{i,j} + k\, b_{i,j}b_{i,j})dA - \frac{\alpha}{2}\int_{D_z} (u_i u_i + k\, b_i b_i)dA, \tag{3.5}$$

$$\begin{aligned}
\frac{I_2}{\alpha} = &-\frac{1}{\nu}\int_z^l\int_{R_\xi} u_i\big[(u_j+v_j)(u_{i,j}+v_{i,j}) - (b_j+c_j)(b_{i,j}+c_{i,j})\big]dxd\xi\\
&-\frac{k}{\hat{\nu}}\int_z^l\int_{R_\xi} b_i\big[(u_j+v_j)(b_{i,j}+c_{i,j}) - (b_j+c_j)(u_{i,j}+v_{i,j})\big]dxd\xi\\
&+\frac{1}{\nu}\int_{R_z} p u_3 dx + \frac{k}{\hat{\nu}}\int_{R_z} \hat{p} b_3 dx.
\end{aligned} \tag{3.6}$$

We note that if we choose $\alpha = 2\lambda$ (λ is the first eigenvalue of the Laplacian), we drop the term I_1 which is nonnegative in view of the Poincare inequality. In order to bound I_2/α, we refer to reference [10].

Inserting the bounds for I_2 into (3.4) and dropping the nonnegative term I_1, we obtain the desired integro-differential inequality for the energy integral

$$\frac{dE}{dz} + a\int_z^l E(\xi)d\xi \le bE(z) + c, \tag{3.7}$$

where

$$\frac{a}{2\lambda} = \left\{1 - \frac{1}{\nu_m}\left(\frac{V_M}{\sqrt{\lambda}} + \frac{Q_s}{\lambda^{1/4}}\right)\sqrt{1+k}\left(1 + \frac{1}{\sqrt{k}}\right)\right\}, \tag{3.8}$$

$$\begin{aligned}\frac{b}{2\lambda} = \sqrt{\frac{C}{\lambda}} + \frac{1}{\nu_m}\Bigg[\frac{V_M}{\lambda}\sqrt{\frac{(1+k)}{k}} + \frac{\sqrt{E(0)}}{\lambda^{3/4}}\left\{\frac{1}{\sqrt{2}} + \sqrt{C}(1+\sqrt{(1+k)k^{-1}})\right\}\\ + \left(\frac{Q_s}{\lambda^{3/4}} + \frac{V_M}{\lambda}\right)\left\{\frac{1}{2} + \frac{1}{\sqrt{k}} + \sqrt{C}(1+\sqrt{1+k}) + 2\sqrt{\frac{(1+k)C}{k}}\right\}\Bigg]\end{aligned} \tag{3.9}$$

$$\begin{aligned}\frac{c}{2\lambda} = 2\sqrt{k+1})\frac{Q_s}{\nu_m}\left\{\sqrt{C}\left(\frac{Q_s}{\lambda^{3/4}} + 2\frac{V_M}{\lambda}\right) + \frac{V_M}{\lambda}\right\}\sqrt{E(0)}\\ + 2\frac{Q_s}{\nu_m}\left(\frac{Q_s}{\lambda^{1/4}} + 2\frac{Q_sV_M}{\sqrt{\lambda}}\right)\sqrt{1+k}\sqrt{E(0)}\,l,\end{aligned} \tag{3.10}$$

where all the appropriate quantities are defined in [2, 10]. To ensure decay, we require

$$a > 0. \tag{3.11}$$

This condition yields a restriction on the flow. We note that b and c involve $\sqrt{E(0)}$. It is shown in [7, 10] that (3.7) is integrated to give

$$E(z) \le \hat{E}(0)e^{-\sigma z} + \frac{c}{\sigma} \quad \text{for } 0 \le z \le l, \tag{3.12}$$

where

$$\sigma = \frac{1}{2}(\sqrt{b^2+4a} - b), \tag{3.13}$$

$$\hat{E}(0) \le \frac{\sqrt{b^2+4a}}{\sigma}E(0) + \frac{c\sqrt{b^2+4a}}{\sigma^2}. \tag{3.14}$$

We note from (3.11) that if we were to assume that v_i and the fully developed flow are identical and $b_i = 0$, we then would recover Horgan and Wheeler's result [7, Eq.(5.4)]

$$\frac{\hat{v}_s}{\nu\sqrt{\lambda}} < 1. \tag{3.15}$$

To make (3.12) explicit we need a bound for $\hat{E}(0)$ or $E(0)$ in terms of data, which we omit here.

Theorem. *If u_i and b_i are classical solutions of the problem* (2.19), *the energy integral E(z) defined by* (3.1) *is bounded by* (3.12).

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