Trends in Mathematics
Information Center for Mathematical Sciences
Volume 8, Number 2, December, 2005, Pages 33–36

ON THE POSITION OF SPIKE LAYERS FOR MOUNTAIN PASS SOLUTIONS OF SINGULARLY PERTURBED NONLINEAR DIRICHLET PROBLEMS

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Abstract. We consider the following singularly perturbed nonlinear elliptic problem on a bounded domain Ω

$$\varepsilon^2\Delta u - u + f(u) = 0, \quad u > 0 \quad \text{in} \quad \Omega, \quad u = 0 \text{ on } \partial\Omega$$

where a nonlinearity f is of subcritical growth and Ω is a bounded domain in $\mathbf{R}^n$. For cerain f, there exists a mountain pass solution u_ε which exhibits a spike layer as $\varepsilon \to 0$. We characterize the location of the spike layer for possibly general f.

Let Ω be a bounded domain in $\mathbf{R}^n$ with $\partial\Omega \in C^2$. We are interested in the following singularly perturbed nonlinear elliptic problem on Ω

$$\varepsilon^2\Delta u - u + f(u) = 0, \quad u > 0 \quad \text{in} \quad \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{1}$$

A solution of (1) corresponds to a critical point of a functional $\Gamma^\varepsilon(u) = \frac{1}{2}\int_\Omega \varepsilon^2|\nabla u|^2 + u^2 dx - \int_\Omega F(u)dx$ on $H_0^{1,2}(\Omega)$. Here, $F(t) = \int_0^t f(s)ds$. The following equation corresponds to a limiting equation to equation (1) as $\varepsilon \to 0$;

$$\Delta u - u + f(u) = 0, \quad u > 0 \text{ in } \mathbf{R}^n, \quad \text{and} \quad \lim_{|x|\to\infty} u(x) = 0. \tag{2}$$

We assume that a function $f : \mathbf{R} \to \mathbf{R}$ satisfies the following conditions

- **(f1):** $f \in C(\mathbf{R}, \mathbf{R}), f(t) = 0$ for $t \le 0$ and $\lim_{t\to 0} f(t)/t = 0$;
- **(f2):** there exists $p \in (1, \frac{n+2}{n-2})$ such that $\limsup_{t\to\infty} f(t)/t^p < \infty$;
- **(f3):** for $F(t) \equiv \int_0^t f(s)ds$, there exists $\mu > 2$ such that $0 < \mu F(t) < f(t)t$ for $t > 0$.

Under conditions (f1),(f2) and (f3), there exists a mountain pass solution v_ε of (1) which is positive on Ω. Then, we can deduce from comparison principles and an energy estimate that for a maximum point x_ε of v_ε, there exist constant $C, c > 0$, independent of $\varepsilon > 0$ satisfying $v_\varepsilon(x) \le C\exp(-\frac{c\text{dist}(x,x_\varepsilon)}{\varepsilon}), x \in \Omega$. Thus, v_ε exhibits a spike layer as $\varepsilon \to 0$. Then, a natural concern is the location of the maximum points x_ε of the solution v_ε for small $\varepsilon > 0$. Ni and Wei proved in [10] that

$$\lim_{\varepsilon\to 0} \text{dist}(x_\varepsilon, \partial\Omega) = \max_{x\in\Omega} \text{dist}(x, \partial\Omega) \tag{3}$$

Key words and phrases. This work was supported by grant No. R01-2004-000-10055-0 from the Basic Research Program of the Korea Science & Engineering Foundation.
singularly perturbed, mountain pass solution, spike layer, limiting problem, distance function.

under the following additional conditions

(f1′): $f \in C^{0,1}(\mathbf{R}, \mathbf{R})$;
(f4): $f(t)/t$ is non-decreasing on $(0, \infty)$;
(f5): for a least energy solution U of (2), if $\Delta V - V + f'(U)V = 0$ and $V \in H^{1,2}(\mathbf{R}^n)$, then, $V = a_1 \frac{\partial^1 U}{\partial x_1} + \cdots + a_n \frac{\partial^n U}{\partial x_n}$ for some $a_1, \cdots, a_n$.

In an interesting paper [3], del Pino and Felmer showed that the asymptotic behaviour (3) can be obtained in a simple manner even without conditions (f1′) and (f5). Their approach in [3] depends strongly on the monotonicity condition (f4). Developing further the approach in [3], we have proved the asymptotic behaviour (3) even more without condition(f4). Thus, we need just conditions (f1),(f2) and (f3) to show the asymptotic behaviour (3). One of basic ideas in [3] is to consider a pass $tu, t \in (0, \infty)$ for each $u \in H_0^{1,2}(\Omega)$; a function $g(t) = \Gamma^\varepsilon(tu)$ has only maximum critical points on $(0, \infty)$ when (f4) holds. We can say that a pass $\{tu | t \in [0, \infty)\}$ is made by deforming the range of u. A main idea or a different view point with [3] in our approach is to deform the domain of u instead of the range of u.

1. Statement of a main result and some remarks

For $u \in C_0^\infty(\Omega)$, we define $\|u\|^\varepsilon \equiv (\int_\Omega \varepsilon^2 |\nabla u|^2 + u^2 dx)^{1/2}$, and for $v \in C_0^\infty(\mathbf{R}^n)$, $\|u\| = (\int_{\mathbf{R}^n} |\nabla v|^2 + v^2 dx)^{1/2}$. Let $H^\varepsilon(\Omega)$ and $H^{1,2}(\mathbf{R}^n)$ be completions of $C_0^\infty(\Omega)$ and $C_0^\infty(\mathbf{R}^n)$ with respect to a norm $\| \cdot \|^\varepsilon$ and $\| \cdot \|$, respectively. For $u \in H^\varepsilon(\Omega)$, we define an energy functional

$$\Gamma^\varepsilon(u) = \frac{1}{2} \int_\Omega \varepsilon^2 |\nabla u|^2 + u^2 dx - \int_\Omega F(u) dx$$

and for $u \in H_0^{1,2}(\mathbf{R}^n)$,

$$\Gamma(u) = \frac{1}{2} \int_{\mathbf{R}^n} |\nabla u|^2 + u^2 dx - \int_{\mathbf{R}^n} F(u) dx.$$

In a classical paper [2], Berestycki and Lions proved that there exists a least energy solution $U \in C^2(\mathbf{R}^n)$ of (2) such that $U(x) = U(|x|)$,

$$|D^\alpha U(x)| \leq C \exp(-c|x|), \qquad x \in \mathbf{R}^n \tag{4}$$

for some $C, \delta > 0$ and any $|\alpha| \leq 2$, We define $\mathcal{L}$ the set of a least energy solution U of (2) satisfying $U(0) = \max_{x \in \mathbf{R}^n} U(x)$. It is well known [4] that any $U \in \mathcal{L}$ is radially symmetric and $\nabla U(x) \cdot x < 0$ for $|x| > 0$ if f is Lipschitz continuous.

The following result has been obtained.

Theorem 1.1. *Assume that f satisfies the conditions (f1),(f2) and f(3). Then, for sufficiently small $\varepsilon > 0$, there exists a mountain pass solution u_ε of (1) such that for some constant $C, c > 0$ and $x_\varepsilon \in \Omega$ with $\max_{x \in \Omega} u_\varepsilon(x) = u_\varepsilon(x_\varepsilon)$, $u_\varepsilon(x) \leq C \exp(-\frac{c|x - x_\varepsilon|}{\varepsilon})$. Furthermore, if $n \geq 3$,*

$$\lim_{\varepsilon \to 0} dist(x_\varepsilon, \partial\Omega) = \max_{x \in \Omega} dist(x, \partial\Omega)$$

and for a transformed function $w_\varepsilon(x) = u_\varepsilon(\varepsilon(x - x_\varepsilon))$ and any $\varepsilon_m > 0$ with $\lim_{m \to \infty} \varepsilon_m = 0$, a sequence $\{w_{\varepsilon_m}\}_m$ converges, up to a subsequence, uniformly to a least energy solution $U \in \mathcal{L}$ of (2).

Remark 1.1. *For $N = 1$, we can easily prove the result in Theorem 1.1 under more weak conditions as in* [1] *and* [7]. *More precisely, if f satisfies (f1) and there exists $T > 0$ satisfying $T < f(T), \frac{T^2}{2} = F(T)$ and $\frac{t^2}{2} > F(t), t \in (0,T)$, then for sufficiently small $\varepsilon > 0$, there exists a mountain pass solution u_ε of (1) with $\Omega = (-a,a)$ such that $u_\varepsilon(x) = u_\varepsilon(-x)$, $u'_\varepsilon(t) < 0$ for $t \in (0,a)$, $u_\varepsilon(0) = \max_{t\in(-a,a)} u_\varepsilon(t)$, $\lim_{\varepsilon\to 0} u_\varepsilon(0) = T$, and $u_\varepsilon(t) \le C\exp(-c\frac{|t|}{\varepsilon})$ for some $C, c > 0$, independent of $\varepsilon > 0$. We should note that even when f satisfies above condition, there can be many other solutions depending on the behaviour of f on (T,∞); on the other hand, if $\Omega = \mathbf{R}$, there exists a unique solution of (1) (see* [8]*).*

For $N = 2$, our approach in this paper for the proof of Theorem 1.1 does not work well since for certain domain deformations $\{y(x,s)\}_{s\in\mathbf{R}}$, a path $W_s \equiv u_\varepsilon(y(\cdot,s))$ in $H_0^{1,2}(\Omega)$ is not a fine approximating mountain path if $N = 2$.

Remark 1.2. *For the limiting problem (2), Berestychi and Lions proved in* [2] *an existence of a least energy solution under conditions (f1),(f2) and the following condition*

(f3′): *for some $T > 0$, $\int_0^T f(s)ds > \frac{T^2}{2}$.*

The condition (f3′) is also a necessary condition for the existence of a solution to (2) when $N \ge 2$. The author believes that the result of Theorem 1.1 holds under more weak conditions (f1),(f2) and (f3′). On the other hand, we should note that our proof on the characterization of a concentration point of a mountain pass solution u_ε in Theorem 1.1 just use condition (f1) and (f2) and the boundedness of $\{\varepsilon^{-N}\int_\Omega \varepsilon^2|\nabla u_\varepsilon|^2 + (u_\varepsilon)^2 dx\}_\varepsilon$. Thus, if we show the existence of a mountain pass solution u_ε and the boundedness of $\{\varepsilon^{-N}\int_\Omega \varepsilon^2|\nabla u_\varepsilon|^2 + (u_\varepsilon)^2 dx\}_\varepsilon$, we can characterize a concentration point of a mountain pass solution as in Theorem 1.1 under conditions (f1), (f2) and (f3′) following our new approach. If Ω is star-shaped, it follows from Pohozaev identity that $\{\varepsilon^{-N}\int_\Omega \varepsilon^2|\nabla u_\varepsilon|^2+(u_\varepsilon)^2 dx\}_\varepsilon$ is bounded if their energies $\{\varepsilon^{-N}\Gamma^\varepsilon(u_\varepsilon)\}_\varepsilon$ is bounded; for the mountain pass solutions, their energies are bounded. Thus, if a domain is convex, any mountain pass solution u_ε if it exits satisfies the properties in the Theorem.

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