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ASYMPTOTIC BEHAVIOR OF NAVIER-STOKES FLOW ON AN EXTERIOR DOMAIN

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1. Introduction

We survey the asymptotic behavior of weak solutions of the Navier–Stokes equations on an exterior domain: in $\Omega \times (0, \infty)$,

$$u_t - \Delta u + (u \cdot \nabla)u + \nabla p = 0, \tag{1.1}$$

$$\nabla \cdot u = 0 \tag{1.2}$$

with initial-boundary conditions

$$u(x, 0) = u_0 \quad \text{for } x \in \Omega, \qquad u(x, t) = 0 \quad \text{for } (x, t) \in \partial\Omega \times (0, \infty),$$

where $n \geq 2$, and $\Omega \subset \mathbb{R}^n$ is an open subset of $\mathbb{R}^n$, with its complement Ω^c is a compact set. Here, $u \stackrel{def}{=} (u_1, \dots, u_n)$ and p denote the velocity and pressure, respectively, while u_0 is a given initial velocity satisfying (1.2).

The decay problem for weak solutions of the Navier-Stokes equations was first proposed by Leray [23] for the Cauchy problem in $\mathbb{R}^3$. If the domain Ω itself or one direction of the domain is bounded, then the decay rate of solutions is exponential. It is obtained easily with the help of Poincare's inequality. For a domain which is not bounded in any direction, it has a different story. The typical domains are the whole space $\mathbb{R}^n$, the half space $\mathbb{R}^n_+$, and the exterior domains. The asymptotic behaviour has two directions, the temporal and the spatial decays.

The temporal decays has long history, but the spatial decays has short one. In the temporal decays, Masuda [24] gave the explicit estimate for the decay of solutions in exterior domains for the first time. Schonbek [27, 28, 29, 30] worked the decay problem in $\mathbb{R}^n$. She obtained the lower and the upper bounds. Kajikiya and Miyakawa [18], and Wiegner [34] discussed the same problem. However, all of the above results except Kajikiya and Miyakawa [18] rely on the theory of the Fourier transform. In the case of the half space the Fourier transform method does not work well.

Borchers and Miyakawa [9] studied the decay problem in $\mathbb{R}^n_+$ using the semigroup theory. Bae and Choe [3] also showed the decay rate if the solution lies in an appropriate weighted space.

In general, to study the decays for the Navier–Stokes equations, the estimates for the solutions of the Stokes equations are essential. If the spatial domain is $\mathbb{R}^n$,

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then the Stokes solution u is reduced to that of the heat equation with initial data u_0. Then for all $1 \le r \le \infty$,

$$\|\nabla u(t)\|_{L^r} \le Ct^{-1/2}\|u_0\|_{L^r} \qquad \text{for } t > 0 \tag{1.3}$$

holds. For $1 < r < \infty$, (1.3) is valid for half spaces (Ukai [33]). For $1 < r < n$, (1.3) is also valid for exterior domains (Borchers and Miyakawa [9]). Schonbek [29, 30] also considered the decay rates if the average is zero, $\int_{\mathbb{R}^n} |u_0|^2|x|dx < \infty$, and under some restrictions on u_0. In Bae and Choe [3], we estimated the decay rate if the solution lies in an appropriate weighted space; for $1 < r \le q < \infty$,

$$\|u(t)\|_{L^q(\mathbb{R}^n_+)} \le Ct^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{2}}\left(\int_{\mathbb{R}^n_+} |y_n|^r|u_0(y)|^r\,dy\right)^{1/r}.$$

Giga, Matsui and Shimizu [12] first showed (1.3) for the critical space $r = 1$ in $\mathbb{R}^n_+$. Since the projection associated with the Helmholtz decomposition is not bounded in L^1 type spaces if the domain has nonempty boundaries, it is not easy to obtain such result for $r = 1$ and $r = \infty$. In [12], they estimated the rates in the Hardy space, and then as a corollary, they obtained (1.3) for $r = 1$. In [2], we also obtain estimates for $r = 1$ in the half spaces. However, the the time decay rate is not $t^{-1/2}$ but t^{-1}. In [1], we estimated u itself, not ∇u in L^1 and L^∞.

In the spatial asymptotic behaviour, Farwig and Sohr [10, 11] showed the spatial decays for the exterior problems:

$$\|\,|x|^{\alpha/2}u\|_{L^2}^2 + \int_0^t \|\,|x|^{\alpha/2}\nabla u\|_{L^2}^2 d\tau \le \begin{cases} C(u_0,\alpha) & \text{if } 0 \le \alpha < 1/2, \\ C(u_0,\alpha)t^{\alpha'/2-1/4} & \text{if } 1/2 \le \alpha < \alpha' < 1, \\ C(u_0)(t^{1/4}+t^{1/2}) & \text{if } \alpha = 1, \end{cases} \tag{1.4}$$

where $1/\alpha + 1/\alpha' = 1$. In He [13] and He and Xin [15], they obtained the decay rates for $0 \le \alpha \le 3/2$. Schonbek and Schonbek [31] studied the decay properties of $\||x|^{\alpha/2}u\|_{L^2}$ for $0 \le \alpha \le 3/2$, when u is smooth.

For the strong solution, Takahashi [32] showed that if u is smooth, then

$$t^\beta|x|^\alpha|u(x,t)| \le C$$

for all $\alpha, \beta \ge 0$ with $\alpha + 2\beta < 3$. Miyakawa [25] obtained in $\mathbb{R}^n$ that for $1 \le \gamma \le n+1$, there is a solution u such that

$$|u(x,t)| \le C(1+|x|)^{-\alpha}(1+t)^{-\beta/2} \quad \text{for all } \alpha, \beta \ge 0 \text{ with } \alpha + \beta = \gamma.$$

He and Xin [15] also obtained a temporal-spatial rate with the hypothesis $\|u_0\|_1 + \|u_0\|_2$ is small.

In Bae and Jin [4], we improved He and Xin's result upto $\alpha < 5/2$. In Bae and Jin [5], we obtain the lower bounds like Schonbek [29, 30], and Miyakawa and Schonbek [26]. However, we include weights for the temporal-spatial decays

$$C_0(1+t)^{-\frac{5}{4}+\frac{\alpha}{2}} \le \|(1+|x|^2)^{\alpha/2}\mathbf{u}(\cdot,t)\|_{L^2(\mathbb{R}^n)} \le C_1(1+t)^{-\frac{5}{4}+\frac{\alpha}{2}}$$

for $0 \le \alpha \le 2$. We also obtain spatial asymptotic behaviour for the case of the strong solution.

In this paper we focus the spatial decay problem on an exterior problem. When regularity estimates or decay estimates are concerned in whole space $\mathbb{R}^3$ or interior of a domain, the pressure representation in terms of velocity function has been useful. From the pressure representation, we have seen that the effect of pressure p is almost the same as the square of velocity $|\mathbf{u}|^2$.

The situation is not simple when a domain with nonempty boundary is involved. For example, if we try to have energy estimates, we might meet the following integral:

$$\int_\Omega \phi(\mathbf{u}\cdot\nabla)p dx = -\int_\Omega p(\mathbf{u}\cdot\nabla)\phi dx,$$

where ϕ may be a cut-off function for the localization if we consider partial regularity estimates at the boundary, or may be weight $(1+|x|^2)$ if we consider spatial decay estimates.

As it is seen in the above identity, the pressure term must be treated. Unfortunately, the pressure has non-local property and we have not enough information of the pressure on the boundary. This fact makes it difficult to derive norm estimates when the boundary is involved.

To derive spatial-temporal decay estimates of weak and strong solutions for Navier-Stokes equations, we suggest a new idea treating energy estimates for the exterior domain as fluid region, and we can avoid the computations involving with the pressure term.

Throughout this work, weak solution means a suitably weak solution which is constructed by Cafarelli, Kohn and Nirenberg.

In [15], He and Xin have considered exterior domain problem, by removing pressure term in solution representation, they have obtained decay rate results for the strong solution, that is, for small data. They obtained the spatial decay of a strong solution:

$$(1+|x|^2)^{\frac{3}{2}(\frac{1}{7}-\frac{1}{p})}\mathbf{u} \in L^\infty(0,\infty;L^p(\Omega)) \qquad \text{for } 7 < p \le \infty.$$

2. Temporal Decays

Masuda [24] gave the decays in exterior domain with $t^{-1/8}$. Heywood [16] obtained the rate $t^{-1/2}$.

Borchers and Miyakawa in [7] and [8] showed that there is a weak solution u such that

(i) $\|u(t)\|_{L^2} \to 0$,
(ii) if $\|e^{-tA}u_0\|_{L^2} = O(t^\alpha)$ for some $\alpha > 0$, then

$$\|u(t)\|_{L^2} = \begin{cases} O(t^{-\alpha}) & \text{if } \alpha < n/4, \\ O(t^{-n/4}) & \text{if } \alpha \ge n/4, \end{cases},$$

where A is the Stokes operator. They used the Fourier analysis for closed linear operators in Banach space and extended those of Schonbek [27, 28, 29, 30], and Kajikiya and Miyakawa [18], and Borchers and Miyakawa [7].

The temporal decay rate of the solution (weak for $1 < p \le 2$, strong for $2 < p \le \infty$) of the Navier-Stokes equation is known as follows (see [7] and [8] and [22]); if $\mathbf{u}_0 \in L^r \cap L^2$, with $1 \le r < 2$, then

$$\|\mathbf{u}(t)\|_{L^2(\Omega)} \le c(1+t)^{-\frac{3}{2}(\frac{1}{r}-\frac{1}{2})}, \tag{2.1}$$

and if $\mathbf{u}_0 \in L^r \cap L^2$, with $1 < r < 2$, then for $r \le q < 2$,

$$\|\mathbf{u}(t)\|_{L^q(\Omega)} \le c(1+t)^{-\frac{3}{2}(\frac{1}{r}-\frac{1}{q})}. \tag{2.2}$$

In Iwashita [17], He and Miyakawa [14], it is shown that strong solutions exist in L^q for all times provided the initial data are small. Furthermore, it is also shown that the the decay rates of L^q-norms of $\mathbf{u}$ and $\nabla\mathbf{u}$, respectively;

$$\begin{aligned} &\|\mathbf{u}(t)\|_{L^q} \le c(1+t)^{-\frac{3}{2}(\frac{1}{r}-\frac{1}{q})}, \quad \text{for } 1\le r\le q\le\infty, 1\le r<\infty,\ q>1, \\ &\|\nabla\mathbf{u}(t)\|_{L^q} \le c(1+t)^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{r}-\frac{1}{q})}, \quad \text{for } 1\le r\le q\le 3. \end{aligned} \tag{2.3}$$

For the spatial estimate, we use the above estimates (2.1), (2.2) for weak solution and (2.3) for strong solutions, as preliminary ones. For our proofs, by the well known temporal decay rate (2.1), it is enough to estimate $\|\,|x|\mathbf{u}\|_2$.

3. Spatial Decays

We state our main theorems.

Theorem 3.1. *Let $\mathbf{u}_0 \in L^r(\Omega) \cap L^2(\Omega)$ with $1 \le r < 6/5$, and $\nabla\cdot\mathbf{u}_0 = 0$. Let $\mathbf{v}_0 = N * (\phi\nabla\times\mathbf{u}_0)$ with $\nabla\mathbf{v}_0 \in L^2(\mathbb{R}^3)$. Here, $\phi(x) = |x|^2\chi(|x|)$, where χ is a cut off function in $[0,\infty)$ with $\chi = 0$ on $|x| \le 1$ and $\chi = 1$ on $|x| \ge 2$ and $N(x) = 1/(4\pi|x|)$.*

Then there is a weak solution $\mathbf{u}$ of the Navier-Stokes equation satisfying the following asymptotic property;
for any sufficiently small $\delta > 0$ there is a positive constant c_δ depending on δ and independent of t such that for $1 \le r < \frac{6}{5}$,

$$\|(1+|x|)\mathbf{u}(t)\|_{L^2(\Omega)} \le c_\delta(1+t)^{\frac{5}{4}-\frac{3}{2r}+\delta}$$

for any $t > 0$.

Theorem 3.2. *Let $1 \le r < 6/5$ and let $2 \le p < \infty$. Let $\mathbf{u}_0 \in L^r \cap L^2$ with $\nabla\cdot\mathbf{u}_0 = 0$ and $\nabla\times\mathbf{v}_0 \in L^1 \cap L^p$, where $\mathbf{v}_0$ is the same defined in theorem 3.1. Suppose that $\mathbf{u}$ is a strong solution of the Navier-Stokes equations. Then we have the following spatial-temporal decay rates:*

$$\|(1+|x|)\mathbf{u}\|_{L^p} \le c(1+t)^{\frac{1}{2}-\frac{3}{2}(\frac{1}{r}-\frac{1}{p})}.$$

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