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ON GROUND STATES OF QUASILINEAR ELLIPTIC EQUATIONS

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1. Introduction

In this note, we consider the quasilinear elliptic equation

$$\Delta_p u + K(x)u^q = 0, \tag{1.1}$$

where $n > p > 1$, $q > p - 1$,

$$\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u),$$

and K is a locally Hölder continuous function in $\mathbf{R}^n\backslash\{0\}$. By a ground state of (1.1), we point to a $\mathbf{C}^1(\mathbf{R}^n\backslash\{0\})$ nonnegative nontrivial weak solution in $\mathbf{R}^n$ which tends to 0 at ∞.

This problem appears in the theory of quasi-regular and quasi-conformal mappings, and in the study of non-Newtonian fluids. Media with $p > 2$, $p = 2$ and $p < 2$ are called dilatant fluids, Newtonian fluids and pseudoplastics respectively. See the references in [13] for the physical background.

In order to understand the role of K in the existence of weak solutions of (1.1), we first assume that K vanishes rapidly at ∞. More precisely, if K is a smooth function vanishing like $|x|^l$ at ∞ for $l < -p$, then (1.1) possesses infinitely many positive solutions tending positive constants at ∞. For $l \geq -p$, this asymptotic property is no longer valid (see (1.5) and (1.6) below). However, as an improvement of Liouville's theorem for p-Laplace equation, a basic nonexistence of ground states in $W^{1,p}_{\mathrm{loc}}(\mathbf{R}^n) \cap \mathbf{C}(\mathbf{R}^n)$ is derived to the following quasilinear inequality

$$-\Delta_p u \geq c|x|^l u^q$$

for $c > 0$ and $l > -p$, if

$$q \leq \frac{(p-1)(n+l)}{n-p}.$$

Moreover, the nonexistence of ground states of the equation

$$\Delta_p u + c|x|^l u^q = 0, \tag{1.2}$$

holds for

$$q < \frac{(p-1)n + p + pl}{n-p}$$

and $l \geq 0$. In [14], the equation

$$\Delta_p u + u^{\frac{(p-1)n+p}{n-p}} = 0$$

admits the one-parameter family of positive ground states given by

$$U_d(x) = \frac{d}{(1 + D(d^{\frac{p}{n-p}}|x|)^{\frac{p}{p-1}})^{\frac{n-p}{p}}}$$

with $D = D_{p,n} = \frac{p-1}{(n-p)n^{1/(p-1)}}$ and $U_d(0) = d > 0$. Similarly, every positive radial solution of (1.2) holds the behavior

$$|x|^{\frac{n-p}{p-1}} u_\alpha(|x|) \to C > 0$$

as $|x| \to \infty$ if

$$q = \frac{(p-1)n + p + pl}{n-p}.$$

A radial solution of (1.2) satisfies the equation

$$(|u_r|^{p-2}u_r)_r + \frac{n-1}{r}|u_r|^{p-2}u_r + cr^l u^q = 0, \tag{1.3}$$

where $u(x) = u(|x|)$ and $r = |x|$. For $l > -p$, (1.3) with $u(0) = \alpha > 0$, has a unique positive solution $u \in \mathbf{C}^1((0,\epsilon)) \cap \mathbf{C}([0,\epsilon))$ for small $\epsilon > 0$ such that $|u_r|^{p-2}u_r \in \mathbf{C}^1([0,\epsilon))$ (see [7, 9]). By $u_\alpha(r)$ we denote the unique local solution with $u_\alpha(0) = \alpha > 0$. Note that $r^{(n-p)/(p-1)}u_\alpha(r)$ always increases as r increases if $u_\alpha > 0$ on $(0,\infty)$. However, if

$$q < \frac{(p-1)n + p + pl}{n-p},$$

then every u_α has a finite zero. In other words, (1.3) has no ground states. In the opposite case

$$q > \frac{(p-1)n + p + pl}{n-p},$$

every local solution is to be a ground state which is a *slowly decaying solution*: $u_\alpha(r) > 0$ on $(0,\infty)$ and $r^{(n-p)/(p-1)}u_\alpha(r) \to \infty$ as $r \to \infty$ [7, 9]. On the other hand, if u is a weak radial solution, then

$$-(r^{n-1}|u_r|^{p-2}u_r)_r = cr^{n-1+l}u^q \tag{1.4}$$

and

$$r^{n-1}|u_r|^{p-2}u_r = -c\int_0^r s^{n-1+l}u^q(s)\,ds.$$

Then, we have

$$r^{n-1}|u_r|^{p-2}u_r \le -cu^q(r)\frac{r^{n+l}}{n+l}.$$

Hence

$$\frac{u_r}{u^{q/(p-1)}} \le -Cr^{\frac{1+l}{p-1}}$$

and

$$u(r) \le Cr^{-m} \tag{1.5}$$

for $m = \frac{p+l}{q-(p-1)}$ when $l > -p$. If (1.4) holds near ∞ for $l = -p$, then

$$u(r) \le C(\log r)^{-\frac{p-1}{q-(p-1)}} \tag{1.6}$$

near ∞. Importantly, (1.4) with $l > -p$ has the scaling invariance:

$$u_\alpha(r) = \alpha u_1(\alpha^{\frac{1}{m}} r).$$

Now, it is natural to look for a singular solution which is invariant under the scaling. That is,

$$U(x) = L|x|^{-m},$$

where

$$L = L(n,p,q,l,c) = \left[m^{p-1}(n-1-(m+1)(p-1))\frac{1}{c} \right]^{\frac{1}{q-(p-1)}}.$$

In fact, this singular solution is defined for $l > -p$ and $q > \frac{(p-1)(n+l)}{n-p}$. Setting $V(t) = r^m u(r)$, $t = \log r$, we see the radial version of (1.1):

$$(p-1)(mV - V')^{p-2}(V'' - mV') - \xi(mV - V')^{p-1} + k(t)V^q = 0, \tag{1.7}$$

where $\xi = n-1-(m+1)(p-1) = \frac{cL^{q-(p-1)}}{m^{p-1}}$ and $k(t) = r^{-l}K(r)$. Furthermore, if $mV - V' > 0$, then

$$(p-1)(V'' - mV') - \xi(mV - V') = -\frac{k(t)V^q}{(mV - V')^{p-2}}$$

and

$$(p-1)V'' + aV' - \xi mV = -\frac{k(t)V^q}{(mV - V')^{p-2}},$$

where $a = n-1-(2m+1)(p-1)$ and $\xi m = \frac{cL^{q-(p-1)}}{m^{p-2}}$. When $k(t) \to c$ as $t \to +\infty$, we suspect that the slowly decaying solutions u_α have the following asymptotic behavior

$$\lim_{r\to\infty} r^m u_\alpha(r) = L.$$

By assuming $V' \to 0$ as $t \to +\infty$, the equation leads to the quadratic characteristic polynomial

$$P(\mu) = (p-1)\mu^2 + a\mu + c\frac{q-(p-1)}{m^{p-2}}L^{q-(p-1)} = 0,$$

which has two roots, $-\lambda_1$ and $-\lambda_2$;

$$\lambda_1 = \lambda_1(n,p,l) = \frac{a - \sqrt{a^2 - 4(p-1)(p+l)(a+m(p-1))}}{2(p-1)},$$

$$\lambda_2 = \lambda_2(n,p,l) = \frac{a + \sqrt{a^2 - 4(p-1)(p+l)(a+m(p-1))}}{2(p-1)}.$$

2. Open Questions

There is a critical exponent $q_c = q_c(n,p,l) > \frac{(p-1)n+p+pl}{n-p}$ such that for $q \geq q_c$,

$$a^2 - 4(p-1)(p+l)(a+m(p-1)) \geq 0$$

and thus, $P(\mu)$ has two negative real roots, $-\lambda_1$ and $-\lambda_2$. For $p=2$, this fact plays the central role in analyzing the separation structure for (1.4) with $q \geq q_c(n,2,l)$. Namely, any two solutions u_α and u_β with $0 < \alpha < \beta$ do not intersect [3]. A fundamental question is whether this is also valid for (1.4) with $q \geq q_c(n,p,l)$. More generally, we investigate sufficient conditions of K verifying separation structure for (1.7). The borderline problem is to study ground states with logarithmic decay of (1.1) when K behaves like $|x|^{-p}$ at ∞ (see [2] for the case $p=2$). With dynamical point of view, one may study stability of ground states regarded as steady states for the quasilinear parabolic equation

$$u_t = \Delta_p u + Ku^q.$$

In the supercritical case of $q > \frac{(p-1)n+p+pl}{n-p}$, uniqueness of singular solutions of (1.1) is an interesting question, even in the radial context while in the subcritical case of $q < \frac{(p-1)n+p+pl}{n-p}$, singularities have been analyzed in [6, 8].

Various quasilinear elliptic equations are not included in this note. For example, the typical form is the equation

$$\Delta_p u - u^{p-1} + Ku^q = 0.$$

See [5, 11, 12, 13, 14] for recent advances on the equation.

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