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ELLIPTIC NEUMANN BOUNDARY VALUE PROBLEMS ON CHEMOTAXIS

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ABSTRACT. The multiplicity of solutions for semilinear elliptic equations with exponential growth nonlinearities is treated. The approach to the problem is a variational method.

1. INTRODUCTION

Let $N \geq 3$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary $\partial\Omega$. Our purpose in this paper is to consider the multiple existence of solutions of problem

$$\text{(P)} \qquad \begin{cases} -\Delta u - \lambda u + ae^u &= \epsilon f \qquad \text{on } \Omega \\ \frac{\partial u}{\partial n} &= 0 \qquad \text{on } \partial\Omega \end{cases}$$

where λ, a and $\epsilon \in \mathbb{R}^+$, $f \in H^{-1}(\Omega)$ and n is the outernormal vector.

Problem (P) is a simplified model of problems occur in Physics([7]), Mathematical biology([5], [8]) and Geometry([1]). In particular, it is related to the stationary system of Keller-Segal[5] describing the aggregation of some organisms(amoeba) sensitive to gradient of a chemical substance, which is written as

$$u_t = \nabla \cdot (\nabla u - u\nabla v) \text{ in } \Omega \times (0, T)$$

$$\tau v_t = \Delta - av + u \text{ in } \Omega \times (0, T)$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega \times (0, T)$$

$$u|_{t=0} = u_0(x) \geq) \text{ in } \Omega$$

$$v|_{t=0} = v_0(x) \geq 0 \text{ in } \Omega.$$

Here,$\tau > 0$ denotes a small constant.

For more information on the structure of the solution set for a stationary system of chemotaxis which is related to the problem (P) is referred to paper[8]. In [6], the authors considered problem (P) with $a = \epsilon = 1$ and $\lambda = \lambda_2$, the smallest non-zero eigenvalue of $-\Delta$ together with Neumann boundary conditions. The authors remarks that the ideas in [6] are not extendable when $\lambda > \lambda_2$ and that the monotonicity was essential to the method of their paper. For more results related to our problem, the reader is referred to papers [2], [3], [4] and [10], and references therein.

In our paper, we assume

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(H) $\qquad \lambda_n < \lambda < \lambda_{n+1}$ for some $n \geq 2$,

Then the result is

Theorem 1. *Let $f \in L^2(\Omega)$. Assume (H). Then there exist positive numbers ε_0 and a_0 such that for each $\varepsilon \in (0, \varepsilon_0)$ and $a \in (0, a_0)$, problem (P) possesses at least two solutions.*

2. Preliminaries

For each $p \in [1, \infty]$, we denote by $|\cdot|_p$ the norm of $L^p(\Omega)$. The norm $\|\cdot\|$ of $H^1(\Omega)$ is defined by $\|u\|^2 = |\bigtriangledown u|_2^2 + |u|_2^2$ for $u \in H^1(\Omega)$. We denote by $\|\cdot\|_*$ the norm of the dual space $H^{-1}(\Omega)$ of $H^1(\Omega)$. The inner product in $L^2(\Omega)$ is denoted by $\langle \cdot, \cdot \rangle$. We denote by $B_r(x)$ the ball in $H^1(\Omega)$ centered at $x \in H^1(\Omega)$ with radius $r > 0$. $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$ be the eigenvalues of the Laplacian on Ω with Neumann boundary condition. We also denote by φ_i an eigenfunction corresponding to the eigenvalue λ_i for each $i \geq 1$. We note that $\lambda_1 = 0$ and $\varphi_1 = c$ for some $c \in \mathbb{R}$. For $i \geq 2$, each eigenfucntion φ_i is sign changing. We define subspaces E_1 and E_2 of $H^1(\Omega)$ by

$$E_1 = \text{span}\{\varphi_1, \varphi_2, \cdots, \varphi_n\} \qquad \text{and} \qquad E_2 = \text{span}\{\varphi_1, \varphi_2, \cdots, \varphi_n\}^{\perp}$$

We also put $H = H^1(\Omega)$ and

$$H_m = \text{span } \{\varphi_1, \cdots, \varphi_m\} \qquad \text{for } m \geq 1.$$

¿From the definition of E_1, there exists $C_\infty > 0$ such that $|\varphi|_\infty \leq C_\infty |\varphi|_2$ for all $\varphi \in E_1$. Throughout the rest of this paper, we fix $f \in L^2(\Omega)$ with $f \geq 0$. Then for $\varepsilon > 0$, we define a functional $I_{\varepsilon f}$ by

$$I_{\varepsilon f}(u) = \frac{1}{2}|\bigtriangledown u|_2^2 - \frac{\lambda}{2}|u|_2^2 + a\int_\Omega e^u - \varepsilon\int_\Omega fu \qquad \text{for } u \in D(I)$$

where

$$D(I) = \left\{u \in H : \int_\Omega e^u < \infty\right\}.$$

We note that each critical point u_m of $I_{\varepsilon f}$ in H_m is a solution of problem

$$\langle -\Delta u - \lambda u + e^u - \varepsilon f, v\rangle = 0 \qquad \text{for all } v \in H_m.$$

We also put

$$J^+_{\varepsilon f}(u) = \frac{1}{2}|\bigtriangledown u|_2^2 - \frac{\lambda}{2}|u|_2^2 + a\int_\Omega e^{u^+} - \int_\Omega \varepsilon fu \qquad \text{for } u \in D(I)$$

for each $u \in H$. It is obvious from the definition that

$$I_{\varepsilon f}(u) \leq J^+_{\varepsilon f}(u) \qquad \text{for all } u \in D(I) \text{ and } \varepsilon > 0.$$

Lemma 1. *There exist $d > 0$, $a_0 > 0$ and $\varepsilon_0' > 0$ such that for each $\varepsilon \in (0, \varepsilon_0')$,*

$$\inf\left\{J^+_{\varepsilon f}(t\varphi) : t \geq 0\right\} \leq a|\Omega| - d$$

for all $\varphi \in E_1$.

Lemma 2. *Let* $(a, \varepsilon) \in (0, a_0,) \times (0, \varepsilon_0')$. *Then there exists a continuous mapping* $\tau : E_1 \diagdown \{0\} \longrightarrow \mathbb{R}^+$ *such that*

$$J^+_{\varepsilon f}(\tau(\varphi)\varphi) = a\,|\Omega| - d.$$

for all $\varphi \in E_1 \setminus \{0\}$.

Lemma 3. *There exists* $\varepsilon_0'' > 0$ *such that for each* $a > 0$ *and* $\varepsilon \in (0, \varepsilon_0'')$,

$$I_{\varepsilon f}(\varphi) \geq a|\Omega| - d/2 \qquad \text{for all } \varphi \in E_1^\perp \cap D(I)$$

Let $\psi : \mathbb{R}^n \longrightarrow E_1$ be an isomorphism. We denote by S^{n-1} the unit sphere of $\mathbb{R}^n$. We define a mapping $\alpha : S^{n-1} \longrightarrow E_1$ by

$$\alpha(z) = \tau(\psi(z))\psi(z) \qquad \text{for } z \in S^{n-1}. \tag{2.1}$$

3. A Priori Estimate

We put $\varepsilon_0 = \min\{\varepsilon_0', \varepsilon_0''\}$. Throughout the rest of this paper, we assume that $(a, \varepsilon) \in (0, a_0) \times (0, \varepsilon_0)$.

Lemma 4. *Let* $\{u_m\} \subset H$ *be a sequence such that for each* $m \geq 1$, $u_m \in H_m$ *is a critical point of* $I_{\varepsilon f}$ *in* H_m. *Then* $\left\{\left\|e^{u_m/2}\right\|\right\}$ *and* $\{\|u_m\|\}$ *are bounded.*

Lemma 5. *Let* $\{u_m\} \subset H$ *be a sequence such that for each* $m \geq 1$, $u_m \in H_m$ *is a critical point of* $I_{\varepsilon f}$ *in* H_m. *Suppose in addition that* $c = \lim_{m \longrightarrow \infty} I_{\varepsilon f}(u_m)$ *exists. Then there exists a subsequence* $\{u_{m_i}\}$ *of* $\{u_m\}$ *and a solution* $u \in H$ *of (P) such that* $I_{\varepsilon f}(u) = c$, $u_{m_i} \longrightarrow u \in H$ *strongly in* $H^1(\Omega)$ *and* $e^{u_{m_i}} \longrightarrow e^u$ *strongly in* $L^2(\Omega)$.

Lemma 6. *For each* $m \geq 1$, *functional* $I_{\varepsilon f}$ *satisfies Palais-Smale condition on* H_m, *i.e., for each sequence* $\{v_i\} \subset H_m$ *such that* $\lim_{i \longrightarrow \infty} \|P_m \bigtriangledown I_{\varepsilon f}(v_i)\| = 0$ *and* $\lim_{i \longrightarrow \infty} I_{\varepsilon f}(v_i) = d \in \mathbb{R}$, *there exists a convergent subsequence* $\{v_{i_k}\} \subset \{v_i\}$ *such that* $v = \lim_{k \longrightarrow \infty} v_{i_k}$ *is a critical point of* $I_{\varepsilon f}$ *in* H_m.

Lemma 7. (1) *There exists* $C_0 > 0$ *such that for each* $\beta > 0$,

$$I_{\epsilon f}(u - \beta) < I_{\epsilon f}(u) + C_0 \qquad \text{for all } u \in D(I):$$

(2) *For each bounded subset* $A \subset D(I)$,

$$\lim_{\beta \longrightarrow \infty} \sup_{u \in A} I_{\epsilon f}(u - \beta) = -\infty.$$

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