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THE VARIATIONAL THEORY OF A CIRCULAR ARCH WITH TORSIONAL SPRINGS AT BOTH EDGES

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ABSTRACT. The behavior of arches are nonlinear and sensitive to the buckling conditions. The existence of equilibrium states of elastic circular arches with rotational resistance at both edges under normal pressure is investigated. Variational formulation is derived using an energy method.

1. INTRODUCTION.

In this article, we study the existence of equilibrium states of buckled circular arches under uniform normal pressure. An arch is an elastic structural member restrained at the bases. The buckling of such a "classical" arch has been considered by Timoshenko, S.P. and Gere, J.M. [7], Vlasov,V.Z.[8] and Simitses, G.J. [5]. In general, the elastic equilibrium equations are derived from a small perturbation from the circular arch, and the buckling pressure is the first eigenvalue of the linearized system. It was found that the buckling pressure depends on the flexural rigidity of the arch, the opening angle, and whether the base is clamped or hinged. Many question still need to be addressed in uniqueness of the solution.

The variational formulation is set up using the energy method based on Hamilton's principal. This principal leads to a minimization of total energy which is used to prove the existence of the solution. The work on variational methods for nonlinear elliptic eigenvalue problem by F.E. Browder [2] serves the proof of the existence of solution.

2. VARIATIONAL FORMULATION WITH NON-ZERO TORSIONAL SPRING CONSTANT.

We consider the equilibria of a spring buckled arch under normal hydrostatic pressure p. Let $x(s)$ and $y(s)$ denote the coordinates of the point s on the cross section, and $\theta(s)$ the angle between the tangent to the cross section and x-axis, where s is arc-length along the cross section of a buckled arch. Then

$$x(s) = \int_{-\alpha}^{s} \cos\theta(s)ds \tag{1a}$$

$$y(s) = \int_{-\alpha}^{s} \sin\theta(s)ds, \tag{1b}$$

and, for a fixed angle α,

$$\theta : [-\alpha, \alpha] \longrightarrow R, \tag{2}$$

satisfying the constant base positions

$$\int_{-\alpha}^{\alpha} \sin\theta(s)ds = 0, \tag{3a}$$

$$\int_{-\alpha}^{\alpha} \cos\theta(s)ds = 2\sin\alpha. \tag{3b}$$

Let $\rho_+ \geq 0$, $\rho_- \geq 0$, and $p \in R$ be constants. The strain energy E is defined by

$$E = \int_{-\alpha}^{\alpha} (1-\theta_s(s))^2 ds, \tag{4}$$

and work done W by

$$\begin{aligned} W &= -p\left[\int_{-\alpha}^{\alpha} (xy_s - yx_s)ds - 2\alpha\right] \\ &= -p\left[\int_{-\alpha}^{\alpha} (x\sin\theta(s) - y\cos\theta(s))ds - 2\alpha\right], \end{aligned} \tag{5}$$

respectively. The work done is considered as the difference in area enclosed by the arch in its deformed and un-deformed states. The potential energy V then is

$$\begin{aligned} V(\theta) &= \rho_-(\theta(-\alpha)+\alpha)^2 + \rho_+(\theta(\alpha)-\alpha)^2 \\ &+ \int_{-\alpha}^{\alpha}\left[(1-\theta_s(s))^2 + p\int_{-\alpha}^{s}\sin(\theta(s)-\theta(\xi))d\xi\right]ds. \end{aligned} \tag{6}$$

Note that the first and second terms in equation (6) imply the energy to keep un-deformed arch at the both edges. If the arch is buckled with clamped bases, then boundary conditions are $\theta(-\alpha) = -\alpha$ and $\theta(\alpha) = \alpha$, which yield first two terms in equation (6) vanish. Moreover, $\rho_- = \rho_+ = 0$ mean the buckling in hinged bases. The transform $z(s) = \theta(s) - s$ yields

$$\int_{-\alpha}^{\alpha} \sin(z(s)+s)ds = 0 \tag{7a}$$

$$\int_{-\alpha}^{\alpha} \cos(z(s)+s)ds = 2\sin\alpha, \tag{7b}$$

and

$$\begin{aligned} V(z) &= \rho_- z(-\alpha)^2 + \rho_+ z(\alpha)^2 \\ &+ \int_{-\alpha}^{\alpha}\left[z_s(s)^2 + p\int_{-\alpha}^{s}\sin[(z(s)-z(\xi)) + (s-\xi)]d\xi\right]ds. \end{aligned} \tag{8}$$

3. Existence of solution.

Here assume that $0 \leq \rho_-$, $0 \leq \rho_+$, and $\rho_+ + \rho_- > 0$. Let us fix pressure p and angle α. We want to minimize the E

$$\begin{aligned} E(z) &= \rho_- z(-\alpha)^2 + \rho_+ z(\alpha)^2 \\ &+ \int_{-\alpha}^{\alpha}\left[z_s(s)^2 + p\int_{-\alpha}^{s}\sin[(z(s)-z(\xi)+(s-\xi))]d\xi\right]ds. \end{aligned} \tag{9}$$

subject to the constraints

$$\int_{-\alpha}^{\alpha} \sin(z(s)+s)ds = 0 \tag{10a}$$

$$\int_{-\alpha}^{\alpha} \cos(z(s)+s)ds = 2\sin\alpha. \tag{10b}$$

Let $H \equiv W^{1,2}$ and define

$$f_1(u) = \int_{-\alpha}^{\alpha} \cos(u(s)+s)ds \tag{11}$$

$$f_2(u) = \int_{-\alpha}^{\alpha} \sin(u(s)+s)ds, \tag{12}$$

for $u \in H$. Define j and Ψ by

$$j(v) = \int_{-\alpha}^{\alpha}\int_{-\alpha}^{s} \sin[(v(s)-v(\xi))+(s-\xi)]d\xi ds \tag{13}$$

$$\Psi(u,v) = \rho_- u(-\alpha)^2 + \rho_+ u(\alpha)^2 + \int_{-\alpha}^{\alpha} u'^2 ds + pj(v), \tag{14}$$

for $u, v \in H$, and note

$$E(u) = \Psi(u,u). \tag{15}$$

Lemma 3.1. *There exists a constant c such that*

$$||u||_\infty \le c|u|_{1,2} \tag{16}$$

for every $u \in H$.

Lemma 3.2. *The set $S = \{u \in H | f_1(u) = 2\sin\alpha, f_2(u) = 0\}$ is nonempty and weakly closed in H.*

Lemma 3.3. *The function $\Psi(\cdot,\cdot)$ is semi-convex function on $H \times H$.*

Lemma 3.4. *The $E(u) \to \infty$ as $|u|_{1,2} \to \infty$ on H.*

Lemma 3.5. *The $E : H \to R^1$ and $f_i : H \to R^1$, $i = 1,2$, are C^1 on H. Moreover,*

$$\begin{aligned} E'(u)h &= 2\rho_- u(-\alpha)h(-\alpha) + 2\rho_+ u(\alpha)h(\alpha) + \int_{-\alpha}^{\alpha} 2u'h' \\ &+ p\int_{-\alpha}^{\alpha}\int_{-\alpha}^{s} \cos[u(s)-u(\xi)+(s-\xi)](h(s)-h(\xi)) \end{aligned} \tag{17}$$

$$f_1'(u)h = \int_{-\alpha}^{\alpha} -\sin(u+s)h(s) \tag{18}$$

$$f_2'(u)h = \int_{-\alpha}^{\alpha} \cos(u+s)h(s), \tag{19}$$

for all $u, h \in H$.

Theorem 3.1. *There exists a minimizer* $u_0 \in H$ *of*

$$\begin{aligned} E(u) &= \rho_- u(-\alpha)^2 + \rho_+ u(\alpha)^2 \\ &+ \int_{-\alpha}^{\alpha} u'^2 ds + p \int_{-\alpha}^{\alpha}\int_{-\alpha}^{s} \sin[(u(s)-u(\xi)) + (s-\xi)] d\xi ds, \end{aligned} \tag{20}$$

with side conditions

$$\int_{-\alpha}^{\alpha} \sin(u(s)+s) ds = 0, \tag{21a}$$

$$\int_{-\alpha}^{\alpha} \cos(u(s)+s) ds = 2\sin\alpha. \tag{21b}$$

Moreover, for some (μ_1, μ_2) *in* R^2,

$$E'(u_0) = \mu_1 f_1'(u_0) + \mu_2 f_2'(u_0).$$

Theorem 3.2. *If* $u_0 \in W^{1,2}(-\alpha,\alpha) \cap S$ *satisfies*

$$E'(u_0)h = \mu_1 f_1'(u_0)h + \mu_2 f_2'(u_0)h \quad \textit{for all} \quad h \in W^{1,2}(-\alpha,\alpha) \tag{22}$$

for some constants μ_1 *and* μ_2, *then* $u_0 \in C^2[-\alpha,\alpha]$, *and*

$$\begin{aligned} u_0'' &- p\int_{-\alpha}^{s} \cos[u_0(s)-u_0(\xi)+(s-\xi)]d\xi + p\sin\alpha\cos(u_0(s)+s) \\ &= \frac{\mu_1}{2}\sin(u_0(s)+s) - \frac{\mu_2}{2}\cos(u_0(s)+s) \end{aligned} \tag{23}$$

$$u_0'(-\alpha) - \rho_- u_0(-\alpha) = 0 \tag{24a}$$

$$u_0'(\alpha) + \rho_+ u_0(\alpha) = 0. \tag{24b}$$

Remark : In our discussion stable state first was investigated for each pressure and opening angle, and then the Robin boundary conditions were searched. The development of argument was different from many previous work.

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