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POSITIVE SOLUTIONS OF SEMILINEAR INDEFINITE WEIGHT PROBLEMS INVOLVING CRITICAL SOBOLEV EXPONENTS

BONGSOO KO

1. Introduction

For a given open bounded smooth domain Ω in $\mathbf{R}^N$ with $N \geq 3$, there are the existence and the nonexistence results of positive solutions u satisfying the following semilinear elliptic boundary value problems : $-\Delta u = f(x,u) + u|u|^p$ in Ω, $u = 0$ on $\partial\Omega$, where $p = \frac{4}{N-2}$ the critical Sobolev exponent, $f(x,0) = 0$, and $f(x,u)$ is a lower-order permutation of $u|u|^p$ ([2],[5]).

As some different studies from that, we discuss the existence of positive solutions of the following indefinite weight semilinear elliptic boundary value problems:

$$(I_{\lambda\alpha}) \quad \begin{cases} -\Delta u = \lambda g(x) u (1 + |u|^p) \text{ in } \Omega, \\ (1-\alpha)\dfrac{\partial u}{\partial n} + \alpha u = 0 \text{ on } \partial\Omega, \end{cases}$$

where λ and α are real parameters, Ω is also an open bounded domain in $\mathbb{R}^N$, $N \geq 3$, with the smooth boundary $\partial\Omega$. To distinguish the previous problem from ours, we assume that the function g is Lipschitz continuous on $\bar{\Omega}$ and changes sign in Ω.

The existence of positive solutions of the case $0 < p < \frac{4}{N-2}$ has been proved with the parameter either $\alpha \in (0,1)$ or $\int_\Omega g(x)dx \neq 0$ and $\alpha \in (\alpha_0, 0]$ for some constant $\alpha_0 < 0$ ([1]). The main method to prove that result is the well-known constrained minimization method of the functional

$$E_\lambda(u) = \int_\Omega |\nabla u|^2 - \lambda \int_\Omega g u^2 + \frac{\alpha}{(1-\alpha)} \int_{\partial\Omega} u^2 dS_x, \quad (\alpha \neq 1)$$

on the constrained set

$$\{u \in W^{1,2}(\Omega) \,:\, \lambda \int_\Omega g|u|^{p+2} = 1\}.$$

The other case $\alpha = 1$ can be proved by the similar method on the Sobolev space $W_0^{1,2}(\Omega)$. In this paper, we assume that if $\alpha = 1$, the underlying space is $W_0^{1,2}(\Omega)$ automatically.

If $p = \frac{4}{N-2}$, the above constrained set may not be weakly closed and the functional E_λ does not satisfy the Palais-Smale condition. We may feel to have serious difficulties when trying to find critical points by the above standard variational methods. However, we can find a different constrained variational methods from the above one in this paper to get positive solutions of the problem $(I_{\lambda\alpha})$.

In Section 2, we show that a minimizing sequence of the following functional $J_{\lambda\alpha}$ which is induced by the weighted problem $(I_{\lambda\alpha})$: For $\alpha \in [0, 1)$,

$$J_{\lambda\alpha}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_\Omega gu^2 - \frac{\lambda}{p+2}\int_\Omega g|u|^{p+2} + \frac{\alpha}{2(1-\alpha)}\int_{\partial\Omega} u^2 dS_x$$

on the Nehari manifold:

$$M_{\lambda\alpha} = \left\{u \in W^{1,2}(\Omega) \,:\, u \neq 0, < J'_{\lambda\alpha}(u), u >= 0\right\},$$

where

$$< J'_{\lambda\alpha}(u), u >= \int_\Omega |\nabla u|^2 - \lambda\int_\Omega gu^2 + \lambda\int_\Omega g|u|^{p+2} + \frac{\alpha}{1-\alpha}\int_{\partial\Omega} u^2 dS_x,$$

converges to a positive function in $M_{\lambda\alpha}$ which is a classical positive solution of the problem $(I_{\lambda\alpha})$ if $\lambda_\alpha^- < \lambda < \lambda_\alpha^+$, and λ is near to either λ_α^- or λ_α^+, where λ_α^- and λ_α^+ are the principal eigenvalues of the following problem ([3]):

$$(L_\alpha) \begin{cases} -\Delta u = \lambda g(x) u \text{ in } \Omega, \\ (1-\alpha)\dfrac{\partial u}{\partial n} + \alpha u = 0 \text{ on } \partial\Omega. \end{cases}$$

Furthermore, we estimate the length of the intervals about λ in which the existence of positive solutions is guaranteed. To get the similar result for $\alpha = 1$, we can use the following functional

$$J_{\lambda 1}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_\Omega gu^2 - \frac{\lambda}{p+2}\int_\Omega g|u|^{p+2}.$$

on the Nehari manifold $M_{\lambda\alpha}$ in $W_0^{1,2}(\Omega)$.

Since we may expect the blowing up behaviors of positive solutions on the parameter λ on the interval $[\lambda_\alpha^-, \lambda_\alpha^+]$, we will use some parts of the following conditions on the function g to get the existence of positive solutions of $(I_{\lambda\alpha})$ on the interval $[\lambda_\alpha^-, \lambda_\alpha^+] \setminus \{0\}$:

- **(C1):** $g(x) \in C^2(\bar{\Omega})$,
- **(C2):** $\Omega^+ = \{x \in \Omega \,:\, g(x) > 0\}$ and $\Omega^- = \{x \in \bar{\Omega} \,:\, g(x) < 0\}$ are nonempty, and $\bar{\Omega}^+ \cap \bar{\Omega}^- = \Gamma \subset \Omega$,
- **(C3):** $\nabla g(x) \neq 0$ on Γ, and
- **(C4):** If $g(x) > 0$ on $\partial\Omega$, there is a positive number ϵ so that for the subset $S = \{x \in \Omega \,:\, d(x, \partial\Omega) < \epsilon\}$, $g(x) > 0$ for all $x \in S$, and if x_0 is any critical point in S of g, there exists $\rho = \rho(x_0) > N-2$, such that, in a small neighborhood of x_0,

$$g(x) = g(x_0) + \sum_{i=1}^N \rho_i |x_i - x_i^0|^\alpha + o(|x - x^0|^\alpha),$$

where $\rho_i = \rho_i(x^0) \neq 0$, $\sum_{i=1}^N \rho_i \neq 0$, $x^0 = (x_1^0, \cdots, x_N^0)$, and $x = (x_1, \cdots, x_N)$.

First, we use the conditions (C1) through (C4) to prove the existence with the estimate of positive solutions on $\bar{\Omega}$, and then using the upper and lower solution method and solving the associated linear variational equation of $(I_{\lambda\alpha})$ at $u = 0$, we can eliminate the last two conditions (C3) and (C4) in proving the existence of positive solutions of $(I_{\lambda\alpha})$ on the intervals $(\lambda_\alpha^-, 0)$ and $(0, \lambda_\alpha^+)$. This existence is one of different properties for the indefinite weighted problem from the well-known following results: If Ω is a ball, $g = 1, N = 3$ and $\alpha = 1$, then $(I_{\lambda\alpha})$ has a positive solution if and only if $\frac{1}{4}\lambda_1 < \lambda < \lambda_1$, where λ_1 is the principal eigenvalue of $-\Delta$ with the homogeneous Dirichlet boundary condition ([5]).

2. The Main Results

Lemma 2.1. *([1]) Suppose $\alpha \in (0, 1)$ or that $\int_\Omega g dx \neq 0$ and $\alpha \in (\alpha_0, 0]$ so that (L_α) has principal eigenvalues λ_α^- and λ_α^+. For any $\lambda \in (\lambda_\alpha^-, \lambda_\alpha^+)$,*

$$\|u\|_{\lambda\alpha} = \left\{ \int_\Omega [|\nabla u|^2 - \lambda g u^2] dx + \frac{\alpha}{1-\alpha} \int_{\partial\Omega} u^2 dS_x \right\}^{\frac{1}{2}}$$

defines a norm in $W^{1,2}(\Omega)$ which is equivalent to the usual norm for $W^{1,2}(\Omega)$.

Lemma 2.2. *Let $\lambda \in (\lambda_\alpha^-, \lambda_\alpha^+)$, $\lambda \neq 0$ and let*

$$M_{\lambda\alpha} = \left\{ u \in W^{1,2}(\Omega) : u \neq 0, < J'_{\lambda\alpha}(u), u >= 0 \right\},$$

Then $M_{\lambda\alpha}$ is a nonempty subset of $W^{1,2}(\Omega)$.

Lemma 2.3. *Let $\alpha \in (0, 1]$ or $\int_\Omega g dx \neq 0$ if $\alpha = 0$. There are three real numbers $\lambda_\alpha^- < \lambda_1 < \lambda_2 = 0 < \lambda_3 < \lambda_\alpha^+$ such that for any $\lambda \in (\lambda_\alpha^-, \lambda_1) \cup (\lambda_3, \lambda_\alpha^+)$, if $\{u_n\}$ be a minimizing sequence of $J_{\lambda\alpha}$ on $M_{\lambda\alpha}$. Then*

$$\liminf_{n\to\infty} \left| \int_\Omega g u_n^2 \right| > 0.$$

Proposition 2.4 ([4], pp. 6)**.** *Let J be a C^1-functional on a Hilbert space X and let M be a closed subset of X verifying the following property:*

For any $u \in M$ with $J'(u) \neq 0$, there exists, for a small enough $\varepsilon > 0$, a Fréchet differentiable function $s_u : B_\varepsilon(X) \longrightarrow \mathbb{R}^1$ such that, by setting $t_u(\delta) = s_u\left(\delta \frac{J'(u)}{||J'(u)||}\right)$ for $0 \le \delta \le \varepsilon$, we have

$$t_u(0) = 1 \text{ and } t_u(\delta)\left(u - \delta \frac{J'(u)}{||J'(u)||}\right) \in M.$$

If J is bounded below on M, then for any minimizing sequence $\{v_n\}$ in M for J, there exists another minimizing sequence $\{u_n\}$ in M of J such that

$$J(u_n) \le J(v_n), \lim_{n\to\infty} ||u_n - v_n|| = 0$$

and

$$||J'(u_n)|| \le \frac{1}{n}\left(1 + ||u_n|||t'_{u_n}(0)|\right) + |t'_{u_n}(0)|| < J'(u_n), u_n > |,$$

where $<\ ,\ >$ is the inner product in X.

Lemma 2.5. *Let* $\alpha \in (0,1]$ *or that* $\int_\Omega g dx \neq 0$ *for* $\alpha = 0$*. Given* $\lambda \in (\lambda_\alpha^-, \lambda_\alpha^+), \lambda \neq 0$*,* $J_{\lambda\alpha}$ *is bounded below on* $M_{\lambda\alpha}$ *and there exists a minimizing sequence* $\{u_n\}$ *of* $J_{\lambda\alpha}$ *on* $M_{\lambda\alpha}$ *so that*

$$\lim_{n\to\infty} ||J'_{\lambda\alpha}(u_n)||_{\lambda\alpha} = 0$$

and

$$\lim_{n\to\infty} J_{\lambda\alpha}(u_n) = \inf J_{\lambda\alpha}(M_{\lambda\alpha})$$

Theorem 2.6. *Let* $\alpha \in (0,1]$ *or that* $\int_\Omega g dx \neq 0$ *for* $\alpha = 0$*. There are three real numbers* $\lambda_\alpha^- < \lambda_1 < \lambda_2 = 0 < \lambda_3 < \lambda_\alpha^+$ *such that for any* $\lambda \in (\lambda_\alpha^-, \lambda_1) \cup (\lambda_3, \lambda_\alpha^+)$*, the problem* $(I_{\lambda\alpha})$ *has a positive solution.*

Theorem 2.7. *Let* $\alpha \in (0,1]$ *or that* $\int_\Omega g dx \neq 0$ *for* $\alpha = 0$*. If the function* g *satisfies the conditions (C1) and (C2), the problem* $(I_{\lambda\alpha})$ *has a positive solution for all* $\lambda \in (\lambda_\alpha^-, \lambda_\alpha^+) \setminus \{0\}$*.*

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Department of Mathematics Education, Educational Research Institute Cheju National University