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ESTIMATES OF STRESS IN COMPOSITE MATERIALS

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1. Introduction

Composite materials consist of a matrix of a homogeneous material and inclusions of various size and shape. Inclusions may be holes or hard ones. Frequently in composites, the inclusions are very closely spaced and may even touch, see [7], and it is important to know that the gradient of solutions, the stress field, can be arbitrarily large, since large stress can be a cause of development of cracks. Recently very precise estimates of the stress in special cases have been obtained in [4, 5] on which we discuss in this article. We will also discuss about some interesting questions for further research.

To describe the first situation considered in in [4, 5], let B_1 and B_2 are two circular inclusions contained in a matrix which we assume to be the free space $\mathbb{R}^2$. For $i = 1, 2$, we suppose that the conductivity k_i of the inclusion B_i is a constant different from the constant conductivity of the matrix, which is assumed to be 1 for convenience. The conductivity k_i of the inclusion may be 0 or ∞. The zero conductivity indicates that the inclusion is a hole or an insulated inclusion while the infinite conductivity indicates a hard or perfect conductor.

The conductivity problem we consider in this paper is the following transmission problem for a given entire harmonic function H:

$$\begin{cases} \nabla\cdot\Big(1+\displaystyle\sum_{i=1,2}(k_i-1)\chi(B_i)\Big)\nabla u = 0 \quad \text{in } \mathbb{R}^2, \\ u(X)-H(X) = O(|X|^{-1}) \quad \text{as } |X|\to\infty. \end{cases} \tag{1.1}$$

The gradient ∇u of the solution u to (1.1) is the stress field and represents the perturbation of the field ∇H in the presence of inclusions B_1 and B_2. For applications to the theory of composite materials, it is particularly important to consider the case when ∇H is a uniform field, *i.e.*, $H(X) = A\cdot X$ for some constant vector A. The equation (1.1) can be rewritten in the following form to emphasize the

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transmission condition on ∂B_i, $i = 1, 2$:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \setminus (\partial B_1 \cup \partial B_2), \\ u|_+ = u|_- & \text{on } \partial B_i,\ i = 1, 2, \\ \left.\dfrac{\partial u}{\partial \nu}\right|_+ = k_i \left.\dfrac{\partial u}{\partial \nu}\right|_- & \text{on } \partial B_i,\ i = 1, 2, \\ u(X) - H(X) = O(|X|^{-1}) & \text{as } |X| \to \infty. \end{cases}$$

Here and throughout this paper the subscript $\pm$ indicates the limit from outside and inside the domain, respectively. If $k_i = 0$, then the transmission condition should be replaced with $\frac{\partial u}{\partial \nu}|_+ = 0$ on ∂B_i. If $k_i = \infty$, then it should be replaced with $u =$ constant on B_i.

In this situation we are interested in the behavior of the gradient of the solution to the equation (1.1) as the distance between B_1 and B_2 goes to zero.

Another situation considered in [4, 5] is when the inclusion is very close to the boundary. Suppose that Ω, which is a disk of radius ρ, contains an inclusion B, which is a disk of radius r. Suppose that the conductivity of Ω is 1 and that of B is $k \neq 1$. We consider the following Dirichlet problem: for a given $f \in C^{1,\alpha}(\partial\Omega)$, $\alpha > 0$,

$$\begin{cases} \nabla \cdot (1 + (k-1)\chi(B))\nabla u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

In this situation we are interested in the estimates of ∇u when B is very close to the boundary of Ω. The same question for the Neuman problem is also considered.

There have been some important works on the estimates of the stress field in the presence of inclusions. For finite and strictly positive conductivities, it was shown by Bonnetier and Vogelius in [8] that the gradient of u remains bounded for circular touching inclusions of comparable radii. Li and Vogelius showed in [11] that ∇u is bounded independently of the distance between the inclusions B_1 and B_2, provided that the conductivities stay away from 0 and $+\infty$. In fact, the result of [11] is much more general: it holds for arbitrary number of inclusions with arbitrary shape. This result has been recently extended to elliptic systems by Li and Nirenberg in [10]. On the other hand, for two identical perfectly conducting circular inclusions (with $k_1 = k_2 = +\infty$) which are ϵ apart, it has been shown in [6] (see also [12] and [9]) that the gradient generally becomes unbounded as the distance ϵ approaches zero. The rate at which this gradient becomes unbounded has actually been calculated in [6], for a special solution. For this special solution, the rate turns out to be $\epsilon^{-1/2}$.

2. Optimal estimates of the stress

We summarize the main results of [4, 5].

To state the first main result, let us fix some notations. For $i = 1, 2$, let $B_i = B(Z_i, r_i)$, the disk centered at Z_i and of radius r_i. Let R_i, $i = 1, 2$, be the reflection with respect to ∂B_i, *i.e.*,

$$R_i(X) := \frac{r_i^2 (X - Z_i)}{|X - Z_i|^2} + Z_i, \quad i = 1, 2.$$

It is easy to see that the combined reflection R_1R_2 and R_2R_1 have unique fixed points. Let I be the line segment between two fixed points. Let X_j, $j = 1, 2$, be the point on ∂B_j closest to the other disk. We also let

$$r_{\min} := \min(r_1, r_2), \quad r_{\max} := \max(r_1, r_2), \quad r_* := \sqrt{(2r_1r_2)/(r_1 + r_2)}.$$

Finally let

$$\lambda_i := \frac{k_i + 1}{2(k_i - 1)}, \quad i = 1, 2 \quad \text{and} \quad \tau := \frac{1}{4\lambda_1\lambda_2}.$$

The following result was obtained in [4, 5].

Theorem 2.1. *Let $\epsilon :=$ dist(B_1, B_2) and let $\nu^{(j)}$ and $T^{(j)}$, $j = 1, 2$, be the unit normal and tangential vector fields to ∂B_j, respectively. Let u be the solution of (1.1).*

(i) *If ϵ is sufficiently small, there is a constant C_1 independent of k_1, k_2, r_1, r_2, and ϵ such that*

$$\frac{C_1 \inf_{X \in I} |\langle \nabla H(X), \nu^{(j)}(X_j)\rangle|}{1 - \tau + (r_*/r_{\min})\sqrt{\epsilon}} \leq |\nabla u|_+(X_j)|, \quad j = 1, 2, \tag{2.1}$$

provided that $k_1,\ k_2 > 1$, and

$$\frac{C_1 \inf_{X \in I} |\langle \nabla H(X), T^{(j)}(X_j)\rangle|}{1 - \tau + (r_*/r_{\min})\sqrt{\epsilon}} \leq |\nabla u|_+(X_j)|, \quad j = 1, 2, \tag{2.2}$$

provided that $k_1,\ k_2 < 1$.

(ii) *Let Ω be a bounded set containing B_1 and B_2. Then there is a constant C_2 independent of k_1, k_2, r_1, r_2, ϵ, and Ω such that*

$$\|\nabla u\|_{L^\infty(\Omega)} \leq \frac{C_2 \|\nabla H\|_{L^\infty(\Omega)}}{1 - |\tau| + (r_*/r_{\max})\sqrt{\epsilon}}. \tag{2.3}$$

Theorem 2.1 quantifies the behavior of ∇u in terms of the conductivities of the inclusions, their radii, and the distance between them, and reveals many interesting features of the stress. For example, if k_1 and k_2 degenerate to $+\infty$ or zero, then $\tau = 1$ and hence (2.1) and (2.3) read

$$\frac{C_1 \inf_{X \in I} |\langle \nabla H(X), \nu^{(j)}(X_j)\rangle|}{(r_*/r_{\min})\sqrt{\epsilon}} \leq |\nabla u|_+(X_j)|, \quad \|\nabla u\|_{L^\infty(\Omega)} \leq \frac{C_2 \|\nabla H\|_{L^\infty(\Omega)}}{(r_*/r_{\max})\sqrt{\epsilon}}, \tag{2.4}$$

Note that if $H(X) = A \cdot X$ for some constant vector A, which is the most interesting case, then

$$\langle \nabla H(X), \nu^{(j)}(X_j)\rangle = \langle A, \nu^{(j)}(X_j)\rangle,$$

and hence it is not vanishing if the vector A is parallel to $\nu^{(j)}(X_j)$. Thus ∇u blows up at the rate of $\epsilon^{-1/2}$ as the inclusions get closer. It further shows that the gradient blows up at X_j which is the point on ∂B_j closest to the other disk.

Observe that if $r_1 = r_2 = r$, then $r_*/r_{\min} = r_*/r_{\max} = 1/\sqrt{r}$. Thus if the radius of the inclusions are of the same order as that of the distance between them, then ∇u does not blow up.

To state the second result of [4, 5], let X_1 be the point on ∂B closest to $\partial\Omega$ and X_2 be the point on $\partial\Omega$ closest to ∂B, and let R_B and R_Ω are reflections with respect to ∂B and $\partial\Omega$, respectively. Let P_1 and P_2 be fixed points of $R_B R_\Omega$ and $R_\Omega R_B$, respectively, and let J_1 be the line segment between P_1 and X_1 and J_2 that between P_2 and X_2. Let $\mathcal{D}_\Omega(f)$ and $\mathcal{S}_\Omega(g)$ denote the double and single layer potentials:

$$\mathcal{S}_\Omega\phi(X) = \frac{1}{2\pi}\int_{\partial\Omega} \ln|X-Y|\phi(Y)\,d\sigma(Y), \quad X\in\mathbb{R}^2,$$

$$\mathcal{D}_\Omega\phi(X) = \frac{1}{2\pi}\int_{\partial\Omega} \frac{\langle Y-X,\nu_Y\rangle}{|X-Y|^2}\phi(Y)\,d\sigma(Y), \quad X\in\mathbb{R}^2\setminus\partial\Omega.$$

For the Dirichlet problem (1.2) we have the following theorem.

Theorem 2.2. *Let*

$$\epsilon := dist(B,\partial\Omega), \quad \sigma := \frac{k-1}{k+1}, \quad r^* := \sqrt{\frac{\rho-r}{\rho r}},$$

and let u be the solution to (1.2).

(i) *If $k>1$, then there exist constants C_1 independent of k, r, ϵ, and f such that for ϵ small enough,*

$$\frac{C_1 \inf_{X\in J_1} |\langle \nabla\mathcal{D}_\Omega(f)(X), \nu_B(X_1)\rangle|}{1-\sigma+4r^*\sqrt{\epsilon}} \le |\nabla u|_+(X_1)|, \tag{2.5}$$

and

$$\frac{C_1 \inf_{X\in J_2} |\langle \nabla\mathcal{D}_\Omega(f)(X), \nu_\Omega(X_2)\rangle|}{1-\sigma+4r^*\sqrt{\epsilon}} \le |\nabla u|_-(X_2)|. \tag{2.6}$$

Here ν_B and ν_Ω denote the outward unit normal to ∂B and $\partial\Omega$.

(ii) *For any $k\neq 1$, there exists a constant C_2 independent of k, r, and ϵ such that for ϵ small enough,*

$$\|\nabla u\|_{L^\infty(\Omega)} \le \frac{C_2\|f\|_{C^{1,\alpha}(\partial\Omega)}}{1-|\sigma|+r^*\sqrt{\epsilon}}. \tag{2.7}$$

If Z is the center of Ω and if $f(X)=A\cdot X$ for some constant vector A, then $\mathcal{D}_\Omega(f)(X)=\frac{1}{2}A\cdot X$ for $X\in\Omega$ and $\mathcal{D}_\Omega(f)(X) = -\frac{\rho^2 A\cdot X}{2|X-Z|^2}$ for $X\in\mathbb{R}^2\setminus\overline{\Omega}$, and hence we can achieve

$$\langle \nabla\mathcal{D}_\Omega(f)(X), \nu_B(X_1)\rangle \neq 0 \text{ and } \langle \nabla\mathcal{D}_\Omega(f)(X), \nu_\Omega(X_2)\rangle \neq 0 \quad \text{for any } X,$$

by choosing A appropriately. Theorem 2.2 shows that in the case of the Dirichlet problem, if the inclusion is a perfect conductor ($k=\infty$ and hence $\sigma=1$), then

$$\frac{A}{r^*\sqrt{\epsilon}} \le \|\nabla u\|_{L^\infty(\Omega)} \le \frac{B}{r^*\sqrt{\epsilon}},$$

for some constants A and B. Thus ∇u blows up at the rate of $\epsilon^{-1/2}$ as long as the magnitude of r is much larger than that of ϵ. It also shows that the gradient blows up at the points X_1 and X_2. If r is the same order as ϵ, then $r^*\approx\frac{1}{\sqrt{\epsilon}}$ and hence ∇u does not blow up. In fact, it stays bounded and an asymptotic expansion of the solution as $\epsilon\to 0$ can be derived. See [1] for this (and [2, 3]).

A similar theorem is obtained for the Neuman problem: for a given $g \in C^\alpha(\partial\Omega)$ with $\int_{\partial\Omega} g = 0$

$$\begin{cases} \nabla \cdot (1 + (k-1)\chi(B))\nabla u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega \ . \end{cases} \tag{2.8}$$

Theorem 2.3. *Let ϵ, σ, r^* be as in Theorem 2.2.*

(i) *If $k < 1$, then there exist constants C_1 independent of k, r, ϵ, and g such that for ϵ small enough,*

$$\frac{C_1 \inf_{X \in J_1} |\langle \nabla \mathcal{S}_\Omega(g)(X), T_B(X_1)\rangle|}{1 + \sigma + 4r^*\sqrt{\epsilon}} \le |\nabla u|_+(X_1)|, \tag{2.9}$$

and

$$\frac{C_1 \inf_{X \in J_2} |\langle \nabla \mathcal{S}_\Omega(g)(X), T_\Omega(X_2)\rangle|}{1 + \sigma + 4r^*\sqrt{\epsilon}} \le |\nabla u|_-(X_2)|. \tag{2.10}$$

Here T_B and T_Ω denote the positively oriented unit tangent vector field on ∂B and $\partial\Omega$, respectively.

(ii) *For any $k \neq 1$, there exists a constant C_2 independent of k, r, and ϵ such that for ϵ small enough,*

$$\|\nabla u\|_{L^\infty(\Omega)} \le \frac{C_2 \|g\|_{C^\alpha(\partial\Omega)}}{1 - |\sigma| + r^*\sqrt{\epsilon}}. \tag{2.11}$$

Observe that if $g := A \cdot \nu$ on $\partial\Omega$, then $\mathcal{S}_\Omega(g) = -\frac{1}{2}A \cdot X + \text{constant}$. Theorem 2.3 shows that in the case of the Neuman problem, ∇u blows up for an insulator ($k = 0$).

3. Some problems

We now discuss some related problems.

Dependency of blow-up rate on the shape. It is shown that in general the gradient of the solution blows up at the rate of $\epsilon^{-1/2}$ for circular inclusions. It is not clear whether the rate $1/2$ is solely for the circular inclusions or for two dimensional inclusions of general shape. For example, if two planar sides of inclusions are very close to each other, what will be the blow up rate. It is quite interesting and challenging to clarify the dependency of blow-up rate on the shape

Estimates of stress in three dimensions. It would be interesting to derive estimates of the stress for three dimensional composites, especially for composite with spherical inclusions. At this moment it is not at all clear whether the gradient of solution will blow up as two inclusions get closer. If it blows up, what will be the blow-up rate?

Estimates of stress for elastic composites. It is very important to have the same kinds of estimates of the stress for the elastic composites when the shear modulus (or the Young modulus) of the inclusions become either infinity or zero.

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