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STRUCTURAL STABILITY OF ORBITAL INVERSE SHADOWING VECTOR FIELDS

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ABSTRACT. In this paper we give a characterization of the structurally stable vector fields via the notion of orbital inverse shadowing. More precisely it is proved that the C^1 interior of the set of C^1 vector fields with the orbital inverse shadowing property coincides with the set of structurally stable vector fields on a compact smooth manifold. This fact improves the result obtained by K. Moriyasu, K. Sakai and N. Sumi in [13].

1. Introduction

Structurally stable systems (both diffeomorphisms and flows) were the main objects of interest in the global qualitative theory of dynamical systems in the last 30 years. Now we know that structural stability is equivalent to Axiom A combined with the strong transversality condition.

One of the most important properties of a structurally stable system is the shadowing property (also known as the pseudo orbit tracing property). The shadowing property is the key of the analysis of such diffeomorphisms or flows.

Long time ago, various approaches were applied to show that a structurally stable diffeomorphism has the shadowing property. But the fact that a structurally stable flow has the shadowing property was proved by Pilyugin recently (see [15]).

The main difficulty of the shadowing problem for a structurally stable flow is created by the following fact specific for flows. Let p be a nonwandering point of a structurally stable flow. Then the trajectory of p is hyperbolic. Denote by $S(p)$ and $U(p)$ the corresponding "stable" and "unstable" subspaces of the hyperbolic structure. If p_1 is a rest point, and p_2 belongs to a nonsingular nonwandering trajectory, then

$$\dim(S(p_1) + U(p_1)) \neq \dim(S(p_2) + U(p_2)).$$

Hence the hyperbolic structures near rest points and near nonsingular nonwandering trajectories are qualitatively different. Consequently the standard shadowing approaches do not work. In fact, it was proved by K. Sakai [18] that the C^1 interior of the set of diffeomorphisms with the shadowing property coincides with the set of structurally stable diffeomorphisms. However it is still open problem whether the

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above results can be applied to the case of flows; i.e. *is a flow in the C^1 interior of the set of flows (or vector fields) with the shadowing property structurally stable?* For the case of flows, the only known result in this direction is that the C^1 interior of the set of topologically stable flows coincides with the set of structurally stable flows (see [14]).

The concept of inverse shadowing for homeomorphisms "dual" to the shadowing was established by Corless and Pilyugin [3], and Kloeden *et al* [8, 9] redefined this property using the concept of a method. Generally speaking, a homeomorphism has the inverse shadowing property with respect to a class of methods if any trajectory can be uniformly approximated with given accuracy by a δ-pseudotrajectory generated by a method from the chosen class if $\delta > 0$ is sufficiently small. An appropriate choice of the class of admissible pseudotrajectories is crucial here (see [4, 10, 16]).

It was shown by S. Pilyugin [16] that every structurally stable diffeomorphism on a compact smooth manifold has the inverse shadowing property with respect to the class of continuous methods. Recently K. Lee, Z. Lee [11]introduced the notion of inverse shadowing for flows and showed that every expansive flow with shadowing has the inverse shadowing proeprty. Recently Y. Han and K. Lee [6] proved that every structurally stable flow on a compact smooth manifold has the inverse shadowing property with respect to the class of continuous methods.

In this paper we introduce the notion of orbital inverse shadowing for flows and prove that the C^1 interior of the set of vector fields with the orbital inverse shadowing property (or inverse shadowing property) coincides with the set of vector fields satisfying Axiom A and the strong transversality condition.

2. Preliminaries

Let M be a compact smooth n-dimensional manifold with a Riemannian metric d. We denote by T_xM the tangent space of M at x and by TM the tangent bundle of M. Consider a C^1 vector field X on M and the system of differential equations

$$\dot{x} = X(x). \tag{1}$$

For two vector fields X and Y on M, we define

$$d_{C^0}(X, Y) = \max_{p \in M} |X(p) - Y(p)|.$$

In this setting, $DX(p)$ is considered as the derivative of the mapping, and for $p \in M$ we define

$$\|DX(p) - DY(p)\| = \max_{v \in T_pM, |v|=1} |DX(p)v - DY(p)v|.$$

Define the C^1 distance d_{C^1} between two vector fields X and Y on M by

$$d_{C^1}(X, Y) = d_{C^0}(X, Y) + \max_{p \in M} \|DX(p) - DY(p)\|.$$

We denote by $\mathcal{X}^1(M)$ the space of C^1 vector fields on M with the topology induced by d_{C^1}.

Let $\Phi : \mathbb{R} \times M \longrightarrow M$ be the solution flow of system (1). We denote by $D\Phi(t, x)$ be the corresponding variational flow.

For $\delta, T > 0$ we say that a mapping

$$\phi : \mathbb{R} \longrightarrow M$$

is a (δ, T)-*pseudotrajectory* of system (1) if, for any $t \in \mathbb{R}$,

$$d(\Phi(s, \phi(t)), \phi(s + t)) < \delta,$$

for $|s| \leq T$.

For $\delta, T > 0$ we say that a mapping

$$\Psi : \mathbb{R} \times M \longrightarrow M$$

is a (δ, T)-*method* for system (1) if, for any $x \in M$, the map $\Psi_x : \mathbb{R} \longrightarrow M$ defined by

$$\Psi_x(t) = \Psi(t, x), \; t \in \mathbb{R},$$

is a (δ, T)-pseudotrajectory of system (1). Ψ is said to be *complete* if $\Psi(0, x) = x$ for $x \in M$.

Note that a (δ, T)-method for system (1) can be considered as a family of (δ, T)-pseudotrajectories of system (1).

A method Ψ for system (1) is said to be continuous if the map

$$\tilde{\Psi} : M \longrightarrow M^{\mathbb{R}}$$

given by

$$\tilde{\Psi}(x)(t) = \Psi(t, x), \; x \in M \text{ and } t \in \mathbb{R},$$

is continuous under the compact open topology on $M^{\mathbb{R}}$, where $M^{\mathbb{R}}$ denotes the set of all functions from $\mathbb{R}$ into M.

The set of all complete (δ, T)-methods [resp. complete continuous (δ, T)-methods] for system (1) will be denoted by $\mathcal{T}_a(\delta, T, \Phi)$ [resp. $\mathcal{T}_c(\delta, T, \Phi)$]. It is clear that if $Y \in \mathcal{X}^1(M)$ is another vector field which is sufficiently close to X in the C^0 topology then the system

$$\dot{x} = Y(x)$$

induces a complete continuous method for system (1).

Let $\mathcal{T}_h(\delta, T, \Phi)$[resp. $\mathcal{T}_d(\delta, T, \Phi)$] be the set of all complete continuous (δ, T)-methods for system (1) which are induced by C^1 vector fields Y with $d_{C^0}(X, Y) < \delta$ [resp. $d_{C^1}(X, Y) < \delta$].

We say that the system (1) has the *inverse shadowing property* with respect to the class $\mathcal{T}_\alpha$ ($\alpha = a, c, h, d$) if for any $\varepsilon > 0$ and $T > 0$ there exists $\delta > 0$ such that for any (δ, T)-method $\Psi \in \mathcal{T}_\alpha(\delta, T, \Phi)$ and any point $x \in M$ there are $y \in M$ and $\alpha_y \in Rep$ for which

$$d(\Phi(t, x), \; \Psi(\alpha_y(t), y)) < \varepsilon, \; t \in \mathbb{R},$$

where, *Rep* denotes the set of increasing homeomorphisms α mapping $\mathbb{R}$ onto $\mathbb{R}$ with $\alpha(0) = 0$.

3. Orbital Inverse Shadowing for vector fields

DEFINITION 3.1. We say that the system (1) has the *orbital inverse shadowing property* with respect to the class $\mathcal{T}_\alpha$ $(\alpha = a, c, h, d)$ if for any $\varepsilon > 0$ and $T > 0$ there exists $\delta > 0$ such that for any (δ, T)-method $\Psi \in \mathcal{T}_\alpha(\delta, T, \Phi)$ and any point $x \in M$ there is $y \in M$ for which

$$d_H(\overline{O(x, \Phi)}, \overline{O(y, \Psi)}) < \varepsilon,$$

where, d_H is Hausdorff metric and $O(x, \Phi) = \{\Phi(t, x) : t \in \mathbb{R}\}$, $O(y, \Psi) = \{\Psi(t, y) : t \in \mathbb{R}\}$.

Remark 3.2. Let us denote $IS_\alpha(M)$ by the set of vector fields with the inverse shadowing property with respect to the class $\mathcal{T}_\alpha$, and $OIS_\alpha(M)$ by the set of vector fields with the orbital inverse shadowing property with respect to the class $\mathcal{T}_\alpha$, where $\alpha = a, c, h, d$. We denote $IS^\circ_\alpha(M)$ [resp. $OIS^\circ_\alpha(M)$] by the C^1 interior of the set $IS_\alpha(M)$ [resp. $OIS_\alpha(M)$] in $\mathcal{X}^1(M)$, where $\alpha = a, c, h, d$.

Clearly we have the following inclusions:

$$\begin{array}{ccccccc} IS_a(M) & \subset & IS_c(M) & \subset & IS_h(M) & \subset & IS_d(M) \\ \bigcap & & \bigcap & & \bigcap & & \bigcap \\ OIS_a(M) & \subset & OIS_c(M) & \subset & OIS_h(M) & \subset & OIS_d(M). \end{array}$$

A vector field $X \in \mathcal{X}^1(M)$ is called *structurally stable* if there is a C^1 neighborhood $\mathcal{U}(X)$ of $X \in \mathcal{X}^1(M)$ such that every $Y \in \mathcal{U}(X)$ is topologically conjugate to X.

It is proved by C. Robinson [17] that if X satisfies Axiom A and the strong transversality condition, then X is structurally stable. The inverse implication is the famous "stability conjecture" which is proved for diffeomorphisms by R. Mañé [13] and is proved for flows by S. Hayashi [7], L. Wen [19] and S. Gan [5].

The purpose of this paper is to give a characterization of the structurally stable vector fields by making use of the notion of orbital inverse shadowing.

The main result of this paper is the following one. For simplicity, we denote $OIS_d(M)$ by $OIS(M)$.

Main Theorem. The C^1 interior of $OIS(M)$ is characterized as the set of vector fields satisfying Axiom A and the strong transversality condition.

We say that $X \in \mathcal{X}^1(M)$ is *topologically stable in* $\mathcal{X}^1(M)$ if for any $\varepsilon > 0$, there is $\delta > 0$ such that for any $Y \in \mathcal{X}^1(M)$ with $d_{C^0}(X, Y) < \delta$, there is a semiconjugacy (h, τ) from Y to X satisfying $d(h(x), x) < \varepsilon$ for all $x \in M$. It is easy to see that a topologically stable vector field $X \in \mathcal{X}^1(M)$ satisfies the orbital inverse shadowing property with respect to the class $\mathcal{T}_h$. Thus we have the following Corollary which is the main result obtained by Moriyasu *et al* in [13].

Corollary. The C^1 interior of the set of all topologically stable C^1 vector fields on M is characterized as the set of all vector fields satisfying Axiom A and the strong transversality condition.

Before giving the proof of the main theorem, we are going to construct a smooth vector field $X \in IS_h(M)$ which is not topologically stable. This implies that the concept of topological stability for vector fields is surely stronger than that of inverse shadowing.

Our construction is through the suspension. For any diffeomorphism $f \in Diff^1(M)$, there is a corresponding suspension flow $S_f \in \mathcal{X}^1(S_M)$ of f, where S_M is the suspension of M; i.e., the cartesian product of M with the unit circle S^1. Then the followings are straightforward:

(1) The suspension flow S_f of $f \in Diff^1(M)$ is topologically stable if and only if for any $\varepsilon > 0$, there is $\delta > 0$ such that for any $g \in Diff^1(M)$ with $d_{C^0}(f, g) < \delta$, there is a semiconjugacy h from g to f satisfying $d(h(x), x) < \varepsilon$ for all $x \in M$.

(2) The suspension flow S_f of $f \in Diff^1(M)$ has the inverse shadowing property with respect to the class $\mathcal{T}_h$ if and only if f has the inverse shadowing property with respect to the class $\mathcal{T}_h$ (for the notion of the inverse shadowing property f, see [11]).

The property for f in the previous result (1) is a little different from the original definition of topological stability for homeomorphisms. In the original definition, the perturbation should be a homeomorphism instead of a diffeomorphism. However these two notions are pairwise equivalent if the phase space is the unit circle S^1. More precisely, the following result can be proved by the techniques in [20].

Let $f \in Diff^1(S^1)$. Then the following are equivalent:
(i) f is topologically conjugate to a Morse-Smale diffeomorphism.
(ii) f is topologically stable.
(iii) for any $\varepsilon > 0$, there is $\delta > 0$ such that for any $g \in Diff^1(S^1)$ with $d_{C^0}(f, g) < \delta$, there is a semiconjugacy h from g to f satisfying $d(h(x), x) < \varepsilon$ for all $x \in S^1$.

Recently K. Lee and J. Park [11] proved that the shadowing property and the inverse shadowing property for homeomorphisms on the unit circle S^1 are pairwise equivalent.

Now we give our construction. In [20], Yano constructed a homeomorphism f of S^1 which has the shadowing property but is not topologically stable. By a slight modification, f can be constructed as a diffeomorphism. By the result before, we know that the suspension flow S_f of f is not topologically stable, but it has the inverse shadowing property with respect to the class T_h.

Let $\mathcal{X}^*(M)$ be the set of all $X \in \mathcal{X}^1(M)$ with the property that there is a C^1 neighborhood $\mathcal{U}(X) \subset \mathcal{X}^1(M)$ of X such that for every $Y \in \mathcal{U}(X)$, whose singularities and periodic orbits are hyperbolic. Denote $\mathcal{X}^\sharp(M)$ all the systems $X \in \mathcal{X}^*(M)$ with the property that there is a C^1 neighborhood $\mathcal{U}(X)$ of X such that for each $Y \in \mathcal{U}(X)$, the stable manifolds and the unstable manifolds of singularities and periodic orbits of Y_t are all transversal. Recently it is proved by S. Gan [5] that $X \in \mathcal{X}^\sharp(M)$ if and only if X satisfies Axiom A and the strong transversality condition.

Very recently Y. Han and K. Lee [6] proved that every structurally stable vector field satisfies the inverse shadowing property with respect to the class $\mathcal{T}_c$. Therefore we get the following inclusions:

$$\mathcal{X}^\sharp(M) = SS(M) \subset IS_c^\circ(M) \subset OIS_c^\circ(M) \subset OIS^\circ(M),$$

where $SS(M)$ denotes the set of structurally stable vector fields.

To prove the main theorem, it is enough to show the following theorem:

Theorem. $OIS^\circ(M) \subset \mathcal{X}^\sharp(M)$.

The proof of the above theorem is completed by the following three propositions. The techniques in our proof are similar to those in [14].

Proposition A. $OIS^\circ(M) \subset \mathcal{X}^*(M)$.

Proposition B. Let $X \in OIS^\circ(M)$, $p \in Sing(X)$ and $q \in Sing(X) \cup PO(X_t)$. Then the stable manifold of p and the unstable manifold of q are transverse.

Proposition C. Let $X \in OIS^\circ(M)$ and let γ, $\gamma' \in PO(X_t)$. Then the stable manifold of γ and the unstable manifold of γ' are transverse.

4. Some Lemmas

For each vector field $X \in \mathcal{X}^1(M)$, we denote the generated flows by X_t. Throughout this paper, let $Sing(X)$ be the set of all singularities of X, and let $PO(X_t)$ be the set of all periodic orbits (which are not singularities) of the generated flow X_t. We say that $p \in Sing(X)$ is *hyperbolic* if the linear map $D_pX : T_pM \longrightarrow T_pM$ has no eigenvalue λ with $Re(\lambda) = 0$. For a hyperbolic singularity p, we define the stable manifold $W^s(p, X_t)$ and the unstable manifold $W^u(p, X_t)$ of p as following;

$$W^s(p, X_t) = \{x \in M : d(X_t(x), p) \to 0, as \ \ t \to \infty\},$$

$$W^u(p, X_t) = \{x \in M : d(X_t(x), p) \to 0, as \ \ t \to -\infty\}.$$

A point $x \in M$ is called a *non-wandering* point of X if for any neighborhood U of x in M, there is $t \geq 1$ such that $X_t(U) \cap U \neq \emptyset$. The set of all non-wandering points of X is denoted by $\Omega(X_t)$. Clearly, $Sing(X) \cup PO(X_t) \subset \Omega(X_t)$.

Hereafter, we assume that the exponential map $\exp_p : T_pM(1) \longrightarrow M$ is well defined for all $p \in M$, where $T_pM(1) = \{v \in T_pM : \|v\| \leq 1\}$. Let $B_\varepsilon(x) = \{y \in M : d(x, y) \leq \varepsilon\}$ $(\varepsilon > 0)$. To prove the main theorem, we need the following results in [14].

LEMMA 4.1.([14, Lemma 1.1.]) Let $X \in \mathcal{X}^1(M)$ and $p \in Sing(X)$. Then for every C^1 neighborhood $\mathcal{U}(X) \subset \mathcal{X}^1(M)$ of X, there are $\delta_0 > 0$ and $\varepsilon_0 > 0$ such that if $O_\delta : T_pM \longrightarrow T_pM$ is a linear map with $\|O_\delta - D_pX\| < \delta < \delta_0$, then there is $Y^\delta \in \mathcal{U}(X)$ satisfying

$$Y^\delta(x) = \begin{cases} (D_{exp_p^{-1}(x)}) \circ O_\delta \circ exp_p^{-1}(x) & \text{if} \quad x \in B_{\varepsilon_0/4}(p), \\ X(x) & \text{if} \quad x \notin B_{\varepsilon_0}(p). \end{cases}$$

Furthermore, $d_{C^1}(Y^\delta, Y^0) \longrightarrow 0$ as $\delta \longrightarrow 0$. Here Y^0 is vector field for $O_\delta = D_pX$.

Let $X \in \mathcal{X}^1(M)$. For every $x \in M \setminus Sing(X)$, put

$$\hat{\Pi}_x = (Span X(x))^\perp \subset T_xM, \; \Pi_{x,r} = exp_x(\hat{\Pi}_{x,r}) \text{ and } \Pi_x = \Pi_{x,1},$$

where, $\hat{\Pi}_{x,r} = \{v \in \hat{\Pi}_x : \|v\| < r\}$ for $r > 0$. Then, for given $x' = X_{t_0}(x)(t_0 > 0)$, there are $r_0 > 0$ and a C^1 map $\tau : \Pi_{x,r_0} \longrightarrow \mathbb{R}$ such that

$$X_{\tau(y)}(y) \in \Pi_{x'} \text{ for } y \in \Pi_{x,r_0}) \text{ and } \tau(x) = t_0.$$

The flow X_t uniquely defines the Poincaré map $f : \Pi_{x,r_0} \longrightarrow \Pi_{x'}$ by

$$f(y) = X_{\tau(y)}(y) \text{ for } y \in \Pi_{x,r_0}.$$

The map is C^1 embedding whose image is interior to $\Pi_{x'}$ if r_0 is small. We denote the set of all C^1 embeddings from $\Pi_{x,r}$ to $\Pi_{x'}(r > 0)$ by $Emb^1(\Pi_{x,r}, \Pi_{x'})$ and topologize it by using the C^1 topology. If $X_t(x) \neq x$ for $0 < t \leq t_0$ and r_0 is sufficiently small, then $(t, y) \mapsto X_t(y)$ C^1 embeds

$$\{(t, y) \in \mathbb{R} \times \Pi_{x,r} : 0 \leq t \leq \tau(y)\}$$

for $0 < r \leq r_0$. The image

$$\{X_t(y) : y \in \Pi_{x,r} \quad \text{and} \quad 0 \leq \text{t} \leq \tau(\text{y})\}$$

is called a t_0-*time length flow box* and is denoted by $F_x(X_t, r, t_0)$. For $\varepsilon > 0$, let $\mathcal{N}_\varepsilon(\Pi_{x,r})$ be the set of all diffeomorphisms $\varphi : \Pi_{x,r} \longrightarrow \Pi_{x,r}$ such that

$$supp(\varphi) \subset \Pi_{x,\frac{r}{2}} \text{ and } d_{C^1}(\varphi, id) < \varepsilon.$$

Here d_{C^1} is the usual C^1 metric, $id : \Pi_{x,r} \longrightarrow \Pi_{x,r}$ is the identity map and the support of φ is the closure of the set where it differs from *id*.

LEMMA 4.2.([14, Lemma 1.2.]) Let $X \in \mathcal{X}^1(M)$. Suppose $X_t(x) \neq x$ for $0 < t \leq t_0 (x \notin Sing(X))$, and let $f : \Pi_{x,r_0} \longrightarrow \Pi_{x'}$ $(x' = X_{t_0}(x))$ be the Poincaré map ($r_0 > 0$ is sufficiently small). Then, for every C^1 neighborhood $\mathcal{U}(X) \subset \mathcal{X}^1(M)$ of X and $0 < r \leq r_0$, there is $\varepsilon > 0$ with the property that for every $\varphi \in \mathcal{N}_\varepsilon(\Pi_{x,r})$, there exists $Y \in \mathcal{U}(X)$ satisfying

$$\begin{cases} Y(y) = X(y) & \text{if } x \notin F_x(X_t, r, t_0) \\ f_Y(y) = f \circ \varphi(y) & \text{if } y \in \Pi_{x,r}. \end{cases}$$

Here $f_Y : \Pi_{x,r} \longrightarrow \Pi_{x'}$ is the Poincaré map defined by Y_t.

Remark 4.3. Under the same notation and assumption of Lemma 4.2, let $Y^\delta \in \mathcal{U}(X)$ be given by Lemma 4.2 for $\varphi_\delta \in N_\varepsilon(\Pi_{x,r})$ $(\delta > 0)$. If $\varphi_\delta \longrightarrow \varphi$ as

$\delta \longrightarrow 0$ with respect to the C^1 topology, then by the construction of Y^δ, we have $d_{C^1}(Y^\delta, Y) \longrightarrow 0$ as $\delta \longrightarrow 0$.

Let $X \in \mathcal{X}^1(M)$ and suppose $p \in \gamma \in PO(X_t)(X_T(p) = p,\ T > 0)$. If $f : \Pi_{p,r_0} \longrightarrow \Pi_p$ is the Poincaré map ($r_0 > 0$), then $f(p) = p$. We say that γ is *hyperbolic* if p is a hyperbolic fixed point of f. If $\gamma \in PO(X_t)$ is hyperbolic, then the stable manifold $W^s(\gamma, X_t)$ and the unstable manifold $W^u(\gamma, X_t)$ of γ are defined by the usual way. Let $\gamma, \gamma' \in PO(X_t)$ be hyperbolic. We say that γ is transverse to γ' if for any $x \in W^s(\gamma, X_t) \cap W^u(\gamma', X_t)$,

$$T_x M = T_x W^s(\gamma, X_t) + T_x W^u(\gamma', X_t).$$

LEMMA 4.4.([14, Lemma 1.3.])Let $X \in \mathcal{X}^1(M)$, $p \in \gamma \in PO(X_t)$ $(X_T(p) = p)$ and $f : \Pi_{p,r_0} \longrightarrow \Pi_p$ be as above, and let $\mathcal{U}(X) \subset \mathcal{X}^1(M)$ be a C^1 neighborhood of X and $0 < r \le r_0$ be given. Then there are $\delta_0 > 0$ and $0 < \varepsilon_0 < \frac{r}{2}$ such that for a linear isomorphism $O_\delta : \hat{\Pi}_p \longrightarrow \hat{\Pi}_p$ with $\|O_\delta - D_p f\| < \delta < \delta_0$, there is $Y^\delta \in \mathcal{U}(X)$ satisfying

(i) $Y^\delta(x) = X(x)$ if $x \notin F_p(X_t, r, T)$,
(ii) $p \in \gamma \in PO(Y_t^\delta)$,
(iii)
$$g_{Y^\delta}(x) = \begin{cases} exp_p \circ O_\delta \circ exp_p^{-1}(x) & \text{if } x \in B_{\varepsilon_0/4}(p) \cap \Pi_{p,r} \\ f(x) & \text{if } x \notin B_{\varepsilon_0}(p) \cap \Pi_{p,r}, \end{cases}$$
where $g_{Y^\delta} : \Pi_{p,r} \longrightarrow \Pi_p$ is the Poincaré map of Y_t^δ. Furthermore, let Y^0 be the vector field for $O_\delta = D_p f$. Then we have
(iv) $d_{C^1}(Y^\delta, Y^0) \longrightarrow 0$ as $\delta \longrightarrow 0$.

5. Proof of Main Theorem

In this section, we will prove Propositions A-C.

Proof of Proposition A. The proof is divided into two cases.

Case 1. we prove Proposition A for singularities. Let $X \in OIS^\circ(M)$. Suppose that there is an eigenvalue λ of $D_p X$ with $Re(\lambda) = 0$ for some $p \in Sing(X)$. By Lemma 4.1, for any C^1 neighborhood $\mathcal{U}(X) \in OIS^\circ(M)$ of X, there are $\delta_0 > 0$ and $\varepsilon_0 > 0$ such that for every linear isomorphism $O_\delta : T_p M \longrightarrow T_p M$ with $\|O_\delta - D_p X\| < \delta < \delta_0$, there is $Y^\delta \in \mathcal{U}(X)$ satisfying

$$Y^\delta(x) = \begin{cases} (D_{exp_p^{-1}(x)} exp_p) \circ O_\delta \circ exp_p^{-1}(x) & \text{if } x \in B_{\varepsilon_0/4}(p) \\ X(x) & \text{if } x \notin B_{\varepsilon_0}(p). \end{cases}$$

Let $Y^0 \in \mathcal{U}(X)$ be as above for $O_\delta = D_p X$ and denote Y^0 by Y . For $0 < \varepsilon < \frac{\varepsilon_0}{16}$, let $0 < \delta < \min\{\delta_0, \varepsilon\}$ be as in the definition of $OIS(M)$ of Y_t. Pick $0 < \delta' < \delta$ and a linear isomorphism $O_{\delta'} : T_p M \longrightarrow T_p M$ whose any eigenvalue has a non-zero real part such that if $\|O_{\delta'} - D_p X\| < \delta'$ then $d_{C^1}(Y^{\delta'}, Y) < \delta$. Then $Y_t^{\delta'} \in \mathcal{T}_d(\delta, 1, Y)$ and p is a hyperbolic singularity of $Y^{\delta'}$. The restriction $Y_t^{\delta'}|_{B_{\varepsilon_0/4}(p)}$ can be regarded as the flow induced from the hyperbolic linear vector field $O_{\delta'}|_{exp_p^{-1}(B_{\varepsilon_0/4}(p))}$ with respect to the exponential coordinates. Since $d_{C^1}(Y, Y^{\delta'}) < \delta$, that is $(\delta, 1)$-method, therefore for any $x \in M$, there is a $y \in M$ such that

$$d_H(\overline{O(x, Y_t)}, \overline{O(y, Y_t^{\delta'})}) < \varepsilon.$$

By the existence of the λ, we can take $z \in M$ such that $p \notin B_\varepsilon(O(z, Y_t)) \subset B_{\varepsilon_0/8}(p)$ (by reducing ε if necessary). Here $B_\varepsilon(A) = \bigcup_{x\in A} B_\varepsilon(x)$ for $A \subset M$. Let w be an orbital inverse shadowing point of $O(z, Y_t)$. We have $O(w, Y_t^{\delta'}) \subset B_\varepsilon(O(z, Y_t))$. This is a contradiction since $Y_t^{\delta'}|_{B_{\varepsilon_0/4}(p)}$ is regarded as the flow induced from the hyperbolic linear vector field $O_{\delta'}|_{exp_p^{-1}(B_{\varepsilon_0/4}(p))}$; i.e.,

$$p \in N_\varepsilon(O(w, Y_t^{\delta'})) \text{ but } p \notin N_\varepsilon(O(z, Y_t)).$$

Case 2. we prove Proposition A for periodic orbits.

We prove for periodic orbits. Let $\mathcal{U}(X) \subset OIS^\circ(M)$ be a C^1 neighborhood of X and pick $p \in \gamma \in PO(X_t)(X_T(p) = p,\ T > 0)$. The flow X_t defines the Poincaré map $f : \Pi_{p,r_0} \longrightarrow \Pi_p$, (for some $r_0 > 0$). By assuming that there is an eigenvalue λ of $D_p f$ with $|\lambda| = 1$, we shall derive a contradiction. Let $\delta_0 > 0$ and $0 < \varepsilon_0 < r_0$ be given by Lemma 4.4 for the $\mathcal{U}(X)$. Then, for every linear isomorphism $O_\delta : \hat{\Pi}_p \longrightarrow \hat{\Pi}_p$ with $\|O_\delta - D_p f\| < \delta < \delta_0$, there is $Y^\delta \in \mathcal{U}(X)$ such that

- $Y^\delta(x) = X(x) \quad \text{if} \quad x \notin F_p(X_t, r_0, T)$,

$$\bullet g_{Y^\delta}(x) = \begin{cases} exp_p \circ O_\delta \circ exp_p^{-1}(x) & \text{if } \mathbf{x} \in B_{\frac{\varepsilon_0}{4}}(p) \cap \Pi_{p,r_0} \\ f(x) & \text{if } x \notin B_{\varepsilon_0}(p) \cap \Pi_{p,r_0}, \end{cases}$$

- $d_{C^1}(Y^\delta, Y^0) \longrightarrow 0 \quad \text{as} \quad \delta \longrightarrow 0.$

Denote Y^0 by Y. For $0 < \varepsilon < \frac{\varepsilon_0}{16}$, let $0 < \delta < \min\{\delta_0, \varepsilon\}$ be as in the definition of $OIS(M)$ of Y_t. Take a $0 < \delta' < \delta$ and a hyperbolic linear isomorphism $O_{\delta'} : \hat{\Pi}_p \longrightarrow \hat{\Pi}_p$ such that if $\|O_{\delta'} - D_p X\| < \delta'$ then $d_{C^1}(Y, Y^\delta) < \delta$. Then $Y_t^{\delta'} \in \mathcal{T}_d(\delta, 1, Y)$ and γ is a hyperbolic periodic orbit of $Y_t^{\delta'}$. Therefore for any $x \in M$, there is a $y \in M$ such that

$$d_H(\overline{O(x, Y_t)}, \overline{O(y, Y_t^{\delta'})}) < \varepsilon.$$

Note that $g_{Y^{\delta'}}(p) = p$ and the restriction $g_{Y^{\delta'}}|_{B_{\varepsilon_0/4}(p)} \cap \Pi_{p,r_0}$ is regarded as the hyperbolic linear isomorphism $O_{\delta'}|_{exp_p^{-1}(B_{\varepsilon_0/4}(p))} \cap \Pi_{p,r_0}$ with respect to the exponential coordinates. Since $|\lambda| = 1$, we may take $z \in \Pi_{p,r_0}$ such that

$$p \notin B_{2\varepsilon}(\{g_Y^i(z) : i \in \mathbb{Z}\}) \subset B_{\varepsilon_0/4}(p) \cap \Pi_{p,r_0}. \tag{2}$$

Let w be an orbital inverse shadowing point of $O(z, Y_t)$. Set $w' = Y_{t'}^{\delta'}(w) \in \Pi_{p,r_0}$, where $|t'| = \min\{|t| : Y_t^{\delta'}(w) \in \Pi_{p,r_0}\}$. Since

$$d_H(\overline{O(z, Y_t)}, \overline{O(w, Y_t^{\delta'})}) < \varepsilon.$$

we have $g^i_{Y^{\delta'}}(w') \in N_\varepsilon(\{g^i_Y(z) : i \in \mathbb{Z}\})$. But $g_{Y^{\delta'}}|_{B_{\varepsilon_0/4}(p)\cap\Pi_{p,r_0}}$ is hyperbolic linear isomorphism, we get $p \in N_\varepsilon(\{g^i_{Y^{\delta'}}(w')\})$. This fact contradicts to (2).

Proof of Proposition B.

Before the proof of Proposition B, we give the following lemma.

Lemma 5.1. Let $X \in \mathcal{X}^1(M)$. Suppose that $p \in Sing(X) \cup PO(X_t)$ and $q \in Sing(X) \cup PO(X_t)$ are hyperbolic and $p \neq q$. Let $x \in W^s(p, X_t) \cap W^u(q, X_t)$. Then there exist $\varepsilon^\star > 0$ and a neighborhood $B_{\eta^\star}(x)$ of x satisfying:

for any $y \in B_{\eta^\star}(x) - \Delta_s$ [resp. $y \in B_{\eta^\star}(x) - \Delta_u$], there exists $t' > 0$ [resp. $t' < 0$] such that

$$d_H(X_{t'}(y), \overline{O(x, X_t)}) \geq \varepsilon^\star.$$

Here Δ_s [resp. Δ_u)] is the connected component of $W^s(p, X_t)$ [resp. $W^u(q, X_t)$)] in $B_{\eta^\star}(x)$ containing x.

Proof. We will prove the case that q and p are singularities. The proof of the other cases are similar.

Let $r_0 > 0$ be a small constant such that $B_{r_0}(p)$ is a local chart about p. Let $C^u(r_0/2) = B_{r_0/2}(p) \cap W^u_{loc}(p)$. Since $x \in W^s(p, X_t) \cap W^u(q, X_t)$, we have

$$C^u(r_0/2) \cap \overline{O(x, X_t)} = \emptyset.$$

Thus

$$d_H(C^u(r_0/2), \overline{O(x, X_t)}) > 0.$$

Set $\varepsilon_1 = d_H(C^u(r_0/2), \overline{O(x, X_t)})/2$. By the hyperbolicity of p, there exists a neighborhood $B_{\eta_1}(x)$ of x such that for any $y \in B_{\eta_1}(x) - \Delta_s$, there exists $t' > 0$ with $X_{t'}(y) \in N_{\varepsilon_1}(C^u(r_0/2))$. Thus we have

$$\begin{aligned} d_H(X_{t'}(y), \overline{O(x, X_t)}) &\geq d_H(C^u(r_0/2), \overline{O(x, X_t)}) - d_H(X_{t'}(y), C^u(r_0/2)) \\ &\geq 2\varepsilon_1 - \varepsilon_1 = \varepsilon_1. \end{aligned}$$

Similarly we can choose $\varepsilon_2 > 0$ and $\eta_2 > 0$ such that for any $y \in B_{\eta_2}(x) - \Delta_u$, there exists $t' < 0$ satisfying

$$d_H(X_{t'}(y), \overline{O(x, X_t)}) \geq \varepsilon_2.$$

Let $\varepsilon^\star = \min\{\varepsilon_1, \varepsilon_2\}$ and $\eta^\star = \min\{\eta_1, \eta_2\}$. It is easy to see that $\varepsilon^\star$ and $\eta^\star$ satisfy the conclusion. □

Let us start the proof of Proposition B.

Let $X \in OIS^\circ(M)$. Suppose that $x \in W^s(p, X_t) \cap W^u(q, X_t)$, $p \in Sing(X)$, $q \in Sing(X) \cup PO(X_t)$ and $T_xM \neq T_xW^s(p, X_t) + T_xW^u(q, X_t)$. We may assume that x is very near p. Take $r_0 > 0$ small enough so that

- there are the Poincaré maps
  $f : \Pi_{x,r_0} \longrightarrow \Pi_{X_1(x)}$ and $f' : \Pi_{X_{-1}(x),r_0} \longrightarrow \Pi_x$,
- $\{X_t(x) : t < 0\} \cap \Pi_{x,r_0} = \emptyset$ and $\{X_t : t < -1\} \cap \Pi_{X_{-1}(x),r_0} = \emptyset$,
- $W^u_{2r_0}(q, X_t) \cap (\Pi_{x,r_0} \cup \Pi_{X_{-1}(x),r_0}) = \emptyset$,

where $W^u_{2r_0}(q, X_t)$ is the local unstable manifold of q. Let $V^s(x)$ be the connected component of $W^s(p, X_t) \cap \Pi_x$ containing x, and $V^u(x)$ be the connected component of $W^u(q, X_t) \cap \Pi_x$ containing x.

Clearly, we have

$$0 \le \dim V^s(x) \le \dim \Pi_x \text{ and } 0 \le \dim V^u(x) \le \dim \Pi_x.$$

If dim $V^s(x) = \dim \Pi_x$ or $\dim V^u(x) = \dim \Pi_x$, then there is nothing to prove.

Notice that $\hat{\Pi}_x \neq T_x V^s(x) + T_x V^u(x)$. We shall divide the proof into the following two cases:

Case (1.1) $\dim V^s(x) + \dim V^u(x) < \dim \Pi_x$,

Case (1.2) $\dim V^s(x) + \dim V^u(x) \ge \dim \Pi_x$.

Proof for case (1.1). Put $V^u_r(x)$ be the connected component of $V^u(x) \cap B_r(x)$ containing x, and $V^s_r(x)$ be the connected component of $V^s(x) \cap B_r(x)$ containing x for $r > 0$. Choose $0 < r' < \frac{r_0}{4}$ with the following properties:

- for every $\delta > 0$, there exists $\varphi_\delta \in N_{\varepsilon(\delta)}(\Pi_{x,r_0})$ satisfying

$$\varphi_\delta(V^u_{r'}(x)) \cap V^s(x) = \emptyset, \tag{3}$$

- $V^u_{r'}(x) = [\bigcup_{0 \le t \le t_1} X_t(W^u_{2r_0}(q, X_t))] \cap \Pi_{x,r_0} \cap B_{r'}(x)$, where $t_1 > 0$ such that $X_{-t_1}(x) \in W^u_{r_0}(q, X_t)$,
- $V^s_{r'}(x) = [\bigcup_{0 \le t \le t_2} X_{-t}(W^s_{2r_0}(p, X_t))] \cap \Pi_{x,r_0} \cap B_{r'}(x)$, where $t_2 > 0$ such that $X_{t_2}(x) \in W^s_{r_0}(p, X_t)$,
- $B_{r'}(\{X_t(x) : t \le 0\}) \cap \Pi_{x,r_0} = B_{r'}(x) \cap \Pi_{x,r'}$.

For any small enough $\delta > 0$, we can choose a vector field Y^δ given by Lemma 4.2 from (3); i.e., Y^δ satisfies the followings:

$$\begin{cases} Y^\delta(y) = X(y) & \text{if } \; y \notin F_x(X_t, r_0, 1), \\ g(y) = f \circ \varphi_\delta(y) & \text{if } y \in \Pi_{x,r_0}, \\ d_{C^1}(X, Y^\delta) < \delta, \end{cases}$$

where $g : \Pi_{x,r_0} \longrightarrow \Pi_{X_1(x)}$ is the Poincaré map induced by Y^δ_t. We may assume that $F_x(Y^\delta_t, r_0, 1) \cap W^s_{r_0}(p, Y^\delta_t) = \emptyset$ for sufficiently small r_0.

For vector field X and point $x \in W^s(p, X_t) \cap W^u(q, X_t)$, we have the numbers $\varepsilon^\star > 0$ and $\eta^\star > 0$ satisfying the conclusion of Lemma 5.1. Take $\varepsilon > 0$ with $\varepsilon < \min\{\frac{\varepsilon^*}{2}, \frac{\eta}{2}, \frac{r'}{2}\}$. Let $0 < \delta = \delta(\varepsilon) < \varepsilon$ be the corresponding number with repect to ε in the definition of $OIS(M)$ of X. So Y^δ constructed as the above is an element of $\mathcal{T}_d(\delta, 1, X_t)$. For simplicity, we denote Y^δ by Y. So, there is $w \in M$ such that

$$d_H(\overline{O(x, X_t)}, \overline{O(w, Y_t)}) < \varepsilon.$$

Since $X(y) = Y(y)$ if $y \notin F_x(X_t, r_0, 1)$, p is singularity and q is also singularity or periodic orbit if Y_t. Thus $O(w, Y_t)$ through $B_{\eta^*}(x)$. Take w' in the orbit of $O(w, Y_t)$ which is in $B_{\eta^\star}(x) \cap \Pi_{x,r_0}$. If $w' \in V^u_{r'}(x)$, then by the properties in the construction of Y and by Lemma 5.1,

$$d_H(\overline{O(x, X_t)}, \overline{O(w, Y_t)}) \ge \varepsilon^* > \varepsilon.$$

If w' is not in $V^u_{r_0}(x)$, then similarly we have

$$d_H(\overline{O(x, X_t)}, \overline{O(w, Y_t)}) \geq \varepsilon^* > \varepsilon.$$

Consequently we get a contradiction, and so complete of the proof of case (1.1).

Proof for case (1.2). Take a C^1 neighborhood $\mathcal{U}(X) \subset OIS(M)$. Let $V^s(X_{-1}(x))$ [resp. $V^u(X_{-1}(x))$] be the connected component of $W^s(p, X_t) \cap \Pi_{X_{-1}}(x)$ [resp. $W^u(q, X_t) \cap \Pi_{X_{-1}}(x)$] containing $X_{-1}(x)$. Let $f' : \Pi_{X_{-1}(x), r_0} \longrightarrow \Pi_x$ be the Poincaré map. For the above $\mathcal{U}(X)$, let $\mu = \mu(\mathcal{U}(X)) > 0$ be given by Lemma 4.2. Since

$$\hat{\Pi}_x \neq T_x V^s(x) + T_x V^u(x) \text{ and } \dim V^s(x) + \dim V^u(x) \geq \dim \Pi_x,$$

there are $0 < r_1 < \frac{r_0}{4}$, $\tilde{\varphi} \in \mathcal{N}_\mu(\Pi_{X_{-1}(x), r_0})$ and a submanifold $V(X_{-1}(x)) \subset \Pi_{X_{-1}(x), r_0}$ such that

- $V^s(X_{-1}(x)) \cap B_{r_1}(X_{-1}(x)) \subset V(X_{-1}(x))$,
- $\tilde{\varphi}(V^u(X_{-1}(x)) \cap B_{r_1}(X_{-1}(x))) \subset V(X_{-1}(x))$ and $\tilde{\varphi}(X_{-1}(x)) = X_{-1}$,
- $\dim V^s(x) + \dim V^u(x) - \dim \Pi_x < \dim V(X_{-1}(x)) < \dim \Pi_{X_{-1}(x)}$.

Let $Y \in \mathcal{U}(X)$ and $g = f' \circ \tilde{\varphi} : \Pi_{X_{-1}(x), r_0} \longrightarrow \Pi_x$ (since $g(X_{-1}(x)) = x$) be given by Lemma 4.2. Let $V^s(x, Y_t)$ [resp. $V^u(x, Y_t)$] be the connected component of $W^s(p, Y_t) \cap \Pi_x$ [resp. $W^u(q, Y_t) \cap \Pi_x$] containing x. If we put $V(x, Y_t) = f'(V(X_{-1}(x)))$, then we get $\dim V(x, Y_t) < \dim \Pi_x$. It is easy to choose $0 < r_2 < \frac{r_0}{4}$ satisfying

$$V^u(x, Y_t) \cap B_{r_2}(x) \subset V(x, Y_t) \text{ and } V^s(x, Y_t) \cap B_{r_2}(x) \subset V(x, Y_t).$$

By the choice of r_0, we can see that

- the map $f : \Pi_{x, r_0} \longrightarrow \Pi_{X_1(x)}$ is also a Poincaré map for Y_t,
- $q \in Sing(Y) \cup PO(Y_t)$,
- $X_t(x) = Y_t(x)$ for $t \leq 0$.

Since $Y(y) = X(y)$ for $y \notin F_{X_{-1}(x)}(X_t, r_0, 1)$, we have

$$W^s_{2r_0}(p, X_t) = W^s_{2r_0}(p, Y_t) \text{ and } W^u_{2r_0}(q, X_t) = W^u_{2r_0}(q, Y_t).$$

As in the proof of proposition B in [14], by applying Lemma 4.2 for Y and f, for any $\delta > 0$ we can obtain $Z^\delta \in \mathcal{U}(X)$ and $r' < r_2$ such that

- Z^δ is perturbation of $\varphi_\delta \in N_{\varepsilon(\delta)}(\Pi_{x, r_0})$ satisfying $\varphi_\delta(V^u_{r'}(x, Y_t)) \cap V(x, Y_t) = \emptyset$,
- $Z^\delta(y) = Y(y)$ if $y \notin F_x(Y_t, r_0, 1)$,
- $g'(y) = f \circ \varphi_\delta(y)$ if $y \in \Pi_{x, r_0}$,
- $d_{C^1}(Y, Z^\delta) < \delta$,

where $g' : \Pi_{x, r_0} \longrightarrow \Pi_{Y_1(x)}$ is the Poincaré map induced by Z^δ_t.

For C^1 vector field Y and $x \in W^s(p, Y_t) \cap W^u(q, Y_t)$, we have numbers $\varepsilon^\star > 0$ and $\eta^\star$ satisfying Lemma 5.1. Choose a sufficiently small $0 < \varepsilon < \min\{\frac{\varepsilon^*}{2}, \frac{r'}{2}, \frac{\eta}{2}, \frac{\mu}{2}\}$, let $0 < \delta < \varepsilon$ be as in the definition of $OIS(M)$ of Y_t. So Z^δ constructed as the above is an element of $\mathcal{T}_d(\delta, 1, Y_t)$. For simplicity, we denote Z^δ by Z. By the definition of orbital inverse shadowing for the flow Y_t with respect to the point x, there is a $y \in M$ such that

$$d_H(\overline{O(x,Y_t)},\overline{O(y,Z_t)}) < \varepsilon.$$

We can derive a contradiction as in the case (1.1), by applying Lemma 5.1. This completes the proof of Proposition B.

Proof of Proposition C.

Suppose that $\gamma,\gamma' \in PO(X_t)$ are hyperbolic and $x \in W^s(\gamma,X_t) \cap W^u(\gamma',X_t)$. Fix $p \in \gamma$ ($X_T(p) = p$), and let $r_0 > 0$ be sufficiently small so that we can define the Poincaré map $f : \Pi_{p,r_0} \longrightarrow \Pi_p$. Since p is hyperbolic, there are a Df-invariant splitting $\hat{\Pi}_p = E_p^s \oplus E_p^u$ and two constants $C > 0$, $0 < \lambda < 1$ such that $\|Df^m_{|E_p^s}\| < C\lambda^m$ and $\|Df^{-m}_{|E_p^u}\| < C\lambda^m$ for all $m \geq 0$. Let $W_r^\sigma(p,f)$ be the connected component of $W^\sigma(\gamma,X_t) \cap \Pi_{p,r}$ containing p for $\sigma = s,u$ and $0 < r \leq r_0$. Suppose that $x \in W^s_{r_0/2}(p,f) \setminus intW^s_{r_0/2}(p,f)$, and let $T' > 0$ be the number with $f(x) = X_{T'}(x)$ and take $0 < r_1 < \frac{r_0}{4}$ such that $F_p(X_t,r_1,T) \cap F_x(X_t,r_1,T') = \emptyset$.

Before continuing our proof, we will cite the following result from [14].

Lemma 5.2. ([14, Lemma 4.1.]) Under the above notation, for every C^1 neighborhood $\mathcal{U}(X)$ of X, there are $0 < \varepsilon_0 < \frac{r_0}{4}$ and $Y \in \mathcal{U}(X)$ satisfying

(i) $Y(y) = X(y)$ if $y \notin F_p(X_t,r_1,T) \cup F_x(X_t,r_1,T')$,
(ii) $\gamma,\gamma' \in PO(Y_t)$ and $Y_T(p) = p \in \gamma$,
(iii)

$$g(y) = \begin{cases} exp_p \circ D_p f \circ exp_p^{-1}(y) & \text{if } \ y \in B_{\varepsilon_0/4}(p) \cap \Pi_{p,r_0} \\ f(y) & \text{if } \ y \notin B_{\varepsilon_0}(p) \cap \Pi_{p,r_0}, \end{cases}$$

(iv) $g(p) = p$, $x \in W^s_{r_0}(p,g)$ and $T_xW^s_{r_0}(p,g) = T_xW^s_{r_0}(p,f)$,
(v) $T_xW^u(\gamma',Y_t) = T_xW^u(\gamma',X_t)$.

Here $g : \Pi_{p,r_0} \longrightarrow \Pi_p$ is the Poincaré map of Y_t and $W^\sigma_{r_0}(p,g)$ is the connected component of $W^\sigma(\gamma,Y_t) \cap \Pi_{p,r_0}$ containing p ($\sigma = s,u$).

Put $E_x^\sigma(\varepsilon) = \{v \in E_x^\sigma \mid \|v\| \leq \varepsilon\}$ for $\varepsilon > 0$ ($\sigma = s,u$), and $g \in Emb^1(\Pi_{p,r_0},\Pi_p)$, $p = g(p) \in \Pi_p$ and $\varepsilon_0 > 0$ be given by Lemma 5.2. Then $exp_p(E_p^\sigma(\frac{\varepsilon_0}{4})) \subset W^\sigma_{r_0}(p,g)$ and $\dim exp_p(E_p^\sigma(\frac{\varepsilon_0}{4})) = \dim W^\sigma_{r_0}(p,g)$ for $\sigma = s,u$ since ε_0 is small. For convenience, we denote $exp(E_p^\sigma(\varepsilon))$ by $W_\varepsilon^\sigma(p,g)$ for $\sigma = s,u$ and $0 < \varepsilon \leq \frac{\varepsilon_0}{4}$.

Let $X \in OIS^\circ(M)$, suppose that $\gamma,\gamma' \in PO(X_t)$ are hyperbolic and $x \in W^s(\gamma,X_t) \cap W^u(\gamma',X_t)$. Let $p \in \gamma(X_T(p) = p, T > 0)$ and $f : \Pi_{p,r_0} \longrightarrow \Pi_p$ ($r_0 > 0$) be as before. We may assume that

- $x \in W^s_{r_0/2}(p,f) \setminus intW^s_{r_0/2}(p,f)$,
- $W^u_{2r_0}(\gamma',X_t) \cap \Pi_{p,r_0}$.

Here, $W^u_{r_0}(\gamma',X_t)$ is the local unstable manifold of γ'. Fix a C^1 neighborhood $\mathcal{U}(X) \subset OIS^\circ(M)$ of X, and let $0 < \varepsilon_0 < \frac{r_0}{4}$, $Y \in \mathcal{U}(X)$ and g be given by Lemma 5.2. Thus $T_xW^s_{r_0}(p,g) = T_xW^s_{r_0}(p,f)$, $W^u_{2r_0}(\gamma',X_t) = W^u_{2r_0}(\gamma',Y_t)$, and $X_t(x) = Y_t(x)$ for all $t \leq 0$. Clearly, $1 \leq \dim W^s_{r_0}(p,g) \leq \dim \Pi_p$ and $1 \leq \dim(W^u_{r_0}(\gamma',Y_t) \cap \Pi_p) \leq \dim \Pi_p$. If $\dim W^s_{r_0}(p,g) = \dim \Pi_p$ or $\dim(W^u_{r_0}(\gamma',Y_t) \cap \Pi_p = \dim \Pi_p$, then the conclusion is clear. Pick $l > 0$ so large that $g^{l-1}(x) \in W^s_{\varepsilon_0/8}(p, g)$ and set $C^u(g^l(x))$ be the connected coponent of $W^u(\gamma', Y_t) \cap \Pi_p$ containing $g^l(x)$. To simplify the notation, denote $g^l(x)$ by x.

Then we get

$$exp_p^{-1}(C^u(x)) \subset \hat{\Pi}_p \text{ and } T_x C^u(x) = T_x(W^u(\gamma', Y_t) \cap \Pi_p).$$

For a linear subspace E of $\hat{\Pi}_p$ and $\nu > 0$, let

$$E_\nu(x) = \{v + exp_p^{-1}(x) | v \in E \text{ with } \|v\| \le \nu\}$$

be a piece of an affine space running parallel to E. Let $T'', T''' > 0$ be numbers with $Y_{T''}(g^{-1}(x)) = (x)$ and $Y_{T'''}(x) = g(x)$, respectively. Choose a linear subspace $E' \subset \hat{\Pi}_p$ and $0 < \nu_0 < \varepsilon_0/8$ such that

- for every $0 < \nu \le \nu_0$, $exp_p(E'_\nu(x)) \subset B_{\varepsilon_0/4}(p)$,
- $T_x exp_p(E'_{\nu_0}(x)) = T_x C^u(x)$,
- $(F_{g^{-1}(x)}(Y_t, \nu_0, T'') \cup F_x(Y_t, \nu_0, T''')) \cap \gamma = \emptyset$,
- $\{Y_t(g^{-1}(x)) : t < 0\} \cap F_{g^{-1}(x)}(Y_t, \nu_0, T'') = \emptyset$ and $\{Y_t(x) : t < 0\} \cap F_x(Y_t, \nu_0, T''') = \emptyset$,
- $B_{\nu_0}(\{Y_t(x) : t \le 0\}) \cap F_x(Y_t, \nu_0, T''') \cap \Pi_{p,r_0}$ $= B_{\nu_0}(x) \cap F_x(Y_t, \nu_0, T''') \cap \Pi_{p,r_o}$,
- $g^i(W^s_{r_0}(p, g) \cap B_{\nu_0}(g^{-1}(x))) \cap B_{\nu_0}(g^{-1}(x)) = \emptyset$ for $i \ge 1$,
- $Y_{-t}(C^u(g^{-1}(x))) \cap B_{\nu_0}(g^{-1}(x)) = \emptyset$ for all $t > 0$.

Here $C^u(g^{-1}(x))$ is the connected coponent of $W^u(\gamma', Y_t) \cap \Pi_p \cap B_{\nu_0}(g^{-1}(x))$ containing $g^{-1}(x)$.

Lemma 5.3. ([13, Lemma 4.2.]) Fix a C^1 neighborhood $\mathcal{U}(Y) \subset \mathcal{U}(X)$ of Y. Then there are $0 < \nu_1 < \frac{\nu_0}{4}$ and $Y' \in \mathcal{U}(Y)$ such that

(i) $Y'(y) = Y(y)$ if $y \notin F_{g^{-1}(x)}(Y_t, \nu_0, T''')$
(ii) $Y'_{T''}(g^{-1}(x)) = x$,
(iii) $exp_p(E'_{\nu_1}(x)) \subset W^u(\gamma', Y'_t) \cap \Pi_p$ and $T_x exp_p(E'_{\nu_1}(x)) = T_x(W^u(\gamma', Y'_t) \cap \Pi_p)$.

Remark 5.4.

(i) We see that $W^u_{2r_0}(\gamma', Y_t) = W^u_{2r_0}(\gamma', Y'_t)$ and $Y_t(x) = Y'_t(x)$ for $t \le 0$.
(ii) Let $g' : \Pi_{p,r_0} \longrightarrow \Pi_p$ be the Pioncaré map induced by Y'_t. Then by Lemma 5.3(i), $\gamma' \in PO(X_t)$ and $g'(y) = g(y)$, if $y \in \Pi_{p,r_0} \setminus B_{\nu_0}(g^{-1}(x))$. Thus $g'(p) = g(p)$. By lemma 5.3(ii), $g'^i(x) = g(x)$ for all $i \ge 0$.
(iii) By the perturbation used in the above lemma, $W^s_{\varepsilon_0/4}(p, g)$ may be deformed near $g^{-j}(x) \in W^s_{\varepsilon_0/4}(p, g)$ for some $j > 0$. We denote the deformed manifold by $W^s_{\varepsilon_0/4}(p, g')$. Then $x \in W^s_{\varepsilon_0/4}(p, g')$ and $T_x W^s_{\varepsilon_0/4}(p, g') = T_x W^s_{\varepsilon_0/4}(p, g)$.

If $exp_p(E'_{\nu_1}(x))$ does not meet $W^s_{\varepsilon_0/4}(p, g')$ trasverse at x, then the piece $E'_\nu(x)$ of the affine space is not transversal with respect to the local linear stable manifold $E^s_p(\frac{\varepsilon_0}{4})$ of p. To simplify the notation, denote Y', g' and $W^s_{\varepsilon_0/4}(p, g')$ by Y, g, and $W^s_{\varepsilon_0/4}(p, g)$, respectively. If $exp_p(E'_{\nu_1}(x))$ dose not meet $W^s_{\varepsilon/4}(p)$ transversely at x then there is $0 < r' < \frac{\nu_1}{2}$ such that

- such that for every $\delta > 0$, there is $\psi_\delta \in N_{\varepsilon(\delta)}(\Pi_{x,2\nu_1})$ satisfying

$$\begin{cases} \psi_\delta(exp_p(E'_{\nu_1}(x)) \cap B_{r'}(x)) \cap W^s_{r_0}(p) = \emptyset \\ \psi_\delta(y) = y \quad \text{if} \quad y \notin \Pi_{x,\nu_1} \end{cases}$$

- $exp_p(E'_{\nu_1}(x)) \cap B_{r'}(x) = [\bigcup_{0 \le t \le t_2} Y_t(W^u_{2r_0}(\gamma', Y_t))] \cap \Pi_{p,r_0} \cap B_{r'}(x)$,

where $Y_{-t_2}(x) \in W^u_{r_0}(\gamma', Y_t)$.

For C^1 vector field Y and $x \in W^s(\gamma, Y_t) \cap W^u(\gamma', Y_t)$, we have numbers $\varepsilon^\star > 0$ and $\eta^\star > 0$ satisfying the conclusion of Lemma 5.1. Choose a sufficiently small $0 < \varepsilon < \min\{\frac{\varepsilon^*}{2}, \frac{r'}{2}, \frac{\eta^\star}{2}\}$, let $0 < \delta < \varepsilon$ be the corresponding number with respect to ε in the definition of $OIS(M)$ of Y_t. Then by Lemma 4.2, we can construct $Z \in \mathcal{U}(Y)$ for the above perturbation such that

$$\begin{cases} Z(z) = Y(z) & \text{if} \quad z \notin F_x(Y_t, 2\nu_1, T'''), \\ \tilde{g}(z) = g \circ \psi_\delta(z) & \text{if} \quad z \in \Pi_{x,2\nu_1}, \\ d_{C^1}(Y, Z) < \delta. \end{cases}$$

Here $\tilde{g} : \Pi_{x,2\nu_1} \longrightarrow \Pi_{g(x)}$ is the Poincaré map induced by Z_t. Because Z is in $\mathcal{T}_d(\delta, 1, Y)$, there is $y \in M$ such that

$$d_H(\overline{O(x, Y_t)}, \overline{O(y, Z_t)} < \varepsilon.$$

Because $Z(y) = Y(y)$ for $y \notin F_x(Y_t, 2\nu_1, T''')$,

$$W^s_{2r_0}(\gamma, Z_t) = W^s_{2r_0}(\gamma, Y_t) \ \text{ and } \ W^u_{2r_0}(\gamma, Z_t) = W^u_{2r_0}(\gamma, Y_t).$$

Similar to the proof of Proposition B, we can get a contradiction. This completes the proof of Proposition C.

Remark 5.5. The notion of weak inverse shadowing for homeomorphisms was introduced in [1] and it was shown that the weak shadowing property is C^0 generic in the space of homeomorphisms with the C^0 topology.

We can introduce the notion of weak inverse shadowing property for flows as follows : we say that the system (1) has the *weak inverse shadowing property* with respect to the class $\mathcal{T}_\alpha$ ($\alpha = a, c, h, d$) if for any (δ, T)-method $\Psi \in \mathcal{T}_\alpha(\delta, T, \Phi)$ and any point $x \in M$, there is a $y \in M$ for which

$$\{\Psi_t(y) : t \in \mathbb{R}\} \subset N_\varepsilon(\{\Phi_t(x) : t \in \mathbb{R}\}).$$

It is proved in [2] that the C^1 interior of the set of diffeomorphisms with the weak inverse shadowing property (with respect to the class $\mathcal{T}_d$) coincides with the set of Ω-stable diffeomorphisms.

We can easily show that the C^1 interior of the set of vector fields with the weak inverse shadowing property (with respect to the class $\mathcal{T}_d$) is contained in the set of Ω-stable vector fields.

We can easily see that every irrational flows on the 2-dimensional torus has the weak inverse shadowing property (with respect to the class $\mathcal{T}_d$), but it is not Ω-stable.

However we do not know yet whether the C^1 interior of the set of vector fields with the weak inverse shadowing property (with respect to the class $\mathcal{T}_d$) coincides with the set of Ω-stable vector fields.

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