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HOMEOMORPHISMS WITH INVERSE SHADOWING ON ZERO DIMENSIONAL SPACES

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ABSTRACT. In this paper we study the density of homeomorphisms with shadowing property and inverse shadowing property on zero dimensional spaces

1. INTRODUCTION

Inverse shadowing was established by Corless and Pilyugin [2] and also as a part of the concept of bishadowing by diamond etal [5]. Kloeden and Ombach [8] redefined this property using the concept of a δ-method. Generally speaking, a dynamical systems is inverse shadowing with respect to a class of methods if any true orbit can be uniformly approximated with given accuracy by a δ-pseudo orbit generated by a method from the chosen class if $\delta > 0$ is sufficiently small. An appropriate choice of the class of admissible pseudo orbits is crucial here (see[2,4,8,9]).

There are some results about how shadowing and inverse shadowing are generated each other(for more detais, see [4,6,8,9]).

Every pseudo Anosov homeomorphism on compact surfaces is inverse shadowing, but it is not shadowing. And every shift homeomorphism is shadowing, but it is not inverse shadowing.

Then we know followings:

If $dimM \geq 2$, then shadowing is not inverse shadowing.

If $dimM = 1$, then shadowing is inverse shadowing [10].

If $\dim M = 0$, then shadowing is not inverse shadowing.

In this paper, we study the density of homeomorphisms with shadowing and inverse shadowing on zero dimensional spaces which are the Cantor set C and two special sets S and Y where $S = \{0, 1, 1/2, 1/3, \cdots\}$ and $Y = \{a, b\} \cup \{a_i |\ i \in \mathbb{Z}\}$ is a subspace of Euclidean space E^2 such that $\lim_{i\to-\infty} a_i = a$ and $\lim_{i\to\infty} a_i = b$ where $a_i \neq a_j, i \neq j$ and $a \neq b$.

The Cantor set C is the unique zero-dimensional infinite group. N. Aoki [1] proved that every group automorphism of C has shadowing(P.O.T.P). M. Sears [13] proved that $E(C)$ is dense in $Z(C)$. M. Dateyma [3] proved that $SP(C)$ is dense in $Z(C)$. T. Kimura [7] proved that $E(C) \cap SP(C)$ is dense in $Z(C)$ and followings:

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a) $E(S)$ is dense in $Z(S)$
b) $SP(S)$ is dense in $Z(S)$
c) $SP(S) \cap E(S) = \phi$.

J. Park [12] proved that $E(Y) \cap SP(Y)$ is dense in $Z(Y)$.

The aim of this paper is to prove the following two theorems.

Theorem 1.
a) $ISP_0(S) = SP(S)$
b) $ISP_h(S) \cap E(S) = \phi$.

Theorem 2.
a) $ISP_0(Y) \cap SP(Y)$ is dense in $Z(Y)$
b) $ISP_h(Y) \cap E(Y) = \phi$
c) $SP(Y) \setminus ISP_h(Y)$ is dense in $Z(Y)$.

Moreover we give following two examples of inverse shadowing on cantor set C.

Example 1. Let $f_n : D^{\mathbb{Z}} \longrightarrow D^{\mathbb{Z}}$, $n \in \mathbb{N} \cup \{0\}$ and $n < \infty$, be a homeomorphism such that for each $x \in D^{\mathbb{Z}}$,

$$(f_n(x))_i = \begin{cases} x_{j(i)} & \text{if } -n \le i \le n \\ x_i & \text{if } i \notin \{-n, \cdots, n\} \end{cases}$$

where $j(i) \neq j(t)$, $i \neq t$ and $i, t, j(i), j(t) \in \{-n, \cdots, n\}$. Then $f_n \in ISP_0(D^{\mathbb{Z}})$.

Example 2. For any $f \in ISP(C)$ and any $\varepsilon > 0$, we can find a $g \in Z(C)$ such that $d_0(f, g) < \varepsilon$ and g does not have the inverse shadowing property.

2. Inverse Shadowing

Let X be a compact metric space with a metric d, and let $Z(X)$ denote the space of homeomorphisms on X with the C_0-metric d_0, where for $f, g \in Z(X)$,

$$d_0(f, g) = \sup_{x \in X} \{d(f(x),\ g(x)),\ d(f^{-1}(x), g^{-1}(x))\}.$$

A homeomorphism $f \in Z(X)$ will be identified with dynamical system it generates by iteration. f is *expansive* if there is $c > 0$ such that for every $x, y \in X$ with $x \neq y$ there is $n \in \mathbb{Z}$ for which $d(f^n(x), f^n(y)) > c$. We say that $E(X) = \{f \in Z(X) | f \text{ is expansive}\}$.

A *δ-pseudo orbit* of $f \in Z(X)$ is a sequence of points $\varepsilon = \{x_k \in X | k \in \mathbb{Z}\}$ such that $d(f(x_k), x_{k+1}) < \delta$ for all $k \in \mathbb{Z}$. A δ-pseudo orbit $\varepsilon = \{x_k\}$ is said to be ε-shadowed by a point $x \in X$(or an orbit $\{f^k(x) | k \in \mathbb{Z}\}$) if $d(f^k(x), x_k) < \varepsilon$ for all $k \in \mathbb{Z}$. Say that $f \in Z(X)$ is *shadowing* (or pseudo orbit tracing property (P.O.T.P)) if given $\varepsilon > 0$ there exists $\delta > 0$ such that any δ-pseudo orbit of f is ε-shadowed by a point (or an orbit) in X. We say that $SP(X) = \{f \in Z(X) | f \text{ is shadowing}\}$.

Let $X^{\mathbb{Z}}$ be the compact metric space of all two sided sequences $\varepsilon = \{x_k | k \in \mathbb{Z}\}$ in X, endowed with the product topoogy. For a constant $\delta > 0$ and $f \in Z(X)$, let $\Phi_f(\delta)$ dnote the set of all δ-pseudo orbits of f. A mapping $\varphi : X \to \Phi_f(\delta) \subset X^{\mathbb{Z}}$

satisfying $\varphi_0(x) = x,\ x \in X$ is said to be a *δ-method* for f. For convenience, write $\varphi(x)$ for $\{\varphi_k(x)\}_{k\in\mathbb{Z}}$. Say that φ is a *continuous δ-method* for f if φ is continuous. The set of all δ-methods[resp. continuous δ-methods] for f will be denoted by $\mathcal{T}_0(f,\delta)$(resp. $\mathcal{T}_c(f,\delta)$). Every $g \in Z(X)$ with $d_0(f,g) < \delta$ induces a continuous δ-method $\varphi_g : X \to X^{\mathbb{Z}}$ for f by defining $\varphi_g(x) = \{g^k(x) | k \in \mathbb{Z}\}$. Let $\mathcal{T}_h(f,\delta)$ denote the set of all continuous δ-methods φ_g for f which are induced by $g \in Z(X)$ with $d_0(f,g) < \delta$. Then we have $\mathcal{T}_h(f,\delta) \subset \mathcal{T}_c(f,\delta) \subset \mathcal{T}_0(f,\delta)$. $f \in Z(X)$ is said to be *$\mathcal{T}_\alpha$-inverse shdowing* $\alpha = 0,\ c,\ h$, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any f-orbit $\varepsilon = \{x_k\}$ and any δ-method $\varphi \in \mathcal{T}_\alpha(f,\delta)$ there is $y \in X$ such that $d(x_k, \varphi_k(y)) < \varepsilon$ for all $k \in \mathbb{Z}$. We say that $ISP_\alpha(X) = \{f \in Z(X) \mid f$ is inverse shadowing w.r.t. $\mathcal{T}_\alpha(f,\delta)\}$.

Clearly we have the following relations among three notions of inverse shadowing.

$$ISP_o(X) \Rightarrow ISP_c(X) \Rightarrow ISP_h(X).$$

We say that $U_\delta(x) = \{y \mid d(x,y) < \delta,\ y \in X\}$ and for a subset $A \subset X$, $[A]^\circ$ is interior of A.

Theorem 1. Let $\{0, 1, 1/2, 1/3, \cdots\}$. Then

1a) $ISP_0(S) = SP(S)$
1b) $ISP_h(S) \cap E(S) = \phi$.

Proof. 1a): Assume that f is in $SP(S)$. Let $\varepsilon > 0$. Take $n \in \mathbb{N}$ such that $1/n < \varepsilon$. Since $f \in SP(S)$, there is $\delta(\varepsilon) > 0$ satisfying shadowing. Take $\delta = \min\{1/(n^2+n), \delta(\varepsilon)\}$. Take $x \in S$. Then there is two cases:

(1a1) $x \geq 1/n$
(1a2) $x < 1/n$.

Proof of Case (1a1): Since $f \in SP(S)$ and $U_\delta(x) = \{x\}$, any δ-pseudo orbit $\{x_i \mid x_0 = x,\ i \in \mathbb{Z}\}$ is ε-traced by x uniquely. Hence for any δ-method $\varphi \in \mathcal{T}_0(f,\delta)$, $d(x_i,\ \varphi^i(x)) < \varepsilon$.

Proof of Case (1a2): If there is $j \in \mathbb{Z}$ such that $y = f^j(x) \geq 1/n$, then since $f \in SP(S)$ and $U_\delta(y) = \{y\}$, any δ-pseudo orbit $\xi = \{x_i \mid x_j = y, i \in Z\}$ is ε-traced by $f^{-j}(y)$ uniquely. Hence for any δ-method $\varphi \in \mathcal{T}_0(f,\delta)$, there is $z \in S$ with $\varphi^j(z) = y$ such that $d(x_i, \varphi^i(z)) < \varepsilon$.

If for all $i \in \mathbb{Z}$, $f^i(x) < 1/n$, then for any δ-method $\varphi \in \mathcal{T}_0(f,\delta)$, $d(f^i(x), \varphi^i(0)) < \varepsilon$. So $f \in ISP_0(S)$.

Now we assume that $f \in ISP_0(S)$. Let $\varepsilon > 0$. Take $n \in \mathbb{N}$ such that $1/n < \varepsilon$. Since $f \in ISP_0(S)$, there is $\delta(\varepsilon) > 0$ satisfying inverse shadowing. Take $\delta = \min\{1/(n^2+n), \delta(\varepsilon)\}$. Take a δ-pseudo orbit $\xi = \{x_i \mid i \in \mathbb{Z}\}$. Then there is two cases:

(1a3) $x_0 \geq 1/n$
(1a4) $x_0 < 1/n$.

Proof of Case (1a3): Since $f \in ISP_0(S)$ and $U_\delta(x_0) = \{x_0\}$, any δ-pseudo orbit $\xi = \{x_i \mid x_0 = x,\ i \in Z\}$ is ε-traced by x.

Proof of Case (1a4): If there is $j \in \mathbb{Z}$ such that $y = x_j \geq 1/n$, then we regard $\xi = \{x_i \mid i \in Z\}$ as $\{x_{i-j} \mid i \in \mathbb{Z}\}$. Then by Case (1a3), $\{x_{i-j} \mid i \in \mathbb{Z}\}$ is ε-traced by y.

If for all $i \in \mathbb{Z}$, $x_i < 1/n$, then $\xi = \{x_i \mid i \in \mathbb{Z}\}$ is ϵ-traced by 0.

1b): Let $f \in E(S)$ with an expansive constant c. First we contrust a mapping $g_\delta : S \longrightarrow S$ with $g_\delta \in Z(S)$ and $d_0(g_\delta,\ f) < \delta$. We take $n \in \mathbb{N}$ such that $1/n < c$. Take a $\varepsilon > 0$ with $\varepsilon < \min\{c,\ 1/(n^2 - n)\}$. Given a $\delta > 0$. Put $2\delta_1 = \min\{\delta,\ \varepsilon\}$. Take a $k \in \mathbb{N}$ such that $1/k < \delta_1$. Take a set

$$M = \{x_m \mid f(x_m) = 1/m,\ m < k\} \text{ and}$$

$$l = \max\{m \mid f(x_m) = 1/m,\ m < k\} + 1.$$

Define a mapping $g_\delta : S \longrightarrow S$ by

1) if $x \in S - S_l,\ g_\delta(x) = x,$
2) if $x \in M,\ g_\delta(x) = f(x)$
3) if $x \in S_l - M,\ g_\delta(x) \in S_l - M$ and $g_\delta(x) \neq g_\delta(x'), x \neq x', \in S_l - M.$

Then we know that by the construction g_δ,

$$d_0(f,\ g) < \delta_1 < \delta \text{ and } g_\delta \in Z(S).$$

Now we show that $f \in ISP_h(S)$. Since $f \in E(S)$, by Kimura [7], there is $z' \in S_n$ such that z' is not a periodic point for f. Then since $\lim_{|i| \to \infty} f^i(z') = 0$, there is $t \in \mathbb{Z}$ such that $f^t(z') < \delta_1$. Put $z = f^t(z')$. We have the following three cases:

(1b1) $y \in S_n$;
(1b2) $y \in S_l - S_n$;
(1b3) $y \in S - S_l$.

Proof of Case (1b1): For all $y \in S_n$, $d(z_0,\ g^0(y)) > \varepsilon$.

Proof of Case (1b2): Any point y in $S_l - S_n$ is a periodic point for g_δ. Take two sets:

$$A = \{y \in S_l - S_n \mid O(y,\ g_\delta) \subset S_l \setminus S_n\} \text{ and}$$

$$B = \{y \in S_l - S_n \mid O(y,\ g_\delta \cap S_n \neq \phi\}.$$

Then $S_l - S_n = A \cup B$. Take a point $y \in A$. Since $f \in E(S)$, there is $i \in \mathbb{Z}$ such that $f^i(z) \in S_n$ and then

$$d(z_i,\ g_\delta^i(y)) > \varepsilon \text{ for some } i \in \mathbb{Z}.$$

Take a point $y \in B$. Since z is not a periodic point, $\lim_{|i| \to \infty} f^i(z) = 0$ and $y \in B$ is a periodic point, there is $r \in \mathbb{Z}$ such that

$$g_\delta^r(z) \in U_{\delta_1}(0) \quad \text{and} \quad g_\delta^r(y) \in S_n.$$

Then $d(z_r, g_\delta^r(y)) > \varepsilon$ for some $r \in \mathbb{Z}$.

Proof of Case (1b3): Since $f \in E(S)$, there is $i \in \mathbb{Z}$ such that $f^i(z) \in S_n$. Thus

$$d(z_i,\ g_\delta^i(y) > \varepsilon \quad \text{for any} \quad y \in S - S_l.$$

Hence for any $y \in S$,

$$d(z_i,\ g_\delta^i(y)) > \varepsilon \quad \text{for some } i \in \mathbb{Z}.$$

Therefore $f \in ISP_h(S)$.

Theorem 2. Let $Y = \{a, b\} \cup \{a_i | \ i \in \mathbb{Z}\}$ be a subspace of Euclidean space E^2 such that $\lim_{i\to-\infty} a_i = a$ and $\lim_{i\to\infty} a_i = b$ where $a_i \neq a_j, i \neq j$ and $a \neq b$. Then

2a) $ISP_0(Y) \cap SP(Y)$ is dense in $Z(Y)$

2b) $ISP_h(Y) \cap E(Y) = \phi$

2c) $SP(Y) \setminus ISP_h(Y)$ is dense in $Z(Y)$.

Proof. 2a): Without loss of generality, we assume that for each $i \in \mathbb{Z}$,

$$d(a_{i+1},\ a_i) < d(a_i,\ a_{i-1}) \ \text{ if } i > 0 \text{ and}$$

$$d(a_{i-1},\ a_i) < d(a_i,\ a_{i+1}) \ \text{ if } i < 0 \ .$$

Let $f \in Z(Y)$ and $\varepsilon > 0$. We will construct $g \in ISP_0(Y) \cap SP(Y)$ such that $g \in Z(Y)$ and $d_0(f, g) < \varepsilon$. To do this, we consider following two cases:

(2a1) $f(a) = a$ and $f(b) = b$,

(2a2) $f(a) = b$ and $f(b) = a$.

Proof of Case (2a1): For $\varepsilon > 0$, we take $n \in \mathbb{N}$ such that for $|i| \geq n,\ i \in \mathbb{Z}$

$$d(a_i,\ a) < \varepsilon \ \text{ or } d(a_i,\ b) < \varepsilon.$$

Consider following three sets:

$$A_1 = \{a_i \mid i \leq -n\},$$

$$A_2 = \{a_i \mid -n < i < n\} \ \text{ and}$$

$$A_3 = \{a_i \mid n \leq i\}.$$

Since f is a homeomorphism, there is $t \in \mathbb{N}$,

$$f(a_i) \in A_1 \ \text{ for all } i < -t \text{ and } f(a_i) \in A_3 \text{ for all } i > t.$$

Put $k = \max\{t,\ n\}$. Consider a set

$$M = \{a_i \in Y \mid f(a_i) = a_j,\ |j| \leq k\}.$$

We take $l, l' \in \mathbb{Z}$ such that

$$l = \max\{i \in \mathbb{Z} \mid a_i \in M\} \text{ and}$$

$$l' = \min\{i \in \mathbb{Z} \mid a_i \in M\} \ .$$

We consider folowing three sets:

$$B = \{a_i \mid l' \leq i \leq l\} \setminus M,$$

$$B_1 = \{a_i \mid f(a_i) \in A_1,\ a_i \in B\} \text{ and}$$

$$B_2 = \{a_i \mid f(a_i) \in A_3,\ a_i \in B\} \text{ and.}$$

Then we know that $B = B_1 \cup B_2$.

Define a mapping $g : Y \longrightarrow Y$ by

1) $g(a) = a$ and $g(b) = b$;

2) $g(a_i) = f(a_i)$ if $a_i \in M$;

3) $g(a_i) \in \{a_i \mid l' < i < -k\}$ if $a_i \in B_1$,
   $g(a_i) \in \{a_i \mid k < i < l\}$ if $a_i \in B_2$ and
   $g(a_i) \neq g(a_j),\ a_i \neq a_j \in B_s,\ s = 1, 2;$
4) $g(a_i) = a_i$ if $a_i \in Y - \{a_i \mid l' \leq i \leq l\}$.

Then by the construction g, $g \in Z(Y)$ and $d_0(f, g) < \varepsilon$. By the constraction g and the sense of Kimura [7] and Park [12], we know that $g \in SP(Y)$. Now we prove that $g \in ISP_0(Y)$. Let $\varepsilon_0 > 0$. We take a ε_1 with $\varepsilon_1 = \min\{\varepsilon_0, \varepsilon\}$. Take $p \in \mathbb{N}$ such that

$$d(a, a_{-p}) < \varepsilon_1 \text{ and } d(b,\ a_p) < \varepsilon_1.$$

Take $r = \max\{p,\ |l'|,\ l\}$. We consider a $\delta > 0$ with

$$\delta = \min\{d(a_r,\ a_{r+1}),\ d(a_{-r-1},\ a_{-r})\}.$$

We define the sets C_1, C_2, C_3, by

$$C_1 = \{a_i \mid i < -r\},\ C_2 = \{a_i \mid -r \leq i \leq r\} \text{ and}$$

$$C_3 = \{a_i \mid i > r\}.$$

We know that by the construction of g, for each $x \in Y$ and for any δ-method $\varphi \in \mathcal{T}_0(g, \delta)$

$$\varphi(x) = \{\varphi(x)_k = g^k(x)\} \text{ if } x \in C_2 \text{ and}$$

$$d(\varphi(x)_k,\ g^k(x)) < \varepsilon \text{ if } x \in C_1 \text{ or } x \in C_3.$$

Hence for any δ-mehod $\varphi \in \mathcal{T}_0(g, \delta)$,

1) if $z \in C_1$ or $z \in C_3$, then $d(z_k,\ \varphi(y)_k) < \varepsilon_1$
   for each $y \in A_1$ or $y \in A_3$ and for all $k \in \mathbb{Z}$;
2) if $z \in C_2$, then $d(z_k,\ \varphi(z)_k) < \varepsilon_1$ for all $k \in \mathbb{Z}$.

Thus $g \in ISP_0(Y)$.

Proof of Case (2a2): We use here the notions in Case (2a1) as the same meaning but t. We consider $t \in \mathbb{N}$ in Case (2a1) as $f(a_i) \in A_1$ if for all $i > t$ and $f(a_i) \in A_3$ for all $i < -t$. Define a mapping $g : Y \longrightarrow Y$ by

1) $g(a) = b$ and $g(b) = a$;
2) $g(a_i) = f(a_i)$ if $a_i \in M$;
3) $g(a_i) \in \{a_i \mid l' < i < -k\}$ if $a_i \in B_1$,
   $g(a_i) \in \{a_i \mid k < i < l\}$ if $a_i \in B_2$ and
   $g(a_i) \neq g(a_j),\ a_i \neq a_j \in B_s,\ s = 1, 2;$
4) $g(a_{l'-1}) = a_{l+i}, i = 1, 2, \cdots$ and
   $g(a_{l+1}) = a_{l'-i}, i = 1, 2, \cdots$.

By the same way in the proof of Case (2a1), $g \in ISP_0(Y) \cap SP(Y)$ and $d_0(g, f) < \varepsilon$.

2b): Let $f \in E(Y)$ with an expansive constant c. We consider following two cases:

(2b1) $f(a) = a$ and $f(b) = b$,
(2b2) $f(a) = b$ and $f(b) = a$.

Proof of Case (2b1): First we construct a mapping $g_\delta : Y \longrightarrow Y$ such that

$$g_\delta \in Z(Y) \text{ and } d_0(f,\ g_\delta) < \delta.$$

We take $n \in \mathbb{N}$ such that

$$d(a_{-n},\ a) < c \text{ and } d(a_n,\ b) < c.$$

Take a $\varepsilon > 0$ with

$$\varepsilon < \min\{c,\ d(a_n,\ a_{n-1}),\ d(a_{-n},\ a_{-n+1})\}.$$

Consider following three sets:

$$A_1 = \{a_i \mid i \leq -n\},$$
$$A_2 = \{a_i \mid -n < i < n\} \text{ and}$$
$$A_3 = \{a_i \mid n \leq i\}.$$

Since f is a homeomorphism, there is $t \in \mathbb{N}$ such that

$$f(a_i) \in A_1 \text{ for all } i < -t \text{ and } f(a_i) \in A_3 \text{ for all } i > t.$$

Put $q = \max\{t, n\}$. Given a $\delta > 0$. Put

$$2\delta_1 = \min\{\delta,\ d(a,\ a_{-q}),\ d(b, a_q)\}.$$

Take a $k \in \mathbb{N}$ sch that

$$d(a_{-k},\ a) < \delta_1 \text{ and } d(a_k,\ b) < \delta_1.$$

Consider a set

$$M = \{a_i \in \ \mid f(a_i) = a_j,\ |j| \leq k\}.$$

We take $l, l' \in \mathbb{Z}$ such that

$$l = \max\{i \in \mathbb{Z} \mid a_i \in M\} \text{ and}$$
$$l' = \min\{i \in \mathbb{Z} \mid a_i \in M\}.$$

We consider following three sets

$$B = \{a_i \mid l' \leq i \leq l\} \setminus M,$$
$$B_1 = \{a_i \mid f(a_i) \in A_1,\ a_i \in B\} \text{ and}$$
$$B_2 = \{a_i \mid f(a_i) \in A_3,\ a_i \in B\}.$$

Then $B = B_1 \cup B_2$.

Define a mapping $g_\delta : Y \longrightarrow Y$ by

1) $g_\delta(a) = a$ and $g_\delta(b) = b$;
2) $g_\delta(a_i) = f(a_i)$ if $a_i \in M$;
3) $g_\delta(a_i) \in \{a_i \mid l' < i < -k\}$ if $a_i \in B_1$,
   $g_\delta(a_i) \in \{a_i \mid k < i < l\}$ if $a_i \in B_2$ and
   $g_\delta(a_i) \neq g(a_j),\ a_i \neq a_j \in B_s,\ s = 1, 2;$
4) $g_\delta(a_i) = a_i$ if $a_i \in Y - \{a_i \mid l' \leq i \leq l\}$.

Then by the construction g_δ, $g_\delta \in Z(Y)$ and $d_0(f,\ g_\delta) < \delta$. We know that there is a point $a_i \in \{a_i \mid -n < i < n\}$ which is not a periodic point for f by the same way in the proof of Kimura [7]. Let $z' \in \{a_i \mid -n < i < n\}$ be not a periodic point. Then

$$\lim_{i\to\infty} f(z') = a \text{ or } \lim_{i\to\infty} f(z') = b.$$

We assume that $\lim_{i\to\infty} f^(z') = a$. Then there is $i \in \mathbb{Z}$ such that $f^i(z') \in U_{\delta_1}(a)$. Put $z = f^i(z')$. We consider following cases:

(2b1a) $D_1 = \{a_i \mid -n < i < n\}$;
(2b1b) $D_2 = \{a_i \mid l' < i < -n\}$;
(2b1c) $D_3 = \{a_i \mid n \le i \le l\}$;
(2b1d) $D_4 = \{a_i \mid i < l'\}$;
(2b1e) $D_5 = \{a_i \mid i > l\}$.

Proof of Case (2b1a): Let $y \in D_1$. Then $d(z_0,\ g_\delta(y)_0) > \varepsilon$.

Proof of Case (2b1b): Let $y \in D_2$. Then y is a periodic point for g_δ.
We have two sets:

$$F = \{y \in D_2 \mid O(y,\ g_\delta) \subset D_2\} \text{ and}$$

$$G = \{y \in D_2 \mid O(y,\ g_\delta) \cap D_i \neq \phi,\ i = 1, 3.\}.$$

Then $D_2 = F \cup G$. Let $y \in F$. Since $f \in E(Y)$, there is $i \in \mathbb{Z}$ such that $f^i(z) \in D_1 \cup D_3 \cup D_5$. Hence $d(z_i,\ g^i_\delta(y)) > \varepsilon$ for some $i \in \mathbb{Z}$.
Let $y \in G$. Since z is not a periodic point, $\lim_{i\to\infty} f^i(z) = a$ and $y \in G$ is a periodic point, there is $r \in \mathbb{Z}$ such that

$$f^r(z) \in U_{\delta_1}(a) \text{ and } g^r_\delta(y) \in D_1 \cup D_3.$$

Then $d(z_r,\ g^r_\delta(y)) > \varepsilon$ for some $r \in \mathbb{Z}$.

Proof of Case (2b1c): Let $y \in D_3$. We know that $d(z_i,\ g^i_\delta(y)) > \varepsilon$ for some $i \in \mathbb{Z}$ by the Proof of Case (2b1b).

Proof of Case (2b1d): Let $y \in D_4$. Since $f \in E(Y)$, there is $i \in \mathbb{Z}$ such that $f^i(D_1) \cup D_3 \cup D_5$. Thus $d(z_i,\ g^i_\delta(y)) > \varepsilon$ for any $y \in D_4$.

Proof of Case (2b1e): Let $y \in D_5$. We know that $d(z_i,\ g^i_\delta(y)) > \varepsilon$ for some $i \in \mathbb{Z}$, by the Proof of Case (2b1d).

Hence for any $y \in Y$, $d(z_i,\ g^i_\delta(y)) > \varepsilon$ for some $i \in \mathbb{Z}$. Therefore f is not in $ISP_h(Y)$.

Proof of Case (2b2): By the same way of the proof of Case (2a2) in 2a) and Case (2b1), we can define a mapping $g_\delta : Y \longrightarrow Y$ and we know that f is not in $ISP_h(Y)$.

2c): We know that by [10](i.e. $E(Y) \cap SP(Y)$ is dense in $Z(Y)$) and above 2b), $SP(Y) \setminus ISP_h(Y)$ is dense in $Z(Y)$. We complete the proof of our theorem.

3. Two Examples of Inverse Shadowing on Cantor Set

Now we consider inverse shadowing of the Cantor set. To do it, we need followings. Let $D^{\mathbb{Z}} = \Pi\{D_i \mid i \in \mathbb{Z}\}$, where $D_i = \{0, 1\}$ for every $i \in \mathbb{Z}$. We define the metric d on $D^{\mathbb{Z}}$ by

$$d(x,y) = \begin{cases} 1/(\min\{|k| \mid x_k \neq y_k\}), & \text{if } x_0 = y_0 \\ 2, & \text{if } x_0 \neq y_0 \end{cases}$$

for every $x = (x_i),\ y = (y_i) \in D^{\mathbb{Z}}$. Obviously, $(D^{\mathbb{Z}},\ d)$ is homeomorphic to the Cantor set. For a homeomorphism of a compact metrizable space X, it is clear that expansiveness, shadowing and inverse shadowing do not depend on the choice of metrics on X. Thus we may regard $(D^{\mathbb{Z}},\ d)$ as the Cantor set.

For every $i, j \in \mathbb{Z}$ with $i \leq j$, we put $D(i,j) = \Pi\{D_k \mid i \leq k \leq j\}$ and for every $v \in D(i,j)$, we put $c^+(v) = j$ and $c^-(v) = i$. For every $v \in D(i,j)$ and any $n \in \mathbb{N}$ with $i \leq -n$ and $n \leq j$ (or for every $v \in D^{\mathbb{Z}}$ and any $n \in \mathbb{N}$) we put $v_{|n} = (v_{-n}, \cdots, v_n) \in D(-n,n)$. For every $v \in D(i,j)$, we put $A_v = P_{ij}^{-1}(v)$ where $P_{ij} : D^{\mathbb{Z}} \longrightarrow D(i,j)$ is the projection. If a space X is the union of a pairwise disjoint collection $\{X_\lambda \mid \lambda \in \Lambda\}$ of open and closed subsets of X, then we represent X as $X = \oplus\{X_\lambda \mid \lambda \in \Lambda\}$.

Example 1. Let $f_n : D^{\mathbb{Z}} \longrightarrow D^{\mathbb{Z}},\ n \in \mathbb{N}\cup\{0\}$ and $n < \infty$, be a homeomorphism such that for each $x \in D^{\mathbb{Z}}$,

$$(f_n(x))_i = \begin{cases} x_{j(i)} & \text{if } -n \leq i \leq n \\ x_i & \text{if } i \notin \{-n, \cdots, n\} \end{cases}$$

where $j(i) \neq j(t),\ i \neq t$ and $i, t, j(i), j(t) \in \{-n, \cdots, n\}$. Then $f_n \in ISP_0(D^{\mathbb{Z}})$.

Proof. Let $\varepsilon > 0$. Take $\delta > 0$ such that $\delta < \min\{\varepsilon/2, 1/n\}$. Take $p \in \mathbb{N}$ such that $1/p < \delta$ and take a point $x \in D^{\mathbb{Z}}$. For any $y \in U_{1/p}(x), (f_n(y))_i = (f_n(x))_i,\ i \in \{-p+1, \cdots, p-1\}$ for all $n \in \mathbb{Z}$ and so $f_n(y) \in U_{1/p}(f_n(x))$. For any δ-method $\varphi \in \mathcal{T}_0(f_n, \delta)$, we take $y \in U_{1/p}(x)$. Then $(y)_i = (x)_i,\ i \in \{-p+1, \cdots, p-1\}$ and $\varphi^k(y) \in U_{1/p}(x^k = f_n^k(x))$ for all $k \in \mathbb{Z}$. Hence $d(\varphi^k(y),\ x^k) < \varepsilon$ for all $k \in \mathbb{Z}$ and $f_n \in ISP_0(D^{\mathbb{Z}})$.

Example 2. For any $f \in ISP(C)$ and any $\varepsilon > 0$, we can find a $g \in Z(C)$ such that $d_0(f,g) < \varepsilon$ and g does not have the inverse shadowing property.

Proof. We may regard $(D^{\mathbb{Z}},\ d)$ as the Cantor set. By the example 1, $ISP_h(D^Z) \neq \phi$. Take a homeomorphism $f \in Z(D^{\mathbb{Z}})$. Let $\varepsilon > 0$. It is sufficient that we define a homeomorphism $g \in U_\varepsilon(f)$ such that $g \notin ISP_h(D^{\mathbb{Z}})$ where $d_0(f,g) < \varepsilon$. We take $k, n \in \mathbb{N}$ such that

$$1/k < \varepsilon \text{ and } d(f(x), f(y)) < 1/k$$

for every $x, y \in D^{\mathbb{Z}}$ with $d(x,y) < 1/n$, since $f \in Z(D^Z)$.

Claim. For every $v \in D(-n,\ n)$, there are $w(v) \in D(-k,\ k)$ and $q(v) \in D(-l_1,\ l_2)$ for some $l_1,\ l_2 \in \mathbb{N}$ satisfying the following three conditions:

1) $D^{\mathbb{Z}} = \oplus\{A_{q(v)} \mid v \in D(-n,\ n)\}$;
2) $f(A_v) \subset A_{w(v)}$;
3) $-l_1 \leq -k,\ k \leq l_2,$ and $q(v)_i = w(v)_i,\ -k \leq i \leq k$.

Proof of the Claim: From $diam A_v < 1/n$, it follows that $diam f(A_v) < 1/k$. Since $D^{\mathbb{Z}} = \oplus\{A_w \mid w \in D(-k,\ k)\}$ and $d(A_w,\ A'_w) \geq 1/k$, for every $w, w' \in D(-k,\ k)$ with $w \neq w'$ there is $w(v) \in D(-k,\ k)$ such that $f(A_v) \subset A_{w(v)}$. For every $w \in D(-k,\ k)$ list $\{v \in D(-n,\ n) \mid w(v) = w\}$ as $\{v^i(w) \mid 1 \leq i \leq P_w\}$. For every i, $1 \leq i \leq P_w$, we take $q^i(w) \in D(-l_1,\ l_2)$ where $(q^i(w))_j = w_j,\ -k \leq j \leq k$ and $-l_1 \leq -k,\ k \leq l_2$ such that $A_w = \oplus\{A_{q^i(w)} \mid 1 \leq i \leq P_w\}$. Let us set $q(v^i(w)) = q^i(w)$ for every $w \in D(-k,\ k)$ and any i, $1 \leq i \leq P_w$. Then $q(v)$ and $w(v)$ have all the required properties.

Now we shall construct a homeomorphism $g : D^{\mathbb{Z}} \longrightarrow D^{\mathbb{Z}}$. For every $x \in D^{\mathbb{Z}}$, we define $g(x)$ as follows. Let $v = x_{|n} \in D(-n,\ n)$ and $q(v) \in D(-l_1,\ l_2)$.

Case 1. $l_1 + l_2 \geq 2n$ and $l_2 \geq n$. Let us set

$$(g(x))_i = \begin{cases} (q(v))_i & \text{if } -l_1 \leq i \leq l_2 \\ x_{i+1} & \text{if } l_2 + 1 \leq i \\ x_{i+l_1+l_2+2} & \text{if } n - l_1 - l_2 - 1 \leq i \leq -l_1 - 1 \\ x_{i-2n+l_1+l_2+1} & \text{if } i \leq n - l_1 - l_2 - 2 \end{cases}$$

and $M^+(v) = 1$ and $M^-(v) = -2n + l_1 + l_2 + 1$.

Case 2. $l_1 + l_2 < 2n$ and $l_1 \leq n$. Let us set

$$(g(x))_i = \begin{cases} (q(v))_i & \text{if } -l_1 \leq i \leq l_2 \\ x_{i+1} & \text{if } i \leq -n - 2 \\ x_{i+2n+2} & \text{if } -n - 1 \leq i \leq -l_1 - 1 \\ x_{i+2n-l_1-l_2+1} & \text{if } l_2 + 1 \leq i \end{cases}$$

and $M^+(v) = 2n - l_1 - l_2 + 1$ and $M^-(v) = 1$.

Case 3. Otherwise, i.e. ($l_1 + l_2 \geq 2n$ and $l_2 < n$) or ($l_1 + l_2 < 2n$ and $l_1 > n$). In this Case we have $l_2 < n < l_1$. Let us set

$$(g(x))_i = \begin{cases} (q(v))_i & \text{if } -l_1 \leq i \leq l_2 \\ x_{i+n-l_2} & \text{if } l_2 + 1 \leq i \\ x_{i+l_1-n} & \text{if } i \leq -l_1 - 1 \end{cases}$$

and $M^+(v) = n - l_1$ and $M^-(v) = l - n$.

Then it is obvious that $g|_{A_v} : A_v \longrightarrow A_{q(v)}$ is a homeomorphism. By 1) of the claim, g is a homeomorphism from $D^{\mathbb{Z}}$ onto itself. By the construction of g, $g \in U_\varepsilon(f)$.

Finally we shall show that g is not in $ISP_h(D^{\mathbb{Z}})$.

Let us set $m = \max\{-c^-(q(v)),\ c^+(q(v)),\ n \mid v \in D(-n,\ n)\}$. Take ε_1 such that $\varepsilon_1 < 1/(2m)$. Given a $\delta > 0$. Take $p, r \in \mathbb{N}$ such that

$$1/(p - M^+(v) \cdot M^-(v)) < \min\{\varepsilon_1,\ \delta\} \text{ and } (p - 2m) = r \cdot M^+(v) \cdot M^-(v)$$

where $r \geq 2$.

Define a homeomorphism $h \in \mathcal{T}_h(D^{\mathbb{Z}})$ such that $h \in U_\delta(g)$. For each $x \in D^{\mathbb{Z}}$, $(v = x_{|n})$,

$$(h(x))_i = \begin{cases} (g(x))_i & \text{if } i \neq p \text{ or } i \neq -p - M^-(v) \\ x_{-p} & \text{if } i = p \\ x_{p+M^+(v)} & \text{if } i = -p - M^-(v) \end{cases}$$

Then by the construction of h, we know that $d_0(g,\ h) < \delta$. Take a point $z = (\cdots, z_{-1}, z_0, z_1, \cdots)$ where $z_{-2m} = 1$ and $x_i = 0$ if $i \neq -2m$.

For any point $y \in D^{\mathbb{Z}}$,

1) if $(y)_{-2m} = 0$ then since y is not in $U_{\varepsilon_1}(z)$, $d(z^0 = g^0(z), h^0(y)) > \varepsilon_1$;
2) if $(y)_{-2m} = 1$, then $(h^{[r(M^+(v)+M^-(v))]}(y))_{2m} = 1$.

So

$$d((h^{[r(M^+(v)+M^-(v))]}(y)),\ z^{[r(M^+(v)+M^-(v))]}) \geq 1/2m > \varepsilon_1,$$

since $(z^{[r(M^+(v)+M^-(v))]})_{2m} = 0$. Thus g is not in $ISP_h(D^{\mathbb{Z}})$ and our proof is complete.

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