

DYNAMICAL SYSTEMS OVER ALGEBRAIC STRUCTURES

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Contents

1. Four algebraic structures
2. Universal examples
3. More examples
4. Representations
5. Duality
6. L^1 -algebras
7. C^* -algebras
8. Dynamical Systems and crossed products
9. Transitions
10. Open problems

1. FOUR ALGEBRAIC STRUCTURES

- **topological groups:** A topological group is a locally compact Hausdorff topological space G with an element e and two continuous maps $\cdot : G \times G \rightarrow G$ and $-1 : G \rightarrow G$ satisfying:
 - (G1) $(x.y).z = x.(y.z)$,
 - (G2) $x.e = e.x = x$,
 - (G3) $x.x^{-1} = x^{-1}.x = e$,for each $x, y, z \in G$.
- **inverse semigroups:** A unital inverse semigroup is a discrete space S with an element e and two operations $\cdot : S \times S \rightarrow S$ and $*$: $S \rightarrow S$ satisfying
 - (S1) $(x.y).z = x.(y.z)$,
 - (S2) $x.e = e.x = x$,
 - (S3) $x.y.x = x, y.x.y = y$ if and only if $y = x^*$,for each $x, y, z \in S$. Note that the last item implies that $x^{**} = x$. The set $E_S = \{ss^* : s \in S\}$ of idempotents is a commutative subsemigroup of S .
- **topological hypergroups:** A topological hypergroup is a locally compact Hausdorff space K with an element e and two operations $\cdot : M(K) \times M(K) \rightarrow M(K)$ ($M(K)$ = Banach space of all bounded Borel measures on K with the variation norm) and $\bar{\cdot} : K \rightarrow K$ satisfying
 - (H1) $M(K)$ is an algebra under \cdot ,
 - (H2) $\delta_x * \delta_y$ is a probability measure,
 - (H3) The map $(x, y) \mapsto \delta_x * \delta_y$ of $K \times K \rightarrow M(K)$ is continuous in the weak* topology,
 - (H4) The map $(x, y) \mapsto \text{supp}(\delta_x * \delta_y)$ of $K \times K \rightarrow \text{Comp}(K)$ is continuous in the Michael topology,

$$(H5) \quad \delta_e * \delta_x = \delta_x * \delta_e = \delta_x,$$

$$(H6) \quad (\delta_x * \delta_y)^- = \delta_{\bar{y}} * \delta_{\bar{x}},$$

$$(H7) \quad e \text{ is in the support of } \delta_x * \delta_y \text{ if and only if } x = \bar{y},$$

for each $x, y \in K$.

- **topological groupoids:** A topological groupoid is a locally compact space \mathcal{G} with a distinguished set $\mathcal{G}^{(0)}$ and four continuous operations $r, s : \mathcal{G} \rightarrow \mathcal{G}^{(0)}$, $-1 : \mathcal{G} \rightarrow \mathcal{G}$, and $\cdot : \mathcal{G}^{(2)} \subseteq \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ ($\mathcal{G}^{(2)}$ = the set of all pairs $(x, y) \in \mathcal{G} \times \mathcal{G}$ with $s(x) = r(y)$) satisfying
 - (G1) If $(x, y), (y, z) \in \mathcal{G}^{(2)}$ then $(x.y).z = x.(y.z)$,
 - (G2) If $(x, y) \in \mathcal{G}^{(2)}$, then $s(x.y) = s(y)$, $r(x.y) = r(x)$,
 - (G3) If $u \in \mathcal{G}^{(0)}$ then $s(u) = r(u) = u$,
 - (G4) $x.s(x) = r(x).x = x$,
 - (G5) $x.x^{-1} = r(x)$, $x^{-1}.x = s(x)$,
 - (G6) $\mathcal{G}^{(0)}$ is Hausdorff,
 - (G7) $x \in \mathcal{G}$ has a Hausdorff neighborhood,
 for each $x, y, z, u \in \mathcal{G}$.

2. UNIVERSAL EXAMPLES

- **topological groups:**
 - Let X be a non-empty topological space and $G = \text{Aut}(X)$ be the set of all continuous bijections from X onto X , then G is a topological group under the composition of functions and topology of uniform convergence on compact sets.
 - Let H be a Hilbert space and $G = \mathcal{U}(H)$ be the set of all unitary operators on H , then G is a topological group under the composition of operators and WOT.
- **inverse semigroups:**
 - Let X be a non-empty set and $S = \text{End}(X)$ be the set of all injections from X into X , then S is a unital inverse semigroup under the composition of functions and inverse of functions.
 - Let H be a Hilbert space and $S = \mathcal{PI}(H)$ be the set of all partial isometries on H , then S is a unital inverse semigroup under the composition of operators and adjoint operation.
- **topological hypergroups:** I am not aware of any universal example (and I doubt that it could exist!)
- **topological groupoids:**
 - Let X be a non-empty topological space and $\mathcal{G} = C(X)$ be the set of all continuous functions from X into X , then \mathcal{G} is a topological group under the range and domain maps and composition of functions ($f \circ g$ is defined only when $R_f = D_g$) and topology of uniform convergence on compact sets.
 - Let H be a Hilbert space and $\mathcal{G} = C(H)$ be the set of all closed (possibly unbounded) operators on H , then \mathcal{G} is a topological groupoid under the composition of operators and WOT.

3. MORE EXAMPLES

- **topological groups:**

- Abelian groups: $(\mathbb{Z}, +)$, (\mathbb{T}, \cdot) , $(\mathbb{R}, +)$,
- non-Abelian groups: $U(2)$, $SU(3)$, Heisenberg $ax + b$, Fundamental group $\pi_1(\mathcal{M})$.

- **inverse semigroups:**

- $(\mathbb{N}, +)$ (with $n^* = n$),
- Cuntz semigroup S_n generated by n partial isometries $s_1, \dots, s_n \in B(H)$ satisfying $s_i^* s_i = I$ ($1 \leq i \leq n$) and $\sum_{i=1}^n s_i s_i^* = I$.

- **topological hypergroups:**

- finite hypergroup $\mathbb{Z}_\theta(2) = \{0, 1\}$ with

$$\delta_1 * \delta_1 = \theta \delta_0 + (1 - \theta) \delta_1 \quad (0 \leq \theta \leq 1),$$

- commutative infinite join hypergroup \mathbb{N} with $\delta_n * \delta_m = \delta_{\max(n,m)}$ ($n \neq m$) and

$$\delta_n * \delta_n = b_n \prod_{k=1}^{n-1} \frac{b_k}{b^k + 1} \delta_0 + \sum_{j=1}^{n-1} \frac{b_n}{b_j} \prod_{k=1}^{n-1} \frac{b_k}{b^k + 1} \delta_j + (1 - b_n) \delta_n.$$

- commutative class hypergroup $K(G)$ =representatives of conjugacy classes of a finite group G , for instance $K(S_3) = \{c_0, c_1, c_2\}$ with

$$\delta_{c_1} * \delta_{c_1} = \frac{1}{3} \delta_{c_0} + \frac{2}{3} \delta_{c_2}, \quad \delta_{c_1} * \delta_{c_2} = \delta_{c_1}, \quad \delta_{c_2} * \delta_{c_2} = \frac{1}{2} \delta_{c_0} + \frac{1}{2} \delta_{c_2}.$$

- discrete dual hypergroup \hat{G} =the dual space of a compact group G with

$$\delta_\pi * \delta_\rho = \sum_{\tau \in \pi \otimes \rho} \frac{\dim \tau}{\dim \pi \cdot \dim \rho} m_{\tau, \pi, \rho} \delta_\tau,$$

for instance $\hat{S}_3 = \{\chi_0, \chi_1, \chi_2\}$ with

$$\delta_{\chi_1} * \delta_{\chi_1} = \delta_{\chi_0}, \quad \delta_{\chi_1} * \delta_{\chi_2} = \delta_{\chi_2}, \quad \delta_{\chi_2} * \delta_{\chi_2} = \frac{1}{4} \delta_{\chi_0} + \frac{1}{4} \delta_{\chi_1} + \frac{1}{2} \delta_{\chi_2}.$$

- **topological groupoids:**

- group actions: If a (topological) group G acts on a set (topological space) X , then $\mathcal{G} = X \times G$ is a (topological) groupoid with $\mathcal{G}^{(0)} = X$ and $r(x, g) = x$, $s(x, g) = x.g$, $(x, g).(x.g, h) = (x, gh)$.

- vector bundles: If $\pi : \mathcal{E} \rightarrow \mathcal{M}$ is a (smooth) vector bundle over a (smooth) manifold \mathcal{M} , then \mathcal{E} is a groupoid with $\mathcal{E}^{(0)} = \mathcal{M}$, and $s(e) = r(e) = \pi(e)$, $e.f = e + f$.

- Connes holonomy groupoid: For the foliated space $(\mathcal{M}, \mathcal{F})$ the set \mathcal{G} of triples $(x, y, [\alpha])$ where x and y lie on the same leaf L and $[\alpha]$ is the holonomy class of a piecewise smooth curve α from x to y in L , is a (smooth) groupoid with $\mathcal{G}^{(0)} = \mathcal{M}$ and $r(x, y, [\alpha]) = y$, $s(x, y, [\alpha]) = x$, $(y, z, [\beta]).(x, y, [\alpha]) = (x, z, [\alpha.\beta])$, where $\alpha.\beta$ is the concatenation of the curves α and β . This groupoid is not Hausdorff in general.

- tangent groupoid: Let \mathcal{M} be a smooth manifold with tangent bundle $T\mathcal{M}$, then $\mathcal{G} = (\mathcal{M} \times \mathcal{M} \times (0, 1]) \cup T\mathcal{M}$ is a smooth groupoid.

- Cuntz groupoid \mathcal{O}_n [17].

4. REPRESENTATIONS

- **topological groups:** A representation of a topological group G is a pair $\{\pi, \mathcal{H}_\pi\}$ where \mathcal{H}_π is a Hilbert space and $\pi : G \rightarrow \mathcal{U}(\mathcal{H}_\pi)$ is a continuous group homomorphism in WOT. An example of particular interest is the left regular representation $\pi_L : G \rightarrow \mathcal{B}(L^2(G))$ defined by

$$\pi_L(x)\xi(y) = \xi(x^{-1}y) \quad (x, y \in G, \xi \in L^2(G)).$$

This is a faithful representation .

- **(Schur)** Irreducible representations of compact groups are finite dimensional.

- **inverse semigroups:** A representation of an inverse semigroup S is a pair $\{\pi, \mathcal{H}\}$ where \mathcal{H} is a Hilbert space with subspaces \mathcal{H}_e for $e \in E_S$ such that for each $s \in S$, $\pi(s) : \mathcal{H}_{ss^*} \rightarrow \mathcal{H}_{s^*s}$ is a partial isometry [20]. The left regular representation $\pi_L : S \rightarrow \mathcal{B}(\ell^2(S))$ is defined by

$$\pi_L(x)\xi(y) = \begin{cases} \xi(x^*y) & \text{if } xx^* \geq y^*y \\ 0 & \text{otherwise} \end{cases} \quad (\xi \in \ell^2(S), x, y \in S),$$

and it is faithful [22].

- I don't know if irreducible representations of finite inverse semigroups are finite dimensional.

- **topological hypergroups:** A representation of a topological hypergroup K is a $*$ -representation $\{\pi, \mathcal{H}\}$ of the Banach $*$ -algebra $M(K)$ such that $\pi(\delta_e) = I$ and, for each $\xi, \zeta \in \mathcal{H}$, the map $\mu \mapsto \langle \pi(\mu)\zeta, \xi \rangle$ is continuous on $M_+(K)$ with weak topology. Left regular representation has to be defined by a convolution operator. They are not well studied.

- Irreducible representations of compact hypergroups are finite dimensional [8].

- **topological groupoids:** A representation of a topological groupoid \mathcal{G} is a triple $(\pi, \mathcal{H}_\pi, \mu_\pi)$, where $\mathcal{H}_\pi = \{\mathcal{H}_u^\pi\}$ is a bundle of Hilbert spaces, μ_π is a quasi-invariant measure on X (with associated measures ν, ν^{-1}, ν^2 , and ν_0) such that

- (i) $\pi(x) \in \mathcal{B}(\mathcal{H}_{s(x)}^\pi, \mathcal{H}_{r(x)}^\pi) \quad (x \in \mathcal{G})$,
- (ii) $\pi(u) = id_u : \mathcal{H}_u^\pi \rightarrow \mathcal{H}_u^\pi \quad (u \in X)$,
- (iii) $\pi(xy) = \pi(x)\pi(y)$ for ν^2 -a.e. $(x, y) \in \mathcal{G}^{(2)}$,
- (iv) $\pi(x^{-1}) = \pi(x)^{-1}$ for ν -a.e. $x \in \mathcal{G}$,
- (v) $x \mapsto \langle \pi(x)\xi(s(x)), \eta(r(x)) \rangle$ is ν -measurable on \mathcal{G} for each $\xi, \eta \in L^2(\mathcal{G}^{(0)}, \mathcal{H}_\pi, \mu_\pi)$ (see section 7 for a definition of this L^2 -space.)

If the items (i) – (iv) happen everywhere (instead of almost everywhere) and the map in (v) is continuous (instead of measurable), $(\pi, \mathcal{H}_\pi, \mu_\pi)$ is called a continuous representation.

The left regular representation $\pi_L : \mathcal{G} \rightarrow \mathcal{B}(L^2(\mathcal{G}))$ is defined by

$$\pi_L(x)\xi(y) = \xi(x^{-1}y) \quad (x \in \mathcal{G}, y \in \mathcal{G}^{s(x)}, \xi \in L^2(\mathcal{G}^{s(x)})).$$

Note that $L^2(\mathcal{G})$ is defined in the bundle sense. This is a faithful representation.

- Irreducible continuous representations of compact groupoids have finite dimensional fibers [2] (but they are not finite dimensional in general.)

5. DUALITY

• **topological groups:**

- The set \hat{G} of unitary equivalence classes of irreducible representations of a topological group G is locally compact T_0 topological space (not necessarily Hausdorff). \hat{G} is compact (discrete) if G is discrete (compact).

- **(Pontryagin)** The set \hat{G} of characters of an Abelian topological group G is an Abelian topological group under pointwise multiplication and induced weak* topology. $(\hat{G})^\wedge \simeq G$ (Pontryagin duality). \hat{G} is compact (discrete) if and only if G is discrete (compact).

- **(Tannaka-Krein)** When G is compact, one can recover G from \hat{G} (only using the data in \hat{G} we construct the Tannaka group $T(\hat{G})$ and show that $T(\hat{G}) \simeq G$).

• **inverse semigroups:**

- For non-commutative inverse semigroup S , the dual space \hat{S} is not so much studied.

- The set \hat{S} of semi-characters of a unital commutative inverse semigroup S is a commutative compact semigroup under pointwise multiplication and induced weak* topology. $(\hat{S})^\wedge \not\cong S$ in general.

• **topological hypergroups:**

- For non-commutative hypergroup K , the dual space \hat{K} is somehow studied [8].

- The set \hat{K} of characters of a commutative hypergroup K is not a hypergroup in general. Even if \hat{K} is a hypergroup, $(\hat{K})^\wedge \not\cong K$ in general (unless K is discrete). If K is compact (discrete) then \hat{K} is discrete (compact).

• **topological groupoids:**

- For topological groupoid \mathcal{G} , the dual space $\hat{\mathcal{G}}$ is not studied at all.

- For a compact groupoid \mathcal{G} , it is shown that \mathcal{G} could be recovered from $\hat{\mathcal{G}}$ [2] (by constructing the Tannaka groupoid of $\hat{\mathcal{G}}$.)

- A commutative groupoid \mathcal{G} is a bundle of Abelian groups and so not very interesting.

- **(Peter-Weyl Theorem)** Let \mathcal{G} be a compact groupoid, then for each $u, v \in X$, $\mathcal{E}_{u,v}^\pi$ is dense in $C(\mathcal{G}_u^v)$ and

$$L^2(\mathcal{G}_u^v, \lambda_u^v) = \bigoplus_{\pi \in \hat{\mathcal{G}}} \mathcal{E}_{u,v}^\pi,$$

and

$$\{\sqrt{d_u^\pi} \pi_{u,v}^{ij} : \pi \in \hat{\mathcal{G}}, 1 \leq i \leq d_v^\pi, 1 \leq j \leq d_u^\pi\},$$

is an orthonormal basis for $L^2(\mathcal{G}_u^v, \lambda_u^v)$. Each $\pi \in \hat{\mathcal{G}}$ occurs in the right and left regular representation of \mathcal{G} over $L^2(\mathcal{G}_u^v, \lambda_u^v)$ with multiplicity d_u^π [2].

6. L^1 -ALGEBRAS

- **topological groups:** For a topological group G with left Haar measure λ ,

$$L^1(G) = \{f : G \rightarrow \mathbb{C} : f \text{ Borel measurable, } \|f\|_1 = \int_G |f| d\lambda < \infty\}$$

is a Banach $*$ -algebra with respect to the following operations and above norm

$$f * g(x) = \int_G f(y)g(y^{-1}x)d\lambda(y), \quad f^*(x) = \Delta(x^{-1})\bar{f}(x^{-1}).$$

- $L^1(G)$ has a bounded approximate identity. It is amenable (as a Banach algebra) iff G is amenable (as a group.)

- **inverse semigroups:** For an inverse semigroup S ,

$$\ell^1(S) = \{f : S \rightarrow \mathbb{C} : \|f\|_1 = \sum_{s \in S} |f(s)| < \infty\}$$

is a Banach $*$ -algebra with respect to the following operations and above norm

$$f * g(x) = \sum_{st=x} f(s)g(t), \quad f^*(x) = \bar{f}(x^*).$$

- $\ell^1(S)$ fails to have a bounded approximate identity in general. If it is amenable (as a Banach algebra) then E_S is finite. In this case, amenability of $\ell^1(S)$ is equivalent to amenability of all maximal subgroups of S [14]. On the other hand $\ell^1(S)$ is amenable as a Banach $\ell^1(E_S)$ -module iff S is amenable [3].

- **topological hypergroups:** They don't have an L^1 -algebra in general. Some work has been done for measured hypergroups [8].

- **topological groupoids:** There is a bundle version of $L^1(\mathcal{G})$, but it is more convenient to work with continuous functions with compact support in this case. As \mathcal{G} is not Hausdorff in general, one should define this algebra carefully. Indeed $C_c(\mathcal{G})$ is defined as the span of the set of functions $f : \mathcal{G} \rightarrow \mathbb{C}$ which are continuous of compact support on an open Hausdorff subset of \mathcal{G} and zero outside. It is a normed $*$ -algebra with respect to the following operations and norm

$$f * g(x) = \int_{\mathcal{G}^{r(x)}} f(y)g(y^{-1}x)d\lambda^{r(x)}(y), \quad f^*(x) = \bar{f}(x^{-1}),$$

$$\|f\|_I = \max\left(\sup_{u \in \mathcal{G}^{(0)}} \int_{\mathcal{G}^u} |f| d\lambda^u, \sup_{u \in \mathcal{G}^{(0)}} \int_{\mathcal{G}_u} |f| d\lambda_u\right).$$

- Note that elements of $C_c(\mathcal{G})$ might fail to be continuous on whole of \mathcal{G} . Also $C_c(\mathcal{G})$ isn't closed under pointwise multiplication and taking absolute values in general. It has a bounded approximate identity, when \mathcal{G} is r -discrete [14].

7. C^* -ALGEBRAS

- **topological groups:** For $\pi \in \text{Rep}(G)$ and $f \in L^1(G)$ put

$$\tilde{\pi}(f) = \int_G f(x)\pi(x)d\lambda(x),$$

where the right hand side is a Bochner integral, and define

$$\|f\| = \sup_{\pi} \|\tilde{\pi}(f)\|, \quad \|f\|_r = \|\tilde{\pi}_L(f)\|.$$

The completions of $L^1(G)$ with respect to these C^* -norms are called the full and reduced group C^* -algebras and are denoted by $C^*(G)$ and $C_r^*(G)$, respectively.

- $C_r^*(G)$ is nuclear iff G is amenable iff $C^*(G) \simeq C_r^*(G)$.

- **inverse semigroups:** For $\pi \in \text{Rep}(S)$ and $f \in \ell^1(S)$ put

$$\tilde{\pi}(f) = \sum_S f(s)\pi(s),$$

where the right hand is convergent in WOT, and define

$$\|f\| = \sup_{\pi} \|\tilde{\pi}(f)\|, \quad \|f\|_r = \|\tilde{\pi}_L(f)\|.$$

The completions of $\ell^1(S)$ with respect to these C^* -norms are called the full and reduced semigroup C^* -algebras and are denoted by $C^*(S)$ and $C_r^*(S)$, respectively. A few examples are instructive:

- If \mathcal{F} is the free monogenic inverse semigroup, then $C^*(\mathcal{F}) \subseteq \mathcal{B}(\oplus_{n=2}^{\infty} \mathbb{C}^n)$ is the C^* -algebra generated by the infinite truncated shift (and its adjoint).

- If G is a (discrete) group with identity e , P is a subsemigroup of G containing e with $PP^{-1} = G$, and β_x is the compression of $\pi_L(x)$ to $\ell^2(P)$, then the inverse semigroup $S_{G,P}$ is studied in [13]. When $G = \mathbb{R}$ and $P = \mathbb{R} \cup \{0\}$, $C_r^*(S_{G,P})$ is the Wiener-Hopf algebra.

- If A is an $n \times n$ matrix with $0, 1$ entries and nonzero rows and columns and s_1, \dots, s_n are partial isometries on a Hilbert space \mathcal{H} such that, for $p_i = s_i s_i^*$ and $q_i = s_i^* s_i$,

$$p_i p_j = 0 \ (i \neq j), \quad q_i = \sum_j A_{ij} p_j$$

then the C^* -algebra \mathcal{O}_A generated by s_i 's (and their adjoints) is called the Cuntz-Kreiger algebra of A . (Note that \mathcal{O}_A is not independent of the Hilbert space \mathcal{H} on which the partial isometries are represented.) When all entries of A are 1, \mathcal{O}_A is denoted by \mathcal{O}_n and is called the Cuntz algebra. One can associate an inverse semigroup S_A to A with $C^*(S_A) \simeq \mathcal{O}_A$ [14].

For any inverse semigroup S we have the following characterization of nuclearity of reduced semigroup C^* -algebra.

- $C_r^*(S)$ is nuclear iff a family of subsemigroups of S indexed by the spectrum of the commutative C^* -algebra $C^*(E_S)$ are amenable and S is hyperfinite. In this case, $C^*(S) \simeq C_r^*(S)$ [4].

- **topological hypergroups:** The full C^* -algebra of a measured hypergroup could be defined similarly. They are barely studied.

- **topological groupoids:** For $\pi \in \text{Rep}(\mathcal{G})$ and $f \in C_c(\mathcal{G})$ put

$$\langle \tilde{\pi}(f)\xi, \eta \rangle = \int_{\mathcal{G}} f(x) \langle \pi(x)\xi(s(x)), \eta(r(x)) \rangle d\nu_0(x),$$

and define

$$\|f\| = \sup_{\pi} \|\tilde{\pi}(f)\|, \quad \|f\|_r = \|\tilde{\pi}_L(f)\|.$$

Note that $\tilde{\pi}(f)$ is an operator in $\mathcal{B}(L^2(\mathcal{G}^{(0)}, \mathcal{H}_\pi, \mu_\pi))$, where

$$L^2(\mathcal{G}^{(0)}, \mathcal{H}_\pi, \mu_\pi) = \{f : \mathcal{G}^{(0)} \rightarrow \bigcup_u \mathcal{H}_\pi^u : \int_{\mathcal{G}^{(0)}} \|f(u)\|_u^2 d\mu_\pi < \infty\}.$$

The completions of $L^1(\mathcal{G})$ with respect to these C^* -norms are called the full and reduced group C^* -algebras and are denoted by $C^*(\mathcal{G})$ and $C_r^*(\mathcal{G})$, respectively.

- If \mathcal{G} has a continuous Haar system and is measurewise amenable, then $C^*(\mathcal{G}) \simeq C_r^*(\mathcal{G})$ [7]. The converse isn't known.

8. DYNAMICAL SYSTEMS AND CROSSED PRODUCTS

- **topological groups:** A topological group G acts on a C^* -algebra A by automorphisms via a group homomorphism $\alpha : G \rightarrow \text{Aut}(A)$ such that maps $x \mapsto \alpha_x(a)$ are continuous, for each $a \in A$. Then

$$L^1(G, A) = \{f : G \rightarrow A : f \text{ Borel measurable, } \int_G \|f(x)\| d\lambda(x) < \infty\}$$

is a Banach $*$ -algebra with respect to the following operations and above norm

$$f * g(x) = \int_G f(y) \alpha_y(g(y^{-1}x)) d\lambda(y), \quad f^*(x) = \Delta(x^{-1}) \alpha_x(f(x^{-1}))^*.$$

If π is a non-degenerate representation of A and u is a unitary representations of G (on the same Hilbert space \mathcal{H}) with

$$\pi(\alpha_x(a)) = u_x \pi(a) u_x^* \quad (a \in A, x \in G),$$

then $\{\pi, u, \mathcal{H}\}$ is called a covariant representation of the dynamical system (A, G, α) . In this case the Bochner integral

$$(\pi \times u)(f) = \int_G \pi(f(x)) u_x d\lambda(x)$$

defines a non-degenerate $*$ -representation of $L^1(G, A)$ and these exhaust all such representations. Also if

$$\tilde{\pi}(a)\xi(x) = \pi(\alpha_{x^{-1}}(a))\xi(x), \quad \pi_L(x)\xi(y) = \xi(x^{-1}y),$$

for $x, y \in G, a \in A, \xi \in L^2(G, \mathcal{H})$, then $\text{Ind}(\pi) = \tilde{\pi} \times \pi_L$ is the left regular representation induced by π and we have the C^* -norms

$$\|f\| = \sup_{\pi, u} \|(\pi \times u)(f)\|, \quad \|f\|_r = \sup_{\pi} \|\text{Ind}(\pi)(f)\|.$$

The completions of $L^1(G, A)$ with respect to these C^* -norms are called the full and reduced C^* -crossed products of A by G and are denoted by $A \rtimes_\alpha G$ and $A \rtimes_{\alpha, r} G$, respectively. If $\alpha = tr$ is the trivial action we get

$$A \rtimes_\alpha G \simeq A \otimes_{\max} C^*(G), \quad A \rtimes_{\alpha, r} G \simeq A \otimes_{\min} C_r^*(G).$$

- **inverse semigroups:** A unital inverse semigroup S acts on a C^* -algebra A by partial automorphisms via a semigroup homomorphism $\beta : S \rightarrow \text{PAut}(A)$ sending s to the triple (β_s, E_{s^*}, E_s) , where $\beta_s : E_{s^*} \rightarrow E_s$ is an isomorphism between two closed ideals of A (that's what a partial automorphism is). Then

$$L = \{f \in \ell^1(S, A) : f(s) \in E_s\}$$

is a Banach $*$ -algebra with respect to the following operations and ℓ^1 -norm

$$f * g(s) = \sum_{rt=s} \beta_r(\beta_{r^*}(f(r))g(t)), \quad f^*(s) = \beta_s(f(s^*)^*).$$

If π is a non-degenerate representation of A and u is a representations of S (on the same Hilbert space \mathcal{H}) with

$$\pi(\beta_s(a)) = u_s \pi(a) u_{s^*} \quad (s \in S, a \in E_{s^*}),$$

then $\{\pi, u, \mathcal{H}\}$ is called a covariant representation of the dynamical system (A, S, β) . In this case the WOT-convergent sum

$$(\pi \times u)(f) = \sum_{s \in S} \pi(f(s)) u_s$$

defines a non-degenerate $*$ -representation of L and these exhaust all (coherent) representations. Also we have the C^* -norm

$$\|f\| = \sup_{\pi, u} \|(\pi \times u)(f)\|,$$

and if $I = \{f \in L : \|f\| = 0\}$, the completions of L/I with respect to this C^* -norm is called the full C^* -crossed product of A by S and is denoted by $A \rtimes_{\beta} S$. If $\beta = tr$ is the trivial action and $A = \mathbb{C}$ we get $\mathbb{C} \rtimes_{tr} S \simeq C^*(G(S))$, where $G(S)$ is the maximal group homomorphic image of S . Also S naturally acts on $C^*(E_S)$ by

$$\beta_s(\delta_e) = \delta_{ses^*} \quad (s \in S, e \in E_S),$$

and $C^*(E_S) \rtimes_{\beta} S \simeq C^*(S)$ [16].

- **topological hypergroups:** There is no notion of hypergroup C^* -dynamical system, but one can easily define them for measured hypergroups.
- **topological groupoids:** A topological groupoid \mathcal{G} acts on a C^* -bundle $\mathcal{A} = \{A_u\}_{u \in \mathcal{G}^{(0)}}$ by automorphisms via a map $\alpha : \mathcal{G} \rightarrow \text{Aut}(\mathcal{A})$ such that maps $(a, x) \mapsto \alpha_x(a)$ (with $a \in A_{s(x)}$) are continuous and maps $\alpha_x : A_{s(x)} \rightarrow A_{r(x)}$ are C^* -algebra isomorphisms, and $\alpha_{xy} = \alpha_x \alpha_y$, whenever $(x, y) \in \mathcal{G}^{(2)}$. \mathcal{A} is assumed to be continuous [18] or upper-semicontinuous [16]. Pulling back via $r : \mathcal{G} \rightarrow \mathcal{G}^{(0)}$, we get the Banach bundle

$$r^* \mathcal{A} = \{(a, x) : a \in A_{r(x)}\}$$

with set of compactly supported continuous sections $C_c(\mathcal{G}, r^* \mathcal{A})$ endowed with the following operations

$$f * g(x) = \int_{\mathcal{G}^{r(x)}} f(y) \alpha_y(g(y^{-1}x)) d\lambda^{r(x)}(y), \quad f^*(x) = \alpha_x(f(x^{-1}))^*.$$

Put $\|f\| = \sup_{\pi} \|\pi(f)\|$, where π ranges over all $*$ -representations of the normed algebra $C_c(\mathcal{G}, r^* \mathcal{A})$ which are continuous from the inductive limit topology to WOT (automatic when \mathcal{G} is r -discrete.) The completion of $C_c(\mathcal{G}, r^* \mathcal{A})$ in this C^* -norm is called the groupoid crossed product and is denoted by $\mathcal{A} \rtimes_{\alpha} \mathcal{G}$. The reduced C^* -crossed product $\mathcal{A} \rtimes_{\alpha, r} \mathcal{G}$ could be defined similar to the group case. The regular representations used here are exactly the representations induced from $C_0(\mathcal{G}^{(0)}, \mathcal{A})$ via the generalized conditional expectation $P : C_c(\mathcal{G}, r^* \mathcal{A}) \rightarrow C_c(\mathcal{G}^{(0)}, \mathcal{A})$ defined by restriction [7].

- If \mathcal{G} has a continuous Haar system and is measurewise amenable, then $\mathcal{A} \rtimes_{\alpha} \mathcal{G} \simeq \mathcal{A} \rtimes_{\alpha, r} \mathcal{G}$ [7]. The converse isn't known.

9. TRANSITIONS

- **from groupoids to inverse semigroups:** For a topological groupoid \mathcal{G} , let's consider the set \mathcal{G}^{op} consisting of those open Hausdorff $U \subseteq \mathcal{G}$ on which r and s are homeomorphisms with open ranges. If \mathcal{G} is Hausdorff and r -discrete (i.e. s, r are local homeomorphisms) \mathcal{G}^{op} consists of open \mathcal{G} -sets and it is an inverse semigroup under the operations

$$vw = \{xy : (x, y) \in (v \times w) \cap \mathcal{G}^{(2)}\}, \quad v^* = \{x^{-1} : x \in v\}.$$

A subsemigroup S of \mathcal{G}^{op} is full if it is a base for the topology of \mathcal{G} and E_S is upward directed. In this case, if \mathcal{G} acts by α on a C^* -bundle $\mathcal{A} = \{A_u\}_{u \in \mathcal{G}^{(0)}}$, then S acts on $C_0(\mathcal{G}, \mathcal{A})$ by

$$\beta_s(f)(u) = \begin{cases} \alpha_{us}(f(s^*us)) & u \in r(s) \\ 0 & \text{otherwise} \end{cases} \quad (s \in S, f \in B_{s(s)}, u \in \mathcal{G}^{(0)}),$$

where

$$B_e = \{f \in C_0(\mathcal{G}, \mathcal{A}) : f = 0 \text{ off } e\} \quad (e \in E_S).$$

Conversely, if S acts by β on a $C_0(\mathcal{G}^{(0)})$ -algebra A (a C^* -algebra with a faithful non-degenerate homomorphism ϕ from $C_0(\mathcal{G}^{(0)})$ into the algebra $ZM(A)$ of central multipliers of A .) then for

$$I_u = \{f \in C_0(\mathcal{G}^{(0)}) : f(u) = 0\} \quad (u \in \mathcal{G}^{(0)})$$

and $A_u = A/(\phi(I_u)A)$, \mathcal{G} acts on $\mathcal{A} = \bigcup_{u \in \mathcal{G}^{(0)}} A_u$ by

$$\alpha_x(f(s(x))) = \beta_s(f)(r(x)) \quad (x \in s \in S, f \in B_{s(s)}).$$

In this case we have $A \rtimes_{\beta} S \simeq \mathcal{A} \rtimes_{\alpha} \mathcal{G}$. In particular, when $\alpha = tr$, $S = \mathcal{G}^{op}$, and $\mathcal{A} = C_0(\mathcal{G}^{(0)})$,

$$C^*(\mathcal{G}) \simeq C_0(\mathcal{G}^{(0)}) \rtimes_{\beta} \mathcal{G}^{op} [16].$$

- **from inverse semigroups to groupoids:** If S is an inverse semigroup with idempotents $E = E_S$, a filter in E is a subsemigroup $A \leq E$ such that $e \in A, f \in E$ and $f \geq e$ implies $f \in A$. Take $X = \ell^1(E)$ (the maximal ideal space of the commutative Banach algebra $\ell^1(E)$ which could be identified with the space of nonzero characters with w^* -topology,) then for each $x \in X$, $A_x = \{e \in E : x(e) = 1\}$ is a filter in E and these exhaust all possible filters (given a filter A , simply put $x = \chi_A$ to get $A = A_x$.) In particular for $e \in E$, $\bar{e} = \{f \in E : f \geq e\}$ is a filter in E and $e \mapsto \bar{e}$ maps E onto a dense subset of X . For $e \in E$ and $s \in S$ put

$$D_e = \{x \in X : x(e) = 1\}, \quad D_s = D_{ss^*}, \quad R_s = D_{s^*}.$$

Then S acts on X by

$$x.s(e) = x(ses^*) \quad (s \in S, x \in X, e \in E)$$

and the map $x \mapsto x.s =: \beta(s)(x)$ is a homeomorphism from D_s onto R_s . Put

$$\Sigma = \{(x, s) : x \in D_s, s \in S\}$$

and set $(x, s) \sim (y, t)$ iff $x = y, es = st$ and $x(e) = 1$, for some $e \in E$. Then $\mathcal{G}_S = \Sigma / \sim$ is a groupoid under operations

$$[x, s].[x.s, t] = [x, st], \quad [x, s]^{-1} = [x.s, s^*] \quad (s, t \in S, x \in X),$$

where $[x, s]$ is the equivalent class of (x, s) . This is called the universal groupoid of S . It is an r -discrete (Hausdorff, when S is E -unitary) groupoid with $\mathcal{G}_S^{(0)} = X$ and $\mathcal{G}_S^{op} = \{[U, s] : s \in S, U \subseteq D_s \text{ open}\}$ with $S \leq \mathcal{G}_S^{op}$ full.

If S acts on a C^* -algebra A by subquotients [12] through α , then $E \rtimes_{\alpha} A$ is a $C_0(\mathcal{G}_S^{(0)})$ -algebra and \mathcal{G}_S acts on $E \rtimes_{\alpha} A$ by $\tilde{\alpha}$ such that

$$\mathcal{G}_S \rtimes_{\tilde{\alpha}} (E \rtimes_{\alpha} A) \simeq S \rtimes_{\alpha} A, \quad \mathcal{G}_S \rtimes_{\tilde{\alpha}, r} (E \rtimes_{\alpha} A) \simeq S \rtimes_{\alpha, r} A.$$

In particular

$$C^*(\mathcal{G}_S) \simeq C^*(S), \quad C_r^*(\mathcal{G}_S) \simeq C_r^*(S) [12], [14].$$

10. OPEN PROBLEMS

There are non-compact groups (Moore groups) whose irreducible representations are all finite dimensional.

- *Is there a Moore theory for inverse semigroups or topological hypergroups?*

If S is an inverse semigroup, then $C_r^*(S)$ and $VN(S)$ are the C^* -algebra and von-Neumann algebra generate by $\{\pi_L(s) : s \in S\}$ in $\mathcal{B}(\ell^2(S))$, respectively. The necessary and sufficient condition for nuclearity of $C_r^*(S)$ are obtained in [4]. Also in [14], it is shown that if all maximal subgroups of S (indexed by E_S) are amenable, then $VN(S)$ is injective.

- *Characterize all inverse semigroups S for which $VN(S)$ is injective.*

One possible attack is to use the characterization of amenability of \mathcal{G}_S given in [4] and relating injectivity of $VN(S)$ to that of $VN(\mathcal{G}_S)$.

- *For which topological groupoids \mathcal{G} , $C^*(\mathcal{G}) \simeq C_r^*(\mathcal{G})$? What about $\mathcal{G} \rtimes_{\alpha} A \simeq \mathcal{G} \rtimes_{\alpha, r} A$?*

A sufficient condition is that \mathcal{G} has a continuous Haar system and it is measurewise amenable [7].

- *For which inverse semigroups S , $C^*(S) \simeq C_r^*(S)$? What about $S \rtimes_{\alpha} A \simeq S \rtimes_{\alpha, r} A$?*

A sufficient condition is given in [4].

There is a well-established theory of C^* -bundles over (discrete) groups [10] and their associated full and reduced C^* -algebras (which are natural generalizations of C^* -crossed products.)

- *What about C^* -bundles over inverse semigroups?*

There are some attacks, but many things left to be explored.

For a topological group G , there is a rich operator space structure on the Banach algebra $B(G) = C^*(G)^*$ and its closed ideals including the well known Fourier algebra $A(G)$. In particular $A(G)$ is operator amenable iff G is amenable [20].

- *What about inverse semigroups or groupoids?*

The Fourier algebras of inverse semigroups and topological groupoids are studied in [5],[6],[19].

Green has shown that the Mackey-Rieffel machine could be applied to group C^* -crossed products in order to study induced representations from subgroups.

- *Is there a corresponding technique to study induced representations of groupoid or inverse semigroup crossed products?*

For groupoids we have the notion of imprimitivity groupoid by Muhly and Williams. There is a work in progress for induced representations of groupoid crossed products [1].

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