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THE WAITING TIME FOR SOME INTERVAL TRANSFORMATIONS

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ABSTRACT. We discuss the asymptotic behaviour of the waiting time for some transformation on the interval including irrational rotations. Like the first return time formula, logarithm of the waiting time to a ball divided by logarithm of the radius of the ball goes to 1 as the radius goes to 0 for many transformations.

Let T be a transformation from $I = [0, 1)$ onto itself and let $Q_n(x)$ be the subinterval $[\,i/2^n, (i+1)/2^n)$, $0 \le i < 2^n$ containing x. Define $K_n(x) = \min\{j \ge 1 : T^j(x) \in Q_n(x)\}$ and $K_n(x, y) = \min\{j \ge 1 : T^{j-1}(y) \in Q_n(x)\}$. For various transformations defined on I, we show that

$$\lim_{n\to\infty} \frac{\log K_n(x)}{n} = 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{\log K_n(x, y)}{n} = 1 \quad \text{a.e.}$$

Let $Tx = x + \theta \pmod 1$. Then for irrational θ of type η

$$\liminf_{n\to\infty} \frac{\log K_n(x, y)}{n} = 1 \quad \text{and} \quad \limsup_{n\to\infty} \frac{\log K_n(x, y)}{n} = \eta \quad \text{a.e.}$$

Since the set of irrational numbers of type 1 has measure 1, for almost every θ the limit exists and is 1.

1. INTRODUCTION

Let μ be a probability measure on X and $T : X \to X$ be a μ-preserving transformation. For a measurable subset $E \subset X$ with $\mu(E) > 0$ and a point $x \in E$ which returns to E under iteration by T, we define the first return time R_E on E by

$$R_E(x) = \min\left\{j \ge 1 : T^j x \in E\right\}.$$

Kac's lemma[10] states that

$$\int_E R_E(x)d\mu \le 1.$$

If T is ergodic, then the equality holds. Define the waiting time of $E \subset X$ at $x \in X$ by

$$W_E(x) = \min\{j \ge 1 : T^{j-1}x \in E\}.$$

The relation between the first return time and entropy has been studied since Wyner and Ziv's work[24]. For each sample sequence $x = (x_1x_2\dots)$ from ergodic stationary source, let R_n be the first return time of first n-block reappears, i.e.,

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$R_n(x) = \min\{j \geq 1 : x_1 \dots x_n = x_{j+1} \dots x_{j+n}\}$. Ornstein and Weiss[16] showed that for an ergodic stationary process

$$\lim_{n\to\infty} \frac{1}{n} \log R_n(x) = \text{entropy} \tag{1}$$

almost surely. Through the paper 'log' denotes the base 2 logarithm. The waiting time $W_n(x, y)$ is defined by $W_n(x, y) = \min\{j \geq 1 : x_1 \dots x_n = y_j \dots y_{j+n-1}\}$. Marton and Shields[15] proved that for a weak Bernoulli process

$$\lim_{n\to\infty} \frac{1}{n} \log W_n(x, y) = \text{entropy} \tag{2}$$

almost surely with respect to the product measure. But this convergence does not hold for general ergodic processes[21]. For a comprehensive reference see [22].

Let $\mathcal{P}$ be a finite partition of X and $\mathcal{P}_n = \mathcal{P} \vee \cdots \vee T^{-n+1}\mathcal{P}$, where $\mathcal{P} \vee \mathcal{Q} = \{P \cap Q : P \in \mathcal{P}, Q \in \mathcal{Q}\}$. Let $P_n(x)$ be the element of $\mathcal{P}_n$ containing x. Then (1) is rewritten by

$$\lim_{n\to\infty} \frac{1}{n} \log R_{P_n(x)}(x) = h(T, \mathcal{P}), \tag{3}$$

for almost every $x \in X$, where $h(T, \mathcal{P})$ is the entropy with respect to $\mathcal{P}$. Therefore, by the Shannon-McMillan-Brieman theorem, if $h(T, \mathcal{P})$ is positive, then we have

$$\lim_{n\to\infty} \frac{\log R_{P_n(x)}(x)}{-\log \mu(P_n(x))} = 1 \quad \text{a.e.}$$

and if T has Bernoulli property,

$$\lim_{n\to\infty} \frac{\log W_{P_n(x)}(y)}{-\log \mu(P_n(x))} = 1 \quad \text{a.e.}$$

We apply these results to ergodic transforms on the unit interval $I = [0, 1)$. Let $\mathcal{Q}_n$ be a partition of I into sets with points which have the same binary expansion in their first n places, i.e.,

$$\mathcal{Q}_n = \left\{ \left[\frac{j}{2^n}, \frac{j+1}{2^n}\right) : 0 \leq j \leq 2^n - 1 \right\}$$

and $Q_n(x)$ be the element of $\mathcal{Q}_n$ containing $x \in I$.

Define the first return time $K_n(x)$ and the waiting time $K_n(x, y)$ as follows:

$$K_n(x) := R_{Q_n(x)}(x) = \min\{j \geq 1 : T^j(x) \in Q_n(x)\},$$

$$K_n(x, y) := W_{Q_n(x)}(y) = \min\{j \geq 1 : T^{j-1}(y) \in Q_n(x)\}.$$

If $T(x) = 2x \pmod 1$ on I, then $R_n(x)$ and $W_n(x, y)$ are $K_n(x)$ and $K_n(x, y)$, respectively. If $X \subset \mathbb{R}^d$ for some $d \in \mathbb{N}$, then K_n is defined from

$$\bar{\mathcal{Q}}_n = \{[i_1 2^{-n}, (i_1 + 1)2^{-n}) \times \cdots \times [i_d 2^{-n}, (i_d + 1)2^{-n}) : (i_1, \dots, i_d) \in \mathbb{Z}^d\}$$

and $\mathcal{Q}_n = \{X \cap A : A \in \bar{\mathcal{Q}}_n\}$.

Let (X, d) be a metric space and $B(x, r) = \{y : d(x, y) < r\}$. Let $T : X \to X$ be a Borel measurable transformation on a measurable set X and μ be a T-invariant probability measure on X. Boshernitzan[3] showed that for almost all x

$$\liminf_{n\to\infty} n \cdot d(x, T^n(x)) < \infty.$$

This yields that

$$\liminf_{r\to 0^+} \frac{\log R_{B(x,r)}(x)}{-\log \mu(B(x,r))} \le 1.$$

See also [5].

Define the upper and lower pointwise dimension of μ at x by

$$\bar{d}_\mu(x) = \limsup_{r\to 0^+} \frac{\log \mu(B(x,r))}{\log r}, \quad \underline{d}_\mu(x) = \liminf_{r\to 0^+} \frac{\log \mu(B(x,r))}{\log r}.$$

Now we have another recurrence theorem for the decreasing sequence of balls. Let $T : X \to X$ be a Borel measurable transformation on a measurable set $X \subset \mathbb{R}^d$ for some $d \in \mathbb{N}$ and μ be a T-invariant probability measure on X. Barreira and Saussol[1] showed that

$$\limsup_{r\to 0^+} \frac{\log R_{B(x,r)}(x)}{-\log r} \le \bar{d}_\mu(x), \quad \liminf_{r\to 0^+} \frac{\log R_{B(x,r)}(x)}{-\log r} \le \underline{d}_\mu(x).$$

If $\underline{d}_\mu(x) > 0$ for μ-almost every x, then

$$\limsup_{n\to\infty} \frac{\log K_n(x)}{n} \le 1, \quad \limsup_{r\to 0^+} \frac{\log R_{B(x,r)}(x)}{-\log \mu(B(x,r))} \le 1 \tag{4}$$

for μ-almost every x. See [13] for the proof.

Choe[6] conjectured that

$$\lim_{n\to\infty} \frac{\log K_n(x)}{n} = 1 \text{ a.e.}$$

for a transformation which has a positive entropy and showed that the conjecture is true for a piecewise linear transformation (see also [7]). Saussol, Troubetzkoy and Vaienti obtained similar results independently[20]. For a piecewise monotone transformation with a derivative of bounded p-variation, they showed that if the transformation has a positive entropy, then

$$\lim_{r\to 0^+} \frac{\log R_{B(x,r)}(x)}{-\log r} = \text{dimension of } \mu.$$

See also [1] and [2] for other transformations satisfying the limit. Note that the convergence does not hold for irrational rotations. For almost every irrational number θ, Choe and Seo[8] showed that

$$\lim_{n\to\infty} \frac{\log K_n(x)}{n} = \lim_{r\to 0^+} \frac{\log R_{B(x,r)}(x)}{-\log r} = 1$$

for almost every x. They also showed that

$$\liminf_{n\to\infty} \frac{\log K_n(x)}{n} = \liminf_{r\to 0^+} \frac{\log R_{B(x,r)}(x)}{-\log r} < 1$$

for some types of irrational numbers.

For some kinds of expanding maps Philipp showed the following[18]: Let $T(x) = rx \pmod 1$, $r > 1$ or $T(x) = \{1/x\}$ on I and μ be the unique T-invariant absolutely continuous measure. Let $\{I_k\}$ be a sequence of intervals contained in I and $\phi(N) =$

$\sum_{k \le N} \mu(I_k)$. Put $A(N, x)$ be the number of positive integers $k \le N$ such that $T^k(x) \in I_k$. Then he showed that

$$A(N, x) = \phi(N) + O(\phi^{1/2}(N) \log^{3/2+\epsilon} \phi(N)), \quad \epsilon > 0$$

for almost all $x \in I$. For the more about Borel-Cantelli lemmas, see [4] and [9]. This dynamical Borel-Cantelli lemma immediately implies that

$$\lim_{n \to \infty} \frac{1}{n} \log K_n(x, y) = 1 \text{ a.e.}$$

In Section 2 we show that Choe's conjecture is true for a piecewise monotone transformation which has a positive entropy and an absolutely continuous invariant measure with respect to Lebesgue measure. We investigate the limit behavior of $\log K_n(x)/n$ and $\log K_n(x, y)/n$. See [12] for the detail.

In Section 3 the convergence of $\frac{1}{n} \log K_n(x, y)$ for the irrational rotation is considered. Let $0 < \theta < 1$ be an irrational number and $T : [0, 1) \to [0, 1)$ an irrational rotation, i.e.,

$$Tx = x + \theta \pmod 1.$$

Note that T has 0 entropy.

2. The piecewise monotone transformation on I

Through this section, μ is a T-invariant probability measure and λ is Lebesgue measure on I.

Theorem 1. *Let $\mathcal{P}$ be a finite partition of I into intervals and $T : I \to I$ is monotone on each element of $\mathcal{P}$. Suppose (T, μ) is ergodic with $h(T, \mathcal{P}) > 0$ and μ is absolutely continuous with respect to λ. Then*

$$\lim_{n \to \infty} \frac{\log K_n(x)}{n} = 1,$$

for almost every x.

Proof. It suffices to show $\liminf_{n\to\infty} \log K_n(x)/n \ge 1$ by (4). Note that $P_n(x)$ is an interval for every $n \ge 1$ and $x \in I$ since T is monotone on each element of $\mathcal{P}$. Given $\varepsilon > 0$, let $m(n) = \lfloor n/(h(T, \mathcal{P}) + 2\varepsilon) \rfloor, n \ge 1$, where $\lfloor t \rfloor$ is the largest integer less than or equal to t.

Given $\varepsilon > 0$, $P_n(x)$ is called (n, ε)-*typical subset* if $2^{-n(h+\varepsilon)} < \mu(P_n(x)) < 2^{-n(h-\varepsilon)}$. The Shannon-McMillan-Breiman theorem guarantees the Asymptotic Equipartition Property[17]: Let $\mathcal{T}_n(\varepsilon)$ be the union of all (n, ε)-typical subsets. Suppose T is ergodic with $h(T, \mathcal{P}) < \infty$. Then for almost every x there exists $N = N(x, \varepsilon)$ such that $x \in \mathcal{T}_n(\varepsilon)$ if $n > N$, and the number of (n, ε)-typical subsets is between $(1 - \varepsilon)2^{n(h-\varepsilon)}$ and $2^{n(h+\varepsilon)}$.

Let

$$A_n = \{x \in I : Q_n(x) \not\subseteq P_m(x)\}.$$

Since each $P_m(x)$ is an interval and $\lambda(Q_n(x)) = 2^{-n}$,

$$\begin{aligned}\lambda(\mathcal{T}_m(\varepsilon) \cap A_n) &\le 2 \cdot (\text{number of } (m,\varepsilon)\text{-typical subsets}) \cdot 2^{-n} \\ &\le 2 \cdot 2^{m(h(T,\mathcal{P})+\varepsilon)} \cdot 2^{-n} \\ &\le 2 \cdot 2^{-n(1-(h(T,\mathcal{P})+\varepsilon)/(h(T,\mathcal{P})+2\varepsilon))},\end{aligned}$$

where the second inequality is due to the Asymptotic Equipartition Property. By the Borel-Cantelli lemma, $\lambda(\mathcal{T}_m(\varepsilon) \cap A_n \text{ i.o.}) = 0$, so $\mu(\mathcal{T}_m(\varepsilon) \cap A_n \text{ i.o.}) = 0$ since μ is absolutely continuous with respect to λ. Thus, by the Asymptotic Equipartition Property, we have

$$\mu(A_n \text{ i.o.}) = 0.$$

It implies that for almost every x, there exists $N = N(x, \varepsilon)$ such that if $n > N$, then $x \notin A_k$, i.e., $Q_n(x) \subset P_m(x)$. Therefore

$$K_n(x) \ge R_{P_m(x)}(x).$$

By (3) we have

$$\liminf_{n\to\infty} \frac{\log K_n(x)}{n} \ge \liminf_{n\to\infty} \frac{\log R_{P_m(x)}(x)}{m} \cdot \frac{m}{n} = \frac{h(T,\mathcal{P})}{h(T,\mathcal{P}) + 2\varepsilon}.$$

Since $h(T,\mathcal{P}) > 0$ and $\varepsilon > 0$ is arbitrary, we have the result. □

Lemma 2. *Suppose $T : I \to I$ preserves an absolutely continuous measure μ. Then*

$$\liminf_{n\to\infty} \frac{\log K_n(x,y)}{n} \ge 1,$$

for almost every x and y with respect to the product measure $\mu \times \mu$.

Proof. Fix $y \in I$ and let $A_n := \{x \in I : K_n(x,y) < 2^{n(1-\varepsilon)}\}$. Then A_n is equal to the union of $Q_n(x)$ such that $T^j(y) \in Q_n(x)$ for some $j \le 2^{n(1-\varepsilon)}$. Hence $\lambda(A_n) \le 2^{n(1-\varepsilon)} \cdot 2^{-n} = 2^{-n\varepsilon}$. The Borel-Cantelli lemma implies $\lambda(A_n \text{ i.o.}) = 0$, so $\mu(A_n \text{ i.o.}) = 0$, since μ is absolutely continuous with respect to λ. Thus for almost every x and y

$$\liminf_{n\to\infty} \frac{\log K_n(x,y)}{n} \ge 1 - \varepsilon.$$

□

For a fixed $a \in I$, the following lemma implies that $T^n(x)$ returns to the neighborhood of a infinitely often but the speed is not fast.

Lemma 3. *Let $T : I \to I$ be a μ-preserving transformation with $d\mu = \rho\, d\lambda$, $\rho \in L^p(\lambda)$ for some $p > 1$. Then for every $a \in I$,*

$$\lim_{n\to\infty} \frac{\log |T^n(x) - a|}{n} = 0 \quad a.e.$$

For the proof see [12].

Corollary 4. *Let $d_n(x)/2^n$ be the minimal distance from x to the end points of $Q_n(x)$. Then*

$$\limsup_{n\to\infty} \left(\frac{-\log d_n(x)}{n} \right) = 0,$$

for almost every x with respect to λ.

Proof. Let $T(x) = 2x \pmod 1$ on I. Lemma 3 implies

$$\lim_{n\to\infty} \frac{-\log|T^n(x) - 0|}{n} = 0 \text{ and } \lim_{n\to\infty} \frac{-\log|T^n(x) - 1|}{n} = 0.$$

Since $d_n(x) = \min\{|T^n(x) - 0|, |T^n(x) - 1|\}$, we have the result. □

Lemma 5. *Let $\mathcal{P}$ be a finite partition of I into intervals and $T : I \to I$ is monotone on each element of $\mathcal{P}$. Suppose (T, μ) is ergodic with $d\mu = \rho\, d\lambda$ and $h(T, \mathcal{P}) > 0$. Then for almost every x,*

$$\lim_{n\to\infty} \frac{\mu(P_n(x))}{\lambda(P_n(x))} = \rho(x).$$

Proof. Let $r_n = \lambda(P_n(x))$ and $B(x, r_n)$ be the ball of radius r_n. Since T is monotone on each element of $\mathcal{P}$, $P_n(x)$ is an interval for every $n \geq 1$ and $x \in I$. Hence $P_n(x) \subset B(x, r_n)$ and $\lambda(P_n(x)) \geq \frac{1}{3}\lambda(B(x, r_n))$. Thus

$$\begin{aligned} \left| \frac{\mu(P_n(x))}{\lambda(P_n(x))} - \rho(x) \right| &= \frac{1}{\lambda(P_n(x))} \left| \int_{P_n(x)} \rho(y) - \rho(x)\, d\lambda(y) \right| \\ &\leq \frac{3}{\lambda(B(x, r_n))} \int_{B(x,r_n)} |\rho(y) - \rho(x)|\, d\lambda(y). \end{aligned}$$

Since $h(T, \mathcal{P}) > 0$, the Shannon-McMillan-Breiman theorem guarantees $\lim_n r_n = 0$ for almost every x. Therefore

$$\lim_{n\to\infty} \frac{\mu(P_n(x))}{\lambda(P_n(x))} = \rho(x),$$

for all Lebesgue points, so almost every x. □

Theorem 6. *Let $\mathcal{P}$ be a finite partition of I into intervals and $T : I \to I$ is monotone on each element of $\mathcal{P}$. Suppose $(T, \mathcal{P}, \mu)$ has the Bernoulli property with $h(T, \mathcal{P}) > 0$ and μ is absolutely continuous with respect to λ. Then*

$$\lim_{n\to\infty} \frac{\log K_n(x, y)}{n} = 1,$$

for almost every x and y with respect to the product measure $\mu \times \mu$.

Proof. It suffices to show $\limsup_{n\to\infty} \log K_n(x, y)/n \leq 1$ by Lemma 2. Let $d_n(x)/2^n$ be the minimum distance from x to the end points of $Q_n(x)$. Given $\varepsilon > 0$, let $m = \lceil (n(1 + \varepsilon) - \log(\rho(x) - \varepsilon))/(h(T, \mathcal{P}) - \varepsilon) \rceil$, where $\lceil t \rceil$ is the smallest integer greater than or equal to t. For almost every x, Lemma 5 and the Shannon-McMillan-Breiman theorem guarantee that there exists $N = N(x, \varepsilon)$ such that if $n > N$, then

$$\lambda(P_m(x)) < \frac{\mu(P_m(x))}{\rho(x) - \varepsilon} \leq \frac{2^{-m(h(T,\mathcal{P})-\varepsilon)}}{\rho(x) - \varepsilon} \leq 2^{-n(1+\varepsilon)}.$$

By Corollary 4, we have $\lambda(P_m(x)) < d_n(x)/2^n$ for every sufficiently large n. Since T is monotone on each element of $\mathcal{P}$, $P_n(x)$ is an interval for every $n \geq 1$ and $x \in I$. Hence $P_m(x) \subset Q_n(x)$, so $K_n(x, y) \leq W_m(x, y)$. By (2) we have

$$\limsup_{n\to\infty} \frac{\log K_n(x, y)}{n} \leq \limsup_{n\to\infty} \frac{\log W_m(x, y)}{m} \cdot \frac{m}{n} = \frac{(1 + \varepsilon)h(T, \mathcal{P})}{h(T, \mathcal{P}) - \varepsilon}.$$

Since $\varepsilon > 0$ is arbitrary and $h(T, \mathcal{P}) > 0$, we have the result. □

The Shannon-McMillan-Breiman theorem and the Asymptotic Equipartition Property also hold[17] for the countable partition with finite entropy. Thus Theorem 1 and 6 can be extended for a countable partition with finite entropy which consists of intervals.

3. The waiting time for irrational rotations

We need some properties on diophantine approximations. For more details, consult [11] and [19]. For an irrational number $0 < \theta < 1$, we have a unique continued fraction expansion

$$\theta = [a_1, a_2, \cdots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$$

if $a_i \geq 1$ for all $i \geq 1$. Put $p_0 = 0$ and $q_0 = 1$. Choose p_i and q_i for $i \geq 1$ such that $(p_i, q_i) = 1$ and

$$\frac{p_i}{q_i} = [a_1, a_2, \cdots, a_i] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \cfrac{1}{a_i}}}}.$$

We call each a_i an i-th partial quotient and p_i/q_i an i-th convergent.

For $t \in \mathbb{R}$ we define $\| \cdot \|$ by

$$\|t\| = \min_{n \in \mathbb{Z}} |t - n|,$$

i.e., the distance to the nearest integer. Then the denominator q_i and the numerator p_i of an i-th convergent satisfy the following facts: $q_{i+2} = a_{i+2}q_{i+1} + q_i$, $p_{i+2} = a_{i+2}p_{i+1} + p_i$ and $\frac{1}{q_{i+1}+q_i} < \|q_i\theta\| < \frac{1}{q_{i+1}}$ for $i \geq 1$. If $0 < j < q_{i+1}$, then $\|j\theta\| \geq \|q_i\theta\|$.

Definition 7. An irrational number θ, $0 < \theta < 1$, is said to be of type η if

$$\eta = \sup\{\beta : \liminf_{j \to \infty} j^{\beta} ||j\theta|| = 0\}.$$

Note that every irrational number is of type $\eta \geq 1$ since $q_i^{1-\epsilon}\|q_i\theta\| < 1/{q_i}^{\epsilon}$ for every $\epsilon > 0$. The set of irrational numbers of type 1 has measure 1 and include the set of irrational numbers with bounded partial quotients, which is of measure 0. See [11] for the details. There exist numbers of type ∞, called the Liouville numbers.

Let $0 < \theta < 1$ be an irrational number and $T : [0, 1) \to [0, 1)$ an irrational rotation, i.e.,

$$Tx = x + \theta \pmod 1.$$

Then T preserves the Lebesgue measure μ on $X = [0, 1)$.

In [8] for an irrational rotational transformation T it is shown that

$$\limsup_{n \to \infty} \frac{1}{n} \log K_n(x) = 1 \text{ a.e.,} \qquad \liminf_{n \to \infty} \frac{1}{n} \log K_n(x) = \frac{1}{\eta} \text{ a.e.}$$

Khintchine showed that if s is a real number, then

$$\liminf_{n\to\infty} n||n\theta + s|| \leq \frac{1}{\sqrt{5}}$$

(see [19]), which suggests that for all x and y

$$\liminf_{n\to\infty} \frac{1}{n} \log K_n(x,y) \leq 1.$$

Theorem 8. *For an irrational rotation*

$$\liminf_{n\to\infty} \frac{1}{n} \log K_n(x,y) = \liminf_{r\to 0} \frac{\log W_{B(x,r)}(y)}{-\log r} = 1 \ a.e.,$$

and

$$\limsup_{n\to\infty} \frac{1}{n} \log K_n(x,y) = \limsup_{r\to 0} \frac{\log W_{B(x,r)}(y)}{-\log r} = \eta \ a.e.$$

To proof the Theorem 8 we need the distribution of the waiting time for an interval. It is known that the first return time R_E of an irrational rotation T has at most three values if E is an interval[23]. The distribution of the waiting time is obtained by the following propositions. For the complete proof of Theorem 8 see [14].

Proposition 9. *Let T be an irrational rotation and $b \in (0, ||\theta||]$ a fixed number. Let $i \geq 0$ be an integer such that $||q_i\theta|| < b \leq ||q_{i-1}\theta||$ and K an integer which satisfies*

$$K = \max\{k \geq 0 : k||q_i\theta|| + ||q_{i+1}\theta|| < b\}.$$

If i is even, then

$$R_{[0,b)}(x) = \begin{cases} q_i, & 0 \leq x < b - ||q_i\theta||, \\ q_{i+1} - (K-1)q_i, & b - ||q_i\theta|| \leq x < K||q_i\theta|| + ||q_{i+1}\theta||, \\ q_{i+1} - Kq_i, & K||q_i\theta|| + ||q_{i+1}\theta|| \leq x < b. \end{cases}$$

If i is odd, then

$$R_{[0,b)}(x) = \begin{cases} q_{i+1} - Kq_i, & 0 \leq x < b - K||q_i\theta|| - ||q_{i+1}\theta||, \\ q_{i+1} - (K-1)q_i, & b - K||q_i\theta|| - ||q_{i+1}\theta|| \leq x < ||q_i\theta||, \\ q_i, & ||q_i\theta|| \leq x < b. \end{cases}$$

Remark 10. (i) Note that the value at the middle interval is the sum of the other two values.
(ii) $0 \leq K \leq a_{i+1} - 1$ since $||q_{i-1}\theta|| = a_{i+1}||q_i\theta|| + ||q_{i+1}\theta||$.

By the above proposition we obtain the distribution of the waiting time W_E for an interval E.

Proposition 11. *Let T be an irrational rotation and $b \in (0, ||\theta||]$ a fixed number. Let $i \geq 0$ be an integer such that $||q_i\theta|| < b \leq ||q_{i-1}\theta||$ and K an integer which satisfies*

$$K = \max\{k \geq 0 : k||q_i\theta|| + ||q_{i+1}\theta|| < b\}.$$

Then we have

$$\mu\{W_{[0,b)} = k\} = \begin{cases} b & \text{if } 1 \leq k \leq q_i, \\ ||q_i\theta|| & \text{if } q_i < k \leq q_{i+1} - Kq_i, \\ (K+1)||q_i\theta|| + ||q_{i+1}\theta|| - b & \text{if } q_{i+1} - Kq_i < k \leq q_{i+1} - (K-1)q_i. \end{cases}$$

Note that

$$\sum_{i=1}^{\infty} \mu\{W_{[0,b)} = i\} = q_i b + ||q_i\theta||(q_{i+1} - (K+1)q_i) + ((K+1)||q_i\theta|| + ||q_{i+1}\theta|| - b)q_i$$
$$= q_{i+1}||q_i\theta|| + q_i||q_{i+1}\theta|| = 1.$$

Proof. By definition of W_E and R_E, we have

$$\{x : W_E(x) = k+1\} = T^{-1}\{x : W_E(x) = k\} \setminus T^{-1}\{x \in E : R_E(x) = k\}.$$

Since

$$\{x \in E : R_E(x) = k\} \subset \{x : W_E(x) = k\},$$

we have

$$\mu(\{x : W_E(x) = k+1\}) = \mu(\{x : W_E(x) = k\}) - \mu(\{x \in E : R_E(x) = k\})$$

and

$$\mu(\{x : W_E(x) = k\}) = \mu(\{x \in E : R_E(x) \geq k\}).$$

Now use Proposition 9 since $b \leq ||\theta||$. □

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