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A NECESSARY AND SUFFICIENT CONDITION FOR THE EXISTENCE OF DOMINATED SPLITTING WITH A GIVEN INDEX

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ABSTRACT. A C^1 diffeomorphism ϕ on a compact boundaryless manifold is said to exhibit an i-eigenvalue gap if for every periodic point x of ϕ, the modulus of i-th eigenvalue of $D\phi^n(x)$ is strictly less than the modulus of $(i+1)$-th eigenvalue of $D\phi^n(x)$, where n is the period of x. We prove that ϕ has a dominated splitting of index i over the set of preperiodic points if and only if there exists a C^1 neighborhood $\mathcal{U}$ of ϕ such that every $\psi \in \mathcal{U}$ exhibits an i-eigenvalue gap.

1. Introduction

Let M be a d-dimensional compact Riemannian manifold without boundary ($d \geq 2$). Denote by $\mathrm{Diff}^1(M)$ the set of diffeomorphisms endowed with the C^1 topology. Let $\phi \in \mathrm{Diff}^1(M)$ and Λ a compact invariant set of ϕ. A $D\phi$-invariant splitting $T_\Lambda M = E \oplus F$ is called a dominated splitting of index i over Λ, if $\dim E(x) = i, \forall x \in \Lambda$ and there exists $l \in \mathbb{N}$ such that $\forall x \in \Lambda$,

$$\|D\phi^l|_{E(x)}\| \, \|D\phi^{-l}|_{F(\phi^l x)}\| \leq \frac{1}{2}.$$

The study of dominated splittings first appeared in [15, 16]. Subsequently, in studying the stability conjecture, Liao [8, 9, 10] and Mañé [11, 12, 13] began their systematic research on dominated splittings. (In Liao's terminology, dominated splitting is called "semi-hyperbolic splitting".) See also [7] for invariant manifolds related to dominated splittings. Recently, the study of dominated splittings, or partially hyperbolic splittings, attracts more and more people (e.g., see [1, 2, 3, 4, 5, 17, 18, 20]).

To state our result precisely, we introduce some terminologies. Recall a point $x \in M$ is called (C^1) preperiodic, if for any neighborhood $\mathcal{U}$ of ϕ in $\mathrm{Diff}^1(M)$ and any neighborhood U of x in M, there exist $\psi \in \mathcal{U}$ and $y \in U$ such that y is a periodic point of ψ ([19]). We denote by $\mathrm{P}_*(\phi)$ the set of preperiodic points. $\mathrm{P}_*(\phi)$

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is closed and ϕ-invariant. Note that

$$\Omega(\phi) \subset \mathrm{P}_*(\phi) \subset \mathrm{CR}(\phi),$$

where $\Omega(\phi)$ and $\mathrm{CR}(\phi)$ denote the nonwandering set and chain recurrent set of ϕ respectively. Specifically, a point $x \in M$ is called (C^1) i-preperiodic of ϕ, $0 \le i \le d$, if for any neighborhood $\mathcal{U}$ of ϕ in $\mathrm{Diff}^1(M)$ and any neighborhood U of x in M, there exist $\psi \in \mathcal{U}$ and $y \in U$ such that y is a hyperbolic periodic point of ψ with index i. Denote by $\mathrm{P}_*^i(\phi)$ the set of i-preperiodic points of ϕ. $\mathrm{P}_*^i(\phi)$ is closed and ϕ-invariant, for each $0 \le i \le d$. Note that $\mathrm{P}_*^i(\phi)$ and $\mathrm{P}_*^j(\phi)$ may not be disjoint for $i \ne j$.

It is proved in [20] that ϕ is far away from homoclinic tangency if and only if for each i, $1 \le i \le d-1$, there is a dominated splitting of index i over $\mathrm{P}_*^i(\phi)$. But as the author indicated, the following statement is not proved: For any given i, $1 \le i \le d-1$, if ϕ is far away from homoclinic tangency related to hyperbolic periodic orbit of index i then there is a dominated splitting of index i over $\mathrm{P}_*^i(\phi)$.

Motivated by this problem, we find a natural condition for the existence of dominated splitting of a given index, which we call eigenvalue gap (or spectral gap).

Given $\phi \in \mathrm{Diff}^1(M)$ and a periodic point x of ϕ, denote by $\lambda_i(x) = \lambda_i(x, \phi)$ the eigenvalues of $D\phi^n(x)$ such that

$$|\lambda_1(x)| \le |\lambda_2(x)| \le \cdots \le |\lambda_d(x)|,$$

where n is the period of x. And we will call $\lambda_i(x)$ the i-th eigenvalue of $D\phi^n(x)$. Given $1 \le i \le d-1$, ϕ is said to exhibit an i-eigenvalue gap at x, if $|\lambda_i(x)| < |\lambda_{i+1}(x)|$. ϕ is said to exhibit an i-eigenvalue gap if for any periodic point x of ϕ, ϕ exhibits an i-eigenvalue gap at x. Now we state our main results.

Theorem 1.1. *Given $\phi \in \mathrm{Diff}^1(M)$, ϕ has a dominated splitting of index i over $\mathrm{P}_*(\phi)$ if and only if there exists a C^1 neighborhood $\mathcal{U}$ of ϕ such that every $\psi \in \mathcal{U}$ exhibits an i-eigenvalue gap.*

Roughly, Theorem 1.1 says that if a diffeomorphism (C^1) robustly exhibits an i-eigenvalue gap, then it has a dominated splitting of index i.

Assume that Λ is a compact invariant set of ϕ. Denote by $\mathrm{P}_*(\Lambda, \phi)$ the set of points $x \in \Lambda$ such that for any neighborhood $\mathcal{U}$ of ϕ in $\mathrm{Diff}^1(M)$ and any neighborhoods U of Λ and V of x in M, there exist $\psi \in \mathcal{U}$ and $y \in V$ such that y is a periodic point of ψ with $\mathrm{Orb}(x, \psi) \subset U$. It is easily seen that $\mathrm{P}_*(\Lambda, \phi)$ is a compact invariant subset of Λ and when $\Lambda = M$, $\mathrm{P}_*(M, \phi) = \mathrm{P}_*(\phi)$. A local version of Theorem 1.1 is the following:

Theorem 1.2. *Given $\phi \in \mathrm{Diff}^1(M)$ and a compact invariant set Λ of ϕ, ϕ has a dominated splitting of index i over $\mathrm{P}_*(\Lambda, \phi)$ if and only if there exist a C^1 neighborhood $\mathcal{U}$ of ϕ and a neighborhood U of Λ such that for any $\psi \in \mathcal{U}$ and any periodic point x with $\mathrm{Orb}(x, \psi) \subset U$, ψ exhibits an i-eigenvalue gap at x.*

We would like to say some words about the relation of this paper and [3]. In [3], Bonatti, Diaz and Pujals introduced the terminology of periodic linear system and developed lots of fundamental theory for periodic linear system, some of which will be used in our paper. Restricted to the periodic linear systems without transitions, the main difference of this paper and [3] is that complex-diagonalizable systems should be considered in our case, while [3] only has to consider real-diagonalizable systems with the help of transitions.

This paper is organized as follows. In §2, we first introduce the terminology of periodic linear system and state some fundamental results in [3], and then reduce the proof of the main results of the paper to the proof of Proposition 2.2, which is a result about periodic linear systems. Even more, we only have to prove Proposition 2.5, the special case of Proposition 2.2 for simple periodic linear systems, i.e., for systems consisting of only one periodic orbit. In §3, by using the technique of reduction of dimension in [3], we give the proof of Proposition 2.5 by assuming that it is proved for lower dimensional cases ($d \leq 4$). Before proving Proposition 2.5 for $d \leq 4$, we make some preparations in §4 and §5. §4 contains three general results. The first is about the continuation of dominated splittings. The second says that if a simple periodic linear system robustly exhibits an i-eigenvalue gap, then there exists a uniform eigenvalue gap for the system and its small perturbations. And the third one is a sufficient condition for the existence of dominated splitting, involving the control of growth rate of norms and no small angles. In §5, we give several perturbation lemmas for lower order matrices, including the only novelty of the paper, Lemma 5.3, where we prove that for a 3×3 matrix, with a pair of conjugate complex eigenvalues and a real eigenvalue, if the complex eigenvalues are far away from the real axes and the angle between the eigenspace corresponding to the complex eigenvalues and the eigenspace corresponding to the real eigenvalue is arbitrary small, then an arbitrary small perturbation of the matrix will have three eigenvalues with equal modulus. Also, we reformulate, in the form we need, a result of Mañé ([12]) on the estimate of angles. This result will reduce the studying of the case of two pairs of conjugate complex eigenvalues and small angle between two eigenspaces. Finally, in §6, we give the proof of Proposition 2.5 for $d \leq 4$.

To end the introduction, we suggest the experienced reader jump directly to §5, which includes the essence of the paper.

2. Reduction to simple periodic linear systems

In this section, we will introduce the notation of (periodic) linear system and then reduce the proof of Theorem 1.1 and Theorem 1.2 to the proof of Proposition 2.2. At the end of this section, we list some fundamental results in [3] for reference.

Let Σ be a set and $f : \Sigma \to \Sigma$ a 1-1 and onto mapping. Note that Σ equips no topology (or the discrete topology).

Consider a d-dimensional Euclidean bundle $\mathcal{E}$ over Σ. "Euclidean" means that for every $x \in \Sigma$, there is an inner product in the fibre $\mathcal{E}(x)$. Since Σ has discrete topology, the bundle $\mathcal{E}$ is always trivial. If we take arbitrarily an orthonormal basis in $\mathcal{E}(x)$ for every $x \in \Sigma$, $\mathcal{E}$ will be identified with $\Sigma \times \mathbb{R}^d$.

A bounded bundle map $A : \mathcal{E} \to \mathcal{E}$ covering $f : \Sigma \to \Sigma$ is called a d-dimensional linear system (or linear cocyle) over f. Precisely,

$$A(x, v) = (fx, A(x)v),$$

where $A(x)$ is an isomorphism from $\mathcal{E}(x)$ to $\mathcal{E}(fx)$. A is called bounded if

$$\|A\| = \sup_{x\in\Sigma}\{\|A(x)\|,\ \|A(x)^{-1}\|\} < \infty.$$

More precisely, we will call a 4-tuple $(\Sigma, f, \mathcal{E}, A)$ a linear system if A is bounded. Given $K \geq 1$, a linear system is bounded by K if $\|A\| \leq K$. If we denote by $\langle\cdot,\cdot\rangle$ the inner product on $\mathcal{E}$, we will denote a linear system by a 5-tuple $(\Sigma, f, \mathcal{E}, A, \langle\cdot,\cdot\rangle)$ when necessary. When we use $(\Sigma, f, \mathcal{E}, A, \langle\cdot,\cdot\rangle)$ to denote a linear system, $\mathcal{E}$ is considered as a vector bundle instead of Euclidean bundle.

Denote by $\mathrm{GL}(\Sigma, f, \mathcal{E})$ the set of linear systems over $f : \Sigma \to \Sigma$, equipped with the following distance:

$$d(A, B) = \sup_{x\in\Sigma}\{\|A(x) - B(x)\|,\ \|A^{-1}(x) - B^{-1}(x)\|\}.$$

If $d(A, B) \leq \varepsilon$, B is called an ε-perturbation of A.

Remark 2.1. One may define another metric $d'(\cdot,\cdot)$ on $\mathrm{GL}(\Sigma, f, \mathcal{E})$ as follows:

$$d'(A, B) = \sup_{x\in\Sigma}\{\|A(x) - B(x)\|\}.$$

Note that since every $A \in \mathrm{GL}(\Sigma, f, \mathcal{E})$ is bounded, the two metrics $d(\cdot,\cdot)$ and $d'(\cdot,\cdot)$ are locally equivalent, i.e., for every $K \geq 1$, there exists $C \geq 1$ such that if $A, B \in \mathrm{GL}(\Sigma, f, \mathcal{E})$ with $\|A\|,\ \|B\| \leq K$ then

$$d'(A, B) \leq d(A, B) \leq Cd'(A, B).$$

Sometimes we will not distinguish these two metrics.

$f : \Sigma \to \Sigma$ is called periodic if every point in Σ is a periodic point of f. If f is periodic, an element $A \in \mathrm{GL}(\Sigma, f, \mathcal{E})$ is called a periodic linear system (abbreviated as PLS). If Σ consists of only one periodic orbit of f, an element $A \in \mathrm{GL}(\Sigma, f, \mathcal{E})$ is called a simple periodic linear system (abbreviated as SPLS) and we will call the period of the system to be the period of the periodic orbit. In this paper we only consider PLSs and even only SPLSs after this section.

Given a d-dimensional PLS $(\Sigma, f, \mathcal{E}, A)$ and $x \in \Sigma,\ k \in \mathbb{Z}$, define $A^k(x) : \mathcal{E}(x) \to \mathcal{E}(f^k x)$ by

$$A^k(x) = \begin{cases} A(f^{(k-1)}x) \circ \cdots \circ A(fx) \circ A(x), & \text{if } k > 0, \\ I, & \text{if } k = 0, \\ A^{-1}(f^{(k+1)}x) \circ \cdots \circ A^{-1}(f^{-1}x) \circ A^{-1}(x), & \text{if } k < 0. \end{cases}$$

Let $\lambda_j(x) = \lambda_j(x, A),\ j = 1, 2, \cdots, d$, be the eigenvalues of $A^n(x)$ such that

$$|\lambda_1(x)| \leq |\lambda_2(x)| \leq \cdots \leq |\lambda_d(x)|,$$

where n is the period of x. $\lambda_j(x)$ is called the j-th eigenvalue of $A^n(x)$. For some $1 \leq i \leq d-1$, we say A exhibits an i-eigenvalue gap at x, if $|\lambda_i(x)| < |\lambda_{i+1}(x)|$ and

A is said to exhibit an i-eigenvalue gap if A exhibits an i-eigenvalue gap at every point $x \in \Sigma$.

A subbundle $E \subset \mathcal{E}$ is called homogeneous if for each $x \in \Sigma$, $\dim E(x)$ is the same. Subbundles considered in this paper are all homogeneous. Let $(\Sigma, f, \mathcal{E}, A)$ be a d-dimensional PLS. A subbundle F of $\mathcal{E}$ is A-invariant if $A(E(x)) = E(fx)$ for every $x \in \Sigma$. We say two A-invariant subbundles E and F forms a dominated splitting if there exists $l \in \mathbb{N}$ such that

$$\|A^l_{E(x)}\| \, \|A^{-l}_{F(f^l x)}\| \le \frac{1}{2}$$

for every $x \in \Sigma$. And (following [3]) we denote by $E \prec F$, or $E \prec_l F$ if we want to emphasize the role of l. It is easily seen that $E \prec F$ implies $E \cap F = 0$. We say that A has an (l, i)-dominated splitting or l-dominated splitting for some $1 \le i \le d-1$ if there exist two A-invariant subbundles E and F such that $\dim E = i$, $E \prec_l F$ and $E(x) \oplus F(x) = \mathcal{E}(x)$ for every $x \in \Sigma$.

Our main results will be deduced from the following proposition on PLSs.

Proposition 2.2. *For any given $\varepsilon > 0$, $K \ge 1$ and $d \in \mathbb{N}$, there exists an integer $l = l(\varepsilon, K, d) \in \mathbb{N}$ satisfying the following property:*

Let $(\Sigma, f, \mathcal{E}, A)$ be a d-dimensional PLS bounded by K. If there exists $1 \le i \le d-1$ such that every ε-perturbation of A exhibits an i-eigenvalue gap, then A has an (l, i)-dominated splitting.

Since Theorem 1.1 is the special case of Theorem 1.2 for Λ being the whole manifold, we will only show how to deduce Theorem 1.2 from Proposition 2.2.

We need the well-known Franks' lemma, which is the bridge between Theorem 1.2 and Proposition 2.2.

Lemma 2.3. *Let $\phi \in \mathrm{Diff}^1(M)$. Then for any neighborhood $\mathcal{U}$ of ϕ there exist $\varepsilon > 0$ and a neighborhood $\mathcal{U}_0 \subset \mathcal{U}$ of ϕ such that given $\psi \in \mathcal{U}_0$, a finite set $\{x_1, \cdots, x_N\} \subset M$, a neighborhood U of $\{x_1, \cdots, x_N\}$ and linear maps $X_j : T_{x_j}M \to T_{\psi x_j}M$ such that $\|X_j - D\psi(x_j)\| \le \varepsilon$ for all $1 \le j \le N$, then there exists $\overline{\psi} \in \mathcal{U}$ such that $\overline{\psi}x = \psi x$ if $x \in \{x_1, \cdots, x_N\} \cup (M - U)$ and $D\overline{\psi}(x_j) = X_j$ for all $1 \le j \le N$.*

Proof of Theorem 1.2: Given $\phi \in \mathrm{Diff}^1(M)$ and a compact invariant set Λ of ϕ, if ϕ has a dominated splitting of index i over $\mathrm{P}_*(\Lambda, \phi)$, then it is well-known (see also Lemma 4.1) that there exist a C^1 neighborhood $\mathcal{U}$ of ϕ and a neighborhood U of Λ such that for any compact invariant set $\Delta \subset U$ of $\psi \in \mathcal{U}$, ψ has a dominated splitting of index i over Δ. Then it is easily seen that for any periodic point x with $\mathrm{Orb}(x, \psi) \subset U$, ψ exhibits an i-eigenvalue gap at x.

Now suppose there exist a C^1 neighborhood $\mathcal{U}$ of ϕ and a neighborhood U of Λ such that for any $\psi \in \mathcal{U}$ and any periodic point x of ψ with $\mathrm{Orb}(x, \psi) \subset U$, ψ exhibits an i-eigenvalue gap at x. Let ε and $\mathcal{U}_0$ be guaranteed by Lemma 2.3. Now set

$$\begin{aligned}
\Sigma &= \{(x,\psi) : \psi \in \mathcal{U}_0 \text{ and } x \text{ is a periodic point of } \psi \text{ with } \mathrm{Orb}(x,\psi) \subset U\} \\
f(x,\psi) &= (\psi(x),\psi), \quad \forall (x,\psi) \in \Sigma, \\
\mathcal{E}(x,\psi) &= T_xM, \quad \forall (x,\psi) \in \Sigma, \\
A(x,\psi) &= D\psi(x), \quad \forall (x,\psi) \in \Sigma.
\end{aligned}$$

Note that since M is a Riemannian manifold, $\mathcal{E}$ is an Euclidean bundle. If necessary, we may shrink $\mathcal{U}_0$ so that

$$K = \sup\{\|D\psi(x)\|,\ \|D\psi^{-1}(x)\| : x \in M,\ \psi \in \mathcal{U}_0\} < \infty.$$

Now $(\Sigma, f, \mathcal{E}, A)$ is a PLS. And according to Lemma 2.3, every ε-perturbation of A exhibits an i-eigenvalue gap. So according to Proposition 2.2, for some integer $l \in \mathbb{N}$, A has an (l,i)-dominated splitting. Now, according to the continuity of dominated splitting, a well-known limit process (e. g., see [3]) tells us that ϕ has an (l,i)-dominated splitting over $\mathrm{P}_*(\Lambda,\phi)$. □

Remark 2.4. Due to the essence of the uniformity of l, which depends only some constants ε, K, d, and the point-wise convergence topology of $\mathrm{GL}(\Sigma, f, \mathcal{E})$, in order to prove Proposition 2.2, it is enough to prove it for SPLSs, i.e., the minimal PLSs.

Now we restate Proposition 2.2 for SPLSs as following.

Proposition 2.5. *For any given $\varepsilon > 0$, $K \geq 1$ and $d \in \mathbb{N}$, there exists an integer $l = l(\varepsilon, K, d) \in \mathbb{N}$ satisfying the following property:*

Let $(\Sigma, f, \mathcal{E}, A)$ be a d-dimensional SPLS bounded by K. If there exists $1 \leq i \leq d-1$ such that every ε-perturbation of A exhibits an i-eigenvalue gap, then A has an (l,i)-dominated splitting.

Because of the importance of uniformity, we will take great care to ensure the uniformity of constants in our proofs. And we will always state our results in the form of Proposition 2.5, which looks awkward but is clear enough so that one could know the uniformity at a glance. For the same reason, we will also reprove some known results, e. g., Lemma 4.1, Proposition 2.5 for real-diagonalizable case, etc.

So, from now on, we only deal with SPLSs. If we do not make another assumption, we always assume that

Convenient Assumption: $\boldsymbol{\Sigma} = \{x = f^n x, fx, \cdots, f^{n-1}x\}$,

i.e., Σ consists of an n-periodic orbit of f. We should emphasize that the above assumption is just for convenience and **we have no any intention to fix the period of systems.**

Let $A \in \mathrm{GL}(\Sigma, f, \mathcal{E})$ be a d-dimensional SPLS. Then $\lambda_j(y, A)$ does not depend on $y \in \Sigma$ and will be denoted by $\lambda_j(A)$. If A exhibits an i-eigenvalue gap for some $1 \leq i \leq d-1$, then for any $y \in \Sigma$, the invariant subspace corresponding to the first i's eigenvalues of $A^n(y)$ is well-defined and the invariant subspace corresponding to the last $(d-i)$'s eigenvalues is also well-defined. Denote these two subspaces by $E^i(y, A)$ and $F^i(y, A)$ respectively. The corresponding subbundles are denoted by $E^i(A)$ and $F^i(A)$ respectively.

Now we list some results in [3] about SPLSs. In [3], these results are stated for PLSs and by Remark 2.4, we restate them for SPLSs.

Lemma 2.6. [3, Lemma 1.2]

1. Given $\varepsilon > 0$ and $K \geq 1$ there exists $\delta = \delta(\varepsilon, K) > 0$ such that for any SPLS $(\Sigma, f, \mathcal{E}, A)$ bounded by K and every δ-perturbations P and Q of the identity linear system $(\Sigma, id_\Sigma, \mathcal{E}, id_\mathcal{E})$, one has that $P \circ A \circ Q$ is an ε-perturbation of A.

2. Given $K \geq 1$, $C \geq 1$ and $\varepsilon > 0$ there exist $K_1 \geq 1$ and $\varepsilon_0 > 0$ such that for any bundle isomorphism $P : (\mathcal{E}, \langle\cdot,\cdot\rangle) \to (\mathcal{E}, \langle\cdot,\cdot\rangle')$ covering the identity $id_\Sigma : \Sigma \to \Sigma$ with $\|P(x)\|$, $\|P(x)^{-1}\| \leq C$ for every $x \in \Sigma$, where $\mathcal{E}$ is vector bundle over Σ, and any linear system $(\Sigma, f, \mathcal{E}, A, \langle\cdot,\cdot\rangle)$ bounded by K, $(\Sigma, f, \mathcal{E}, B, \langle\cdot,\cdot\rangle')$ is a linear system bounded by K_1, where $B = P \circ A \circ P^{-1}$. Moreover, any ε_0-perturbation of B is conjugate by P to some ε-perturbation of A.

Two metrics $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle'$ on a vector bundle $\mathcal{E}$ is called equivalent if there exists $C \geq 1$, for every $u \in E$, we have

$$\frac{1}{C}\|u\| \leq \|u\|' \leq C\|u\|,$$

where $\|\cdot\|$ and $\|\cdot\|'$ are the norms induced by $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle'$ respectively. The second item in Lemma 2.6 will simplify our discussion. Roughly, it says that two equivalent metrics play the same role in our discussion. Since the constants δ, K_1 are uniform (depends only on ε, K, C), when citing this item, we will not introduce new constants and just say "after a uniformly bounded change of metric on $\mathcal{E}$, we may assume ...".

For any SPLS $(\Sigma, f, \mathcal{E}, A)$ and every A-invariant subbundle F, A naturally induces a SPLS, denoted by (Σ, f, F, A_F), or simply A_F, which will be called the restriction of A on F. Denote by $\mathcal{E}/F$ (or $F^\perp$) the orthogonal complement of F in $\mathcal{E}$. A also induces a linear system $(\Sigma, f, \mathcal{E}/F, A/F)$ on $\mathcal{E}/F$ as follows.

For any $x \in \Sigma$ and $u \in \mathcal{E}(x)/F(x)$, $(A/F)(x)u$ is equal to the orthogonal projection of $A(x)u$ on $\mathcal{E}(fx)/F(fx)$ (of course, along $F(fx)$).

The linear system $(\Sigma, f, \mathcal{E}/F, A/F)$ or A/F is called a quotient linear system of A. With respect to the orthogonal decomposition $\mathcal{E} = F \oplus \mathcal{E}/F$, A has the form:

$$A = \begin{pmatrix} A_F & * \\ 0 & A/F \end{pmatrix}.$$

Lemma 2.7. [3, Lemma 4.1] *For any $\varepsilon > 0$ and $K \geq 1$, there exists $\varepsilon_1 = \varepsilon_1(\varepsilon, K) \in (0, \varepsilon]$ such that for any SPLS $(\Sigma, f, \mathcal{E}, A)$ bounded by K and A-invariant subbundle F of $\mathcal{E}$, we have*

1. Both A_F and A/F are SPLSs bounded by K.

2. For every ε_1-perturbation C of A_F, $P \circ A$ is an ε-perturbation of A, where

$$P = \begin{pmatrix} C \circ A_F^{-1} & 0 \\ 0 & I \end{pmatrix}.$$

Therefore, $(P \circ A)_F = C$ and $(P \circ A)/F = A/F$.

3. For every ε_1-perturbation C of A/F, $Q\circ A$ is an ε-perturbation of A, where

$$Q=\begin{pmatrix} I & 0 \\ 0 & C\circ(A^{-1}/F) \end{pmatrix}.$$

Therefore, $(Q\circ A)_F=A_F$ and $(Q\circ A)/F=C$.

Let E be another A-invariant subbundle with $E\cap F=0$. Denote by P the orthogonal projection on $\mathcal{E}/F$. Denote by E/F the subbundle $P(E)$, which has the same dimension as E. Since both E and F are A-invariant, $(A/F)(E/F)=E/F$. Write

$$A_{E/F}=(A/F)_{E/F}: E/F\to E/F.$$

So, $(\Sigma,f,\mathcal{E},f)$ induces a quotient system $(\Sigma,f,E/F,A_{E/F})$.

With respect to the orthogonal decomposition

$$\mathcal{E}=F\oplus E/F\oplus \mathcal{E}/(E\oplus F),$$

A can be written as

$$A=\begin{pmatrix} A_F & * & * \\ 0 & A_{E/F} & * \\ 0 & 0 & A/(E\oplus F) \end{pmatrix}.$$

An easy consequence of Lemma 2.7 is:

Corollary 2.8. *For any $\varepsilon>0$ and $K\geq 1$, there exists $\varepsilon_2=\varepsilon_2(\varepsilon,K)\in(0,\varepsilon]$ such that for any SPLS $(\Sigma,f,\mathcal{E},A)$ bounded by K and A-invariant subbundles E and F of $\mathcal{E}$ with $E\cap F=0$, we have*

1. $A_{E/F}$ is bounded by K.

2. For every ε_2-perturbation C of $A_{E/F}$, $P\circ A$ is an ε-perturbation of A, where

$$P=\begin{pmatrix} I & 0 & 0 \\ 0 & C\circ(A^{-1}_{E/F}) & 0 \\ 0 & 0 & I \end{pmatrix}.$$

Therefore, $(P\circ A)_F=A_F$, $(P\circ A)/(E\oplus F)=A/(E\oplus F)$ and $(P\circ A)_{E/F}=C$.

3. Reduction of the dimension and the proof of Proposition 2.2

A d-dimensional SPLS $A\in \mathrm{GL}(\Sigma,f,\mathcal{E})$ is called complex-diagonalizable if every eigenvalue of A is simple and $|\lambda_j(A)|=|\lambda_{j+1}(A)|$ implies $\lambda_j(A)$ and $\lambda_{j+1}(A)$ are complex conjugates and called real-diagonalizable if for any $1\leq j\leq d-1$, $|\lambda_j(A)|<|\lambda_{j+1}(A)|$. Denote by $\mathrm{GL}_c(\Sigma,f,\mathcal{E})$ and $\mathrm{GL}_r(\Sigma,f,\mathcal{E})$ the sets of complex-diagonalizable and real-diagonalizable SPLSs respectively. $\mathrm{GL}_c(\Sigma,f,\mathcal{E})$ is dense in $\mathrm{GL}(\Sigma,f,\mathcal{E})$ but $\mathrm{GL}_r(\Sigma,f,\mathcal{E})$ is not.

Let $A\in \mathrm{GL}_c(\Sigma,f,\mathcal{E})$ and the eigenspaces decomposition of $A^n(x)$ be

$$\mathcal{E}(x)=E_1(x)\oplus E_2(x)\oplus\cdots\oplus E_s(x), \tag{3.1}$$

with $\dim E_j(x)=1$ or 2 and if $\dim E_j(x)=1$, the eigenvalue of $A^n_{E_j(x)}$ is a real number μ_j and if $\dim E_j(x)=2$, the eigenvalues of $A^n_{E_j}$ are complex conjugates $\{\mu_j,\ \bar{\mu}_j\}$. Moreover, we assume that $|\mu_1|<|\mu_2|<\cdots<|\mu_s|$. If necessary we will write $\mu_j = \mu_j(A)$. Under these assumptions, the decomposition (3.1) is called the standard eigenspaces decomposition of $A^n(x)$. Let

$$\mathcal{E} = E_1 \oplus E_2 \oplus \cdots \oplus E_s \tag{3.2}$$

be the bundle decomposition induced by the decomposition (3.1) at x. We will call (3.2) the standard eigenspaces decomposition of A and call the s-tuple $(\dim E_1, \dim E_2, \cdots, \dim E_s)$ the type of A, denoted by $t(A)$. Let $t(A) = (t_1, t_2, \cdots, t_2)$. If A exhibits an i-eigenvalue gap, then there exists $1 \le a = a(A) \le s-1$ such that

$$i = \sum_{j=1}^{a} t_j.$$

For any $1 \le j \le a \le k < s$, denote by

$$\begin{aligned} E_{jk}(x) &= (E_j(x) \oplus E_{k+1}(x))/(E_{j+1}(x) \oplus \cdots \oplus E_k(x)), \\ &= E_j(x)/(E_{j+1}(x) \oplus \cdots \oplus E_k(x)) \oplus E_{k+1}(x)/(E_{j+1}(x) \oplus \cdots \oplus E_k(x)) \\ A_{jk}(x) &= A_{E_{jk}}(x). \end{aligned}$$

If $j = k$,

$$E_j(x)/(E_{j+1}(x) \oplus \cdots \oplus E_k(x)) = E_j(x), \qquad E_{k+1}(x)/(E_{j+1}(x) \oplus \cdots \oplus E_k(x)) = E_{k+1}(x).$$

Let E_{jk} be the subbundle of $\mathcal{E}$ with fibre $E_{jk}(x)$. When necessary, we will write

$$E_j(x) = E_j(x, A), \quad E_j = E_j(A), \quad E_{jk}(x) = E_{jk}(x, A), \quad E_{jk} = E_{jk}(A).$$

The following fact is obvious.

Lemma 3.1. *Let $A \in \mathrm{GL}(\Sigma, f, \mathcal{E})$, $B_k \in \mathrm{GL}(\Sigma, f, \mathcal{E})$ be d-dimensional SPLSs such that $\|B_k - A\| \to 0$ as $k \to \infty$. Assume that for some $1 \le i \le d-1$ and $l \in \mathbb{N}$, every B_k has an (l, i)-dominated splitting. Then A has an (l, i)-dominated splitting.*

According to Lemma 3.1 and the denseness of $\mathrm{GL}_c(\Sigma, f, \mathcal{E})$ in $\mathrm{GL}(\Sigma, f, \mathcal{E})$, we only have to prove Proposition 2.5 for $A \in \mathrm{GL}_c(\Sigma, f, \mathcal{E})$.

The following lemma is a combination of Lemma 5.2 and 5.3 in [3], which will help us to reduce the proof of Proposition 2.5 for general dimensional case to lower dimensional ($d \le 4$) case.

Lemma 3.2. *For any given $K \ge 1$ and $l,\ d \in \mathbb{N}$, there exists $L = L(K, l, d) \in \mathbb{N}$ satisfying the following property:*

Let $(\Sigma, f, \mathcal{E}, A)$ be a d-dimensional SPLS bounded by K. And assume that there is an A-invariant direct sum decomposition:

$$\mathcal{E} = E_1 \oplus E_2 \oplus \cdots \oplus E_s$$

If there exists $1 \le a \le s-1$ satisfying that for every $1 \le j \le a \le k \le s-1$,

$$E_j/(E_{j+1} \oplus \cdots \oplus E_k) \prec_l E_{k+1}/(E_{j+1} \oplus \cdots \oplus E_k),$$

then

$$(E_1 \oplus \cdots \oplus E_a) \prec_L (E_{a+1} \oplus \cdots \oplus E_s).$$

Proof We only remark that the proofs given in [3] do work for $\dim E_j$ not necessarily equal to 1. □

In our proof of Proposition 2.5, we will use Lemma 3.2 for $\dim E_j = 1$ or 2.

We also need the following lemma in the reduction procedure.

Lemma 3.3. *Given $\varepsilon > 0$ and $K \geq 1$, there exists $\varepsilon_3 = \varepsilon_3(\varepsilon, K) \in (0, \varepsilon]$ satisfying the following property:*

Given $d \in \mathbb{N}$, $1 \leq i \leq d-1$, let $(\Sigma, f, \mathcal{E}, A)$ be a d-dimensional complex-diagonalizable SPLS bounded by K such that every ε-perturbation of A exhibits an i-eigenvalue gap. Let $t(A) = (t_1, t_2, \cdots, t_s)$. Then for every $1 \leq j \leq a \leq k \leq s-1$ ($a = a(A)$), every ε_3-perturbation of the SPLS $(\Sigma, f, E_{jk}, A_{jk})$ exhibits a t_j-eigenvalue gap .

Proof We use the arguments in [3, Lemma 5.1] to prove the lemma.

Let $\varepsilon_3 = \varepsilon_2(\varepsilon, K)$ be determined by Corollary 2.8. We will show that ε_3 satisfies the lemma.

Let $(\Sigma, f, \mathcal{E}, A)$ be a d-dimensional complex-diagonalizable SPLS bounded by K such that every ε-perturbation of A exhibits an i-eigenvalue gap. Let $t(A) = (t_1, t_2, \cdots, t_s)$ and the standard eigenspaces decomposition of A be (3.2). Write $u = t_j = \dim E_j$ and $v = t_{k+1} = \dim E_{k+1}$.

If the conclusion is not satisfied by the above A, let C be an ε_3-perturbation of A_{jk} such that $|\lambda_u(C)| = |\lambda_{u+1}(C)| \triangleq p$. In the following, we will show that some ε-perturbation of A exhibits no i-eigenvalue gap and this contradiction then finishes the proof.

According to Corollary 2.8, there exists an ε-perturbation B of A such that the eigenvalues of B are

$$\begin{aligned} &\lambda_l(A), && l \in Q, \\ &\lambda_m(C), && m = 1, \cdots, u+v, \end{aligned}$$

where

$$\begin{aligned} Q &= \{1, \cdots, \sigma_{j-1},\ \sigma_j + 1, \cdots, \sigma_k,\ \sigma_{k+1} + 1, \cdots, d\} \subset \mathbb{N}, \\ \sigma_b &= \sum_{c=1}^{b} t_c. \end{aligned}$$

Let $B_\tau = A + \tau(B - A)$, $\tau \in [0, 1]$. So $B_0 = A$ and $B_1 = B$. Every B_τ is an ε-perturbation of A (see Remark 2.1). If denote by

$$C_\tau = (B_\tau)_{E_{jk}(A)},$$

then C_τ is continuous in τ and $C_0 = A_{jk}$, $C_1 = C$. Moreover, the eigenvalues of B_τ are

$$\begin{aligned} &\lambda_l(A), && l \in Q, \\ &\lambda_m(C_\tau), && m = 1, \cdots, u+v. \end{aligned}$$

We may assume that for $\tau \in [0, 1)$, $|\lambda_u(C_\tau)| < |\lambda_{u+1}(C_\tau)|$. Since C_τ is continuous in τ and eigenvalues of matrix are continuous in matrices, $|\lambda_u(C_\tau)|$, $|\lambda_{u+1}(C_\tau)|$ are continuous in τ. We have three cases: 1) $|\lambda_i(A)| \le p \le |\lambda_{i+1}(A)|$, 2) $p < |\lambda_i(A)|$ and 3) $p > |\lambda_{i+1}(A)|$.

In the first case, we have $p = |\lambda_i(B)| = |\lambda_{i+1}(B)|$ and hence B has no i-eigenvalue gap. In the second case, if $j = a$ and $p > |\lambda_{\sigma_{a-1}}|$, then again we have $p = |\lambda_i(B)| = |\lambda_{i+1}(B)|$. If $j = a$ and $p \le |\lambda_{\sigma_{a-1}}|$, then according to the continuity of $|\lambda_{u+1}(C_\tau)|$ with τ, there exists $\tau \in [0,1]$ such that $|\lambda_{u+1}(C_\tau)| = |\lambda_{\sigma_{a-1}}|$. Hence $|\lambda_i(B_\tau)| = |\lambda_{i+1}(B_\tau)|$ and then B_τ has no i-eigenvalue gap. If $j < a$, then $i \in Q$ and according to the continuity of $|\lambda_{u+1}(C_\tau)|$ with τ, there exists $\tau \in [0,1)$ such that $|\lambda_{u+1}(C_\tau)| = |\lambda_i(A)|$. Since in this situation, $\lambda_i(A)$ is also an eigenvalue of B_τ, B_τ has no i-eigenvalue gap. The argument for the third case is similar to the second. This proves this lemma. □

Now we give the proof of Proposition 2.5, after assuming that it has been proven for the special cases: $d = 2,\ 3$ and $d = 4,\ i = 2$, while the proof of the special cases is left to §6.

Proof of Proposition 2.5 after assuming it for $d \le 4$:

Since complex-diagonalizable matrices are dense in the set of matrices, according to Lemma 3.1, we only have to give a proof for complex-diagonalizable SPLSs.

Let $\varepsilon' = \varepsilon_3(\varepsilon, K)$ be determined by Lemma 3.3, $l = l(\varepsilon', K, d)$ determined by Proposition 2.5 for $d = 2,\ 3$ and $d = 4,\ i = 2$ and $L = L(K, l, d)$ determined by Lemma 3.2. We will show that this L satisfies Proposition 2.5.

Let $A \in \mathrm{GL}_c(\Sigma, f, \mathcal{E})$ be a d-dimensional SPLS bounded by K such that every ε-perturbation of A exhibits an i-eigenvalue gap. Denote by $t(A) = (t_1, t_2, \cdots, t_s),\ a = a(A)$. Let

$$\mathcal{E} = E_1 \oplus E_2 \oplus \cdots \oplus E_s$$

be the standard eigenspaces decomposition of A.

According to Lemma 3.2, we only have to prove that for every $1 \le j \le a \le k \le s-1$, one has

$$E_j/(E_{j+1} \oplus \cdots \oplus E_k) \prec_l E_{k+1}/(E_{j+1} \oplus \cdots \oplus E_k). \tag{3.3}$$

By Lemma 3.3, every ε'-perturbation of the SPLS $(\Sigma, f, E_{jk}, A_{jk})$ exhibits a t_j-eigenvalue gap. Since $t_j,\ t_{k+1} = 1$ or 2, (3.3) is satisfied by the choice of l. This finishes the proof of Proposition 2.5. □

4. Some general lemmas

In this section, we list three general lemmas, which will be used in the proof of Proposition 2.5 for $d \le 4$.

The first is about the continuation of dominated splittings for SPLSs. Since it is more or less well-known (e.g., see [13, 17]), we give the sketch of a proof here for emphasizing the uniformity of constants.

Let $\mathcal{E}$ be an Euclidean bundle over Σ and $y \in \Sigma$. Given any two subspaces $X,\ Y \subset \mathcal{E}(y)$ with $X \cap Y = 0$, let $L : X/Y \to X$ be a linear map such that Y is the graph of L, i.e., $Y = \{u + Lu : u \in X/Y\}$. Then the angle between X and Y is defined as $\|L\|^{-1}$ and denoted by $\alpha(X,Y)$. Let $E,\ F$ be two subbundles of $\mathcal{E}$ with $E \cap F = 0$, then define $\alpha(E,F) = \min\{\alpha(E(y),F(y)) :\ y \in \Sigma\}$.

Lemma 4.1. *For any given $K \geq 1$ and $l \in \mathbb{N}$, there exists $\varepsilon_4 = \varepsilon_4(K,l) > 0$, $L = L(K,l) \in \mathbb{N}$ satisfying the following property:*

For any $d \in \mathbb{N}$, let $(\Sigma, f, \mathcal{E}, A)$ be a d-dimensional SPLS bounded by K and assume that A has an (l,i)-dominated splitting $E \oplus F$ for some $1 \leq i \leq d-1$. Then every ε_4-perturbation B of A has an (L,i)-dominated splitting $E(B) \oplus F(B)$. Especially,

$$\alpha(E(B), F(B)) \geq \frac{1}{2(K+\varepsilon_4)^{2L}+1}.$$

Proof Denote by

$$\alpha = 1/(2K^{2l}+1).$$

We show that if $(\Sigma, f, \mathcal{E}, A)$ is a SPLS bounded by K, with an l-dominated splitting $E \prec_l F$, then $\alpha(E,F) \geq \alpha$, which is also a well-known result (e. g., see [3]).

Let $F(y)$ be the graph of linear map $L : \mathcal{E}(y)/E(y) \to E(y)$. Then by definition,

$$\alpha(E(y), F(y)) = \|L\|^{-1}.$$

Given any $0 \neq w \in \mathcal{E}(y)/E(y)$, write $v = w + Lw \in F(y)$. By the definition of l-dominated splitting, we have

$$\begin{aligned} \frac{\|A^l w\|}{\|A^l v\|} &= \frac{\|A^l v - A^l L w\|}{\|A^l v\|} \geq \frac{\|A^l v\| - \|A^l L w\|}{\|A^l v\|} \\ &\geq 1 - \frac{\|A^l L w\|}{\|A^l v\|} \geq 1 - \frac{1}{2} \cdot \frac{\|Lw\|}{\|v\|} \geq \frac{1}{2}. \end{aligned}$$

On the other hand,

$$\frac{\|A^l w\|}{\|A^l v\|} \leq K^{2l} \frac{\|w\|}{\|v\|}.$$

So we have

$$\frac{\|v\|}{\|w\|} \leq 2K^{2l}.$$

So

$$\frac{\|Lw\|}{\|w\|} = \frac{\|v-w\|}{\|w\|} \leq \frac{\|v\| + \|w\|}{\|w\|} \leq 2K^{2l}+1.$$

Hence $\|L\| \leq \alpha^{-1}$ and then $\alpha(E(y), F(y)) \geq \alpha$.

Now, according to item 2 of Lemma 2.6, after a uniformly bounded change of metric, we may assume that the two subbundles E and F are orthogonal everywhere.

Since the method of the proof is more or less well-known, we just give here the sketch of the proof.

For any $a \in (0,1)$ and $y \in \Sigma$, denote by

$$\begin{aligned} C_a^E(y) &= \{v = v_E + v_F :\ v_E \in E(y),\ v_F \in F(y),\ \|v_F\| \leq a\|v_E\|\}, \\ C_a^F(y) &= \{v = v_E + v_F :\ v_E \in E(y),\ v_F \in F(y),\ \|v_E\| \leq a\|v_F\|\} \end{aligned}$$

the a-cone around $E(y)$ and $F(y)$ respectively. And denote by C_a^E, C_a^F the cone fields. Then there exists $\delta = \delta(K,l) \in (0,1)$ such that for any $y \in \Sigma$,

$$A^{-l}(C_\delta^E(y)) \subset C_{2\delta/3}^E(f^{-l}y), \qquad A^l(C_\delta^F(y)) \subset C_{2\delta/3}^F(f^l y)$$

and for any $u \in C_\delta^E(y),\ v \in C_\delta^F(y)$ with $\|u\| = \|v\| = 1$,

$$\frac{\|A^l u\|}{\|A^l v\|} \leq \frac{2}{3}.$$

Then there exists $\varepsilon_4 = \varepsilon_4(\delta, K, l) = \varepsilon_4(K,l)$ such that for any ε_4-perturbation B of A, one has

$$B^{-l}(C_\delta^E(y)) \subset C_{3\delta/4}^E(f^{-l}y), \qquad B^l(C_\delta^F(y)) \subset C_{3\delta/4}^F(f^l y)$$

and for any $u \in C_\delta^E(y),\ v \in C_\delta^F(y)$ with $\|u\| = \|v\| = 1$,

$$\frac{\|B^l u\|}{\|B^l v\|} \leq \frac{3}{4}.$$

Now take $L = 3l$. Then one can find two B-invariant subbundles $E(B) \subset C_\delta^E$ and $F(B) \subset C_\delta^F$ such that $\mathcal{E} = E(B) \oplus F(B)$ is an L-dominated splitting of B. This finishes the sketch of the proof. □

The second lemma says that if a SPLS robustly exhibits an eigenvalue gap, then there exists a uniform gap for its small perturbations.

Lemma 4.2. *For any given $\varepsilon > 0$ and $K \geq 1$, there exist $\varepsilon_5 = \varepsilon_5(\varepsilon, K) \in (0, \varepsilon]$ and $\gamma = \gamma(\varepsilon, K) > 1$ satisfying the following property:*

For any $d \in \mathbb{N}$, let $A \in \mathrm{GL}(\Sigma, f, \mathcal{E})$ be a d-dimensional SPLS bounded by K. And assume that there exists $1 \leq i \leq d-1$ such that every ε-perturbation of A exhibits an i-eigenvalue gap. Then for every ε_5-perturbation B of A,

$$\frac{|\lambda_{i+1}(B)|}{|\lambda_i(B)|} \geq \gamma^n.$$

Proof Let $\varepsilon_5 = \varepsilon_1(\varepsilon/2, K+\varepsilon) \leq \varepsilon/2$ be determined by Lemma 2.7. If the uniform constant $\gamma > 1$ does not exist, then for some $\varepsilon > 0$, $K \geq 1$ and any $k \in \mathbb{N}$, there exists a d_k-dimensional SPLS $(\Sigma_k, f_k, \mathcal{E}_k, A_k)$ bounded by K and integer $1 \leq i_k \leq d_k - 1$ such that every ε-perturbation of A_k exhibits an i_k-eigenvalue gap but for some ε_5-perturbation B_k of A_k,

$$\frac{|\lambda_{i_k+1}(B_k)|}{|\lambda_{i_k}(B_k)|} \leq \left(1 + \frac{1}{k}\right)^{n_k},$$

where n_k is the period of B_k.

Denote by B_{kE} the restriction of B_k to the subbundle $E^{i_k}(B_k)$. Let

$$C_k = \left(\frac{|\lambda_{i_k+1}(B_k)|}{|\lambda_{i_k}(B_k)|}\right)^{\frac{1}{n_k}} B_{kE}.$$

Then for k large enough, C_k is an ε_5-perturbation of B_{kE}. Then according to Lemma 2.7, there is an $\varepsilon/2$-perturbation D_k of B_k such that the eigenvalues of

$D_k^{n_k}(x)$ are,

$$\frac{|\lambda_{i_k+1}(B_k)|}{|\lambda_{i_k}(B_k)|}\lambda_1(B_k),\ \frac{|\lambda_{i_k+1}(B_k)|}{|\lambda_{i_k}(B_k)|}\lambda_2(B_k),\ \cdots,$$
$$\frac{|\lambda_{i_k+1}(B_k)|}{|\lambda_{i_k}(B_k)|}\lambda_{i_k}(B_k),\ \lambda_{i_k+1}(B_k),\ \cdots,\ \lambda_{d_k}(B_k).$$

So D_k has no i_k-eigenvalue gap. But

$$\|D_k - A_k\| \le \|D_k - B_k\| + \|B_k - A_k\| \le \varepsilon.$$

So D_k is an ε-perturbation of A_k, which is a contradiction. □

To prove the third result, we need a result in Linear Algebra considering uniformity of the convergence in the Spectral Radius Theorem for finite-dimensional matrices. Since we could not find a proof in standard books on Linear Algebra, we give a proof here.

Lemma 4.3. *Given $d \in \mathbb{N}$, for any $d \times d$ matrix A, the convergence*

$$\lim_{n\to\infty} \|A^n\|^{\frac{1}{n}} = \rho(A)$$

is locally uniform, where $\rho(A)$ is the largest eigenvalue $|\lambda_d(A)|$ of A in absolute value, i.e., the spectral radius of A. More precisely, there exists $\delta = \delta(A) > 0$ satisfying the following property:

For any given $\varepsilon > 0$, there exists an integer $N \in \mathbb{N}$ such that for any integer $n \ge N$ and matrix B with $\|B - A\| \le \delta$, one has

$$\|B^n\| \le (\rho(B) + \varepsilon)^n.$$

Proof We will only prove the lemma for the case $\rho(A) > 0$. The case $\rho(A) = 0$ can be proved similarly.

For any given matrix A, according to the Spectral Radius Theorem, one has

$$\lim_{n\to\infty} \|A^n\|^{\frac{1}{n}} = \rho(A).$$

Hence for any $\varepsilon > 0$, there exists an integer $N = N(\varepsilon) \in \mathbb{N}$ such that for any integer $n \ge N$, one has

$$\|A^n\| \le (\rho(A) + \varepsilon/4)^n.$$

A simple calculation shows that

$$\|B^n - A^n\| \le \|B - A\| \sum_{j=0}^{n-1} \|A\|^j \|B\|^{n-1-j}.$$

If $\|B - A\| \le 1$ (we always assume this condition in the following and that the variables ε, $\delta \le 1$) and denote by $K = \|A\| + 2$, then

$$\|B^n - A^n\| \le nK^{n-1}\|B - A\|.$$

It is well-known that $\rho(A)$ depends continuously on A (e.g., see [14, Proposition 2.18]). So there exists $\delta > 0$ such that for any $\|B - A\| \le \delta$,

$$\rho(A) - \varepsilon/4 \le \rho(B) \le \rho(A) + \varepsilon/4.$$

Moreover we may assume $\delta = \delta(A, \varepsilon, N) = \delta(A, \varepsilon)$ is small enough so that

$$\|B^N - A^N\| \le \rho(A)(\varepsilon/4)^{N-1}.$$

So

$$\|B^N\| \le \|A^N\| + \rho(A)(\varepsilon/4)^{N-1} \le (\rho(A) + 2\varepsilon/4)^N \le (\rho(B) + 3\varepsilon/4)^N.$$

For any $n \ge N$, write $n = kN + s$, $0 \le s \le N-1$. Therefore

$$\begin{aligned} \|B^n\| &= \|B^{kN+s}\| \le \|B^{kN}\|\|B^s\| \le K^N \|B^N\|^k \\ &\le K^N(\rho(B) + 3\varepsilon/4)^{kN} \le K^{2N}(\rho(B) + 3\varepsilon/4)^n. \end{aligned}$$

Take $N_1 \ge N$ large enough so that for any $n \ge N_1$, we have

$$K^{\frac{2N}{n}} \le \frac{\rho(B) + \varepsilon}{\rho(B) + 3\varepsilon/4}$$

for any $\|B - A\| \le \delta$. We get that for any $\varepsilon > 0$, there exist $\delta > 0, N_1 \in \mathbb{N}$ such that for any $n \ge N_1$ and $\|B - A\| \le \delta$, we have

$$\|B^n\| \le (\rho(B) + \varepsilon)^n.$$

Now let us prove the lemma. Suppose on the contrary, there exists a matrix A, for any $\delta > 0$, there exists $\varepsilon > 0$, for any integer $k \in \mathbb{N}$, there exist $n_k \ge k$ and a matrix B_k with $\|B_k - A\| \le \delta$,

$$\|B_k^{n_k}\| > (\rho(B_k) + \varepsilon)^{n_k}.$$

Since $\|B_k - A\| \le \delta$, $\{\|B_k\|\}_{k=1}^{\infty}$ is bounded. By taking subsequence, we may assume that $B_k \to C$ as $k \to \infty$. Since $n_k \ge k \to \infty$, applying the result of the last paragraph to the matrix C, we get a contradiction. This finishes the proof of the lemma. □

An immediate consequence of Lemma 4.3 is that for every integer $d \in \mathbb{N}$, the convergence in the Spectral Radius Theorem is uniform on any bounded subset of the set of $d \times d$ matrices. A weak version of this lemma is used in [20].

Lemma 4.4. *For any given $\varepsilon > 0$, $K \ge 1$, $\alpha > 0$, $C \ge 1$, $\gamma > 0$ and $d \in \mathbb{N}$, there exists an integer $l = l(\varepsilon, K, \alpha, C, \gamma, d) \in \mathbb{N}$ satisfying the following property:*

Let $(\Sigma, f, \mathcal{E}, A)$ be a d-dimensional SPLS bounded by K. And assume that

C0. there exists $1 \le i \le d-1$ such that every ε-perturbation of A exhibits an i-eigenvalue gap. Moreover, suppose

C1. the angle $\alpha(E^i(y, B), F^i(y, B)) \ge \alpha$ for every $y \in \Sigma$ and every ε-perturbation B of A, and

C2. for every $y \in \Sigma$,

$$\|A^n_{E^i(y,A)}\| \, \|A^{-n}/E^i(f^n y, A)\| \le C\gamma^{-n}. \tag{4.4}$$

(where n is the period of $(\Sigma, f, \mathcal{E}, A)$)

Then A has an (l, i)-dominated splitting.

Proof The proof given here is more or less a standard process to show the existence of dominated splitting (e.g., see [12, 20]).

First, let us define some constants. Let $\delta = \delta(\varepsilon, K+\varepsilon)$ be determined by item 1 of Lemma 2.6. Take $\varepsilon' > 0$ such that $2\varepsilon' + \varepsilon'^2 < \delta$ and $\varepsilon' < \gamma - 1$. And then take integer $m \in \mathbb{N}$ such that

$$\varepsilon'(1+\varepsilon')^m \geq 1 + \frac{2}{\alpha}, \qquad C\left(\frac{1+\varepsilon'}{\gamma}\right)^m < 1.$$

For any d-dimensional SPLS $(\Sigma, f, \mathcal{E}, A)$ bounded by K and satisfying C0, C1 and C2, we will show first that if $n > m$, then A has an (m, i)-dominated splitting.

According to item 2 of Lemma 2.6, after a uniformly bounded change of metric on $\mathcal{E}$, we may assume that $E^i(y, A)$ is orthogonal to $F^i(y, A)$ for every $y \in \Sigma$. And now, $A/E^i(y, A) = A_{F^i(y,A)}$.

We make some explanations for this citing of item 2 of Lemma 2.6. Under the new metric, the constants C and γ in (4.4) are not modified since we do not change the metric on $E^i(A)$ and $F^i(A)$, but the constants ε, K will be changed according to Lemma 2.6 and by the definition of angle between two subspaces, one can also verify that the new α depends only on ε, K and the old α, which determines the bound of two metrics, i.e., the bound of $\|P\|$ in Lemma 2.6.

Let us continue the proof. Suppose on the contrary that for (some given $\varepsilon > 0$, $K \geq 1$, $\alpha > 0$, $C \geq 1$, $\gamma > 0$ and $d \in \mathbb{N}$ and) some d-dimensional SPLS A bounded by K and satisfying C0, C1 and C2 and for some $y \in \Sigma$ with period $n > m$, one has

$$\|A^m_{E^i(y)}\|\,\|A^{-m}_{F^i(f^m y)}\| > \frac{1}{2}. \tag{4.5}$$

We will construct a perturbation B of A such that $B(f^j y) = P_j \circ A(f^j y) \circ Q_j$ for $j = 0, 1, \cdots, n-1$, where

$$\|Q_j - I\|,\ \|Q_j^{-1} - I\|,\ \|P_j - I\|,\ \|P_j^{-1} - I\| \leq \delta$$

and

$$\alpha(E^i(f^m y, B), F^i(f^m y, B)) < \alpha.$$

Then according to Lemma 2.6, B is an ε-perturbation of A. So the above inequality contradicts the condition $C1$ in the lemma.

By (4.5), we can take two unit vectors $w \in E^i(y, A), v \in F^i(y, A)$ such that

$$\frac{\|A^m(y)(w)\|}{\|A^m(y)(v)\|} > \frac{1}{2}.$$

Take a linear map $L : F^i(y, A) \to E^i(y, A)$ such that $Lv = \varepsilon' w$ and $\|L\| = \varepsilon'$. And define $\overline{L} : F^i(y, A) \to E^i(y, A)$ by

$$\overline{L} = (1+\varepsilon')^n A^n_{E^i(y,A)} \circ L \circ A^{-n}_{F^i(y,A)}. \tag{4.6}$$

Then according to the choice of ε',

$$\|\overline{L}\| \leq (1+\varepsilon')^n \|L\| C\gamma^{-n} < \|L\| \leq \varepsilon'.$$

Define $P, S : \mathcal{E}(y) \to \mathcal{E}(y)$ by

$$P = \begin{pmatrix} I & L \\ 0 & I \end{pmatrix}, \quad S = \begin{pmatrix} I & \overline{L} \\ 0 & I \end{pmatrix},$$

where the matrix is respect to the decomposition

$$\mathcal{E}(y) = E^i(y, A) \oplus F^i(y, A).$$

According to our choice of metric, this is an orthogonal decomposition. In the following, all matrices are respect to this kind of decomposition.

Now define $T_j : \mathcal{E}(f^j y) \to \mathcal{E}(f^j y)$, $j = 0, 1, \cdots, n-1$, by

$$T_j = \begin{pmatrix} (1+\varepsilon')I & 0 \\ 0 & I \end{pmatrix}.$$

Now define the perturbation B of A by

$$\begin{aligned} B(y) &= T_1 \circ A(y) \circ P, \\ B(f^j y) &= T_{j+1} \circ A(f^j y), \quad j = 1, 2, \cdots, n-2, \\ B(f^{n-1} y) &= S^{-1} \circ T_0 \circ A(f^{n-1} y). \end{aligned}$$

According to the choice of ε', one can easily verify that B is an ε-perturbation of A.

Now let us calculate $E^i(y, B)$, $F^i(y, B)$ and $\alpha(E^i(f^m y, B),\ F^i(f^m y, B))$.

Denote by

$$T^j = \begin{pmatrix} (1+\varepsilon')^j I & 0 \\ 0 & I \end{pmatrix}.$$

Then

$$\begin{aligned} B^n(y) &= S^{-1} \circ T^n \circ A^n(y) \circ P \\ &= \begin{pmatrix} I & -\overline{L} \\ 0 & I \end{pmatrix} \begin{pmatrix} (1+\varepsilon')^n I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A^n_{E^i(y,A)} & 0 \\ 0 & A^n_{F^i(y,A)} \end{pmatrix} \begin{pmatrix} I & L \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} (1+\varepsilon')^n A^n_{E^i(y,A)} & 0 \\ 0 & A^n_{F^i(y,A)} \end{pmatrix}. \end{aligned}$$

The reason for the last equality is the definition (4.6) of $\overline{L}$. So

$$E^i(y, B) = E^i(y, A), \qquad F^i(y, B) = F^i(y, A).$$

But

$$\begin{aligned} B^m(y) &= T^m \circ A^m(y) \circ P \\ &= \begin{pmatrix} (1+\varepsilon')^m I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A^m_{E^i(y,A)} & 0 \\ 0 & A^m_{F^i(y,A)} \end{pmatrix} \begin{pmatrix} I & L \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} (1+\varepsilon')^m A^m_{E^i(y,A)} & (1+\varepsilon')^m A^m_{E^i(y,A)} \circ L \\ 0 & A^m_{F^i(y,A)} \end{pmatrix}. \end{aligned}$$

Define $L_m : F^i(f^m y, A) \to E^i(f^m y, A)$ by

$$L_m = (1+\varepsilon')^m A^m_{E^i(y,A)} \circ L \circ A^{-m}_{F^i(f^m y, A)}.$$

Then $F^i(f^m y, B)$ is the graph of L_m and $E^i(f^m y, B) = E^i(f^m y, A)$. Hence

$$\alpha(E^i(f^m y, B),\ F^i(f^m y, B)) = \|L_m\|^{-1}.$$

Since

$$\|L_m\| \geq \frac{\|L_m(A^m v)\|}{\|A^m v\|} = \varepsilon'(1+\varepsilon')^m/2 > \frac{1}{\alpha},$$

we have $\alpha(E^i(f^m y, B), F^i(f^m y, B)) < \alpha$, a desired contradiction.

Finally, let $\gamma_1 = \sqrt[3]{\gamma(\varepsilon, K)}$ be determined by Lemma 4.2. And by Lemma 4.3, take $l = km$ and $k = jm!$ large enough so that $\gamma_1^k \geq 2$ and

$$\|C\|^k \leq (\gamma_1 \rho(C))^k$$

for every matrix C with $\|C\| \leq K^m$. One can easily verify that

$$\|A^l_{E^i(y)}\|\,\|A^{-l}_{F^i(f^l y)}\| \leq \frac{1}{2} \tag{4.7}$$

for every d-dimensional SPLS $(\Sigma, f, \mathcal{E}, A)$ with period $\leq m$ and every $y \in \Sigma$. Then l satisfies the demand of the lemma. This finishes the proof of the lemma. $\square$

5. Some perturbation lemmas for lower order ($d \leq 4$) matrices

In this section, we will present four auxiliary lemmas, which deal with the perturbations of 2×2, 3×3 and 4×4 matrices. Three of them, Lemma 5.1, 5.3 and 5.5, involve angles between two eigenspaces. Lemma 5.1, considers 2×2 matrices with two real eigenvalues, which is contained in [5]. Lemma 5.3, is new, which considers 3×3 matrices with a pair of conjugate complex eigenvalues and a real eigenvalue. And Lemma 5.5 is, in some sense, a special case of Lemma II.9 in [12]. In fact, our proof follows the method in [12]. Lemma 5.2 gives a condition for 2×2 matrices with complex eigenvalues to be perturbed to one with real eigenvalues.

Lemma 5.1. [5, Fact 4.13] *For every $\delta > 0$, $\gamma > 1$, there exists $\alpha = \alpha(\delta, \gamma) > 0$ such that for every 2×2 matrix Q with the form*

$$Q = \begin{pmatrix} 1 & * \\ 0 & \mu \end{pmatrix},$$

such that $|\mu| > \gamma$ and the angle between the eigenspace E of 1 and the eigenspace F of μ is $\leq \alpha$, then there is a matrix

$$P = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$$

with $\|P - I\| \leq \delta$ such that the modulus of the two eigenvalues of $P \circ Q$ are equal to $|\mu|^{\frac{1}{2}}$.

We give the motivation for the following three lemmas. ¿From Linear Algebra, we know that for every linear isomorphism $A : \mathbb{R}^2 \to \mathbb{R}^2$ with a pair of conjugate complex eigenvalues $\{e^{i\theta},\ e^{-i\theta}\}$, under a proper orthonormal basis and for some $C \geq 1$, A has the form

$$R_{\theta, C} = P_C^{-1} \circ R_\theta \circ P_C = \begin{pmatrix} \cos\theta & -C\sin\theta \\ C^{-1}\sin\theta & \cos\theta \end{pmatrix}, \tag{5.8}$$

where

$$P_C = \begin{pmatrix} 1 & 0 \\ 0 & C \end{pmatrix}, \qquad R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Hence, after a change of metric, bounded by C, A has the form R_θ under a proper orthonormal basis of the new metric.

Lemma 5.2 controls the bound of C. Once C is uniformly bounded and we assume $C = 1$ after a uniformly bounded change of metric. So we only have to deal with R_θ, which is the object of Lemma 5.3 and 5.5 for dim = 3 and dim = 4 respectively.

Lemma 5.2. *For any $\delta > 0$, there exists $\eta = \eta(\delta) > 0$ such that if $|C^{-1}\sin\theta| \le \eta$ then exists a 2×2 matrix P with $\|P - I\| \le \delta$ so that $P \circ R_{\theta, C}$ is real-diagonalizable.*

Proof If $|\tan\theta| < 1$, then take $\eta = \delta$ and

$$P = \begin{pmatrix} 1 & 0 \\ -C^{-1}\tan\theta & 1 \end{pmatrix}.$$

Otherwise, take $\eta = \delta/8$ and

$$P = \begin{pmatrix} 1 & 0 \\ 4C^{-1}/\sin\theta & 1 \end{pmatrix}.$$

□

Lemma 5.3. *For every $\delta > 0$, $\eta > 0$ and $\gamma > 1$, there exists $\alpha = \alpha(\delta, \eta, \gamma) > 0$ such that for every matrix Q with the form*

$$Q = \begin{pmatrix} \cos\theta & -\sin\theta & p \\ \sin\theta & \cos\theta & q \\ 0 & 0 & \mu \end{pmatrix},$$

such that $|\mu| > \gamma$, $|\sin\theta| \ge \eta$ and the angle between the eigenspace E of $\{e^{i\theta},\ e^{-i\theta}\}$ and the eigenspace F of μ is $\le \alpha$, then there is a matrix P with the form

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{pmatrix}$$

such that $\|P - I\| \le \delta$ and the modulus of the three eigenvalues of $P \circ Q$ are all equal to $|\mu|^{\frac{1}{3}}$.

Proof Let $(x, y, 1)^T$ be an eigenvector of Q corresponding to the eigenvalue μ. By the definition of angle between two subspaces, $\alpha(E, F) = \frac{1}{\sqrt{x^2+y^2}}$. An easy calculation shows that

$$\begin{pmatrix} x \\ y \end{pmatrix} = (\mu - R_\theta)^{-1} \begin{pmatrix} p \\ q \end{pmatrix}.$$

And it is easy to see that

$$p^2 + q^2 \ge (|\mu| - 1)^2(x^2 + y^2) = (|\mu| - 1)^2 \alpha(E, F)^{-2}.$$

Let $\det(\lambda - P \circ Q) = \lambda^3 + a_2\lambda^2 + a_1\lambda - \mu$. If we can take two numbers a, b such that $a_2 = a_1 = 0$, then the modulus of the three eigenvalues of $P \circ Q$ are all equal to $|\mu|^{\frac{1}{3}}$. Solving the equations $a_2 = a_1 = 0$, we get

$$\begin{aligned} a &= \frac{q\cos 2\theta - p\sin 2\theta - p\mu\sin\theta}{(p^2+q^2)\sin\theta}, \\ b &= \frac{-p\cos 2\theta - q\sin 2\theta - q\mu\sin\theta}{(p^2+q^2)\sin\theta}. \end{aligned}$$

An easy computation shows that

$$a^2 + b^2 \leq \frac{4(|\mu|+1)^2}{(p^2+q^2)\eta^2} \leq \frac{4}{\eta^2}\frac{(|\mu|+1)^2}{(|\mu|-1)^2}\alpha(E,F)^2 \leq \frac{4}{\eta^2}\frac{(\gamma+1)^2}{(\gamma-1)^2}\alpha(E,F)^2.$$

¿From the above formula, one can take $\alpha = \alpha(\delta, \eta, \gamma)$ easily. □

Remark 5.4. Note that the perturbation PQ of Q in Lemma 5.1 and 5.3 has no eigenvalue gap of any type.

Lemma 5.5. *For every $\delta > 0$ and $\gamma > 1$, there exists $\alpha = \alpha(\delta, \gamma) > 0$ such that for every matrix Q with the form*

$$Q = \begin{pmatrix} R_\theta & D \\ 0 & \mu R_\phi \end{pmatrix},$$

where D is a 2×2 matrix, such that $\mu > \gamma$ and the angle between the eigenspace E of $\{e^{i\theta},\ e^{-i\theta}\}$ and the eigenspace F of $\{\mu e^{i\phi},\ \mu e^{-i\phi}\}$ is $\leq \alpha$, then there is a matrix

$$P = \begin{pmatrix} I & 0 \\ C & I \end{pmatrix} \tag{5.9}$$

with $\|P - I\| \leq \delta$ (where C is a 2×2 matrix) such that the matrix $P \circ Q$ has a real eigenvalue equal to $\sqrt{\mu}$. Especially, $P \circ Q$ has at most one pair of conjugate complex eigenvalues.

Proof (Following [12, Lemma II.9]) Let $L : E^\perp \to E$ be a linear map such that the graph of L is equal to F, i.e., $F = \{y + Ly : y \in E^\perp\}$. Since $Q(F) = F$, we have that for any $y \in E^\perp$, there exists $y' \in E^\perp$ such that

$$\begin{pmatrix} R_\theta & D \\ 0 & \mu R_\phi \end{pmatrix} \begin{pmatrix} Ly \\ y \end{pmatrix} = \begin{pmatrix} Ly' \\ y' \end{pmatrix}.$$

So

$$R_\theta L + D = \mu L R_\phi.$$

Hence

$$L = \mu^{-1} R_\theta L R_{-\phi} + \mu^{-1} D R_{-\phi}.$$

We get

$$\|L\| \leq \mu^{-1}\|L\| + \mu^{-1}\|D\|.$$

Hence

$$\|L\| \leq \frac{1}{\mu - 1}\|D\|.$$

Since $\alpha(E,F)=\|L\|^{-1}$, we have

$$\|D\| \geq \frac{\mu-1}{\alpha(E,F)}.$$

Denote $\beta=\sqrt{\mu}>1$. We will find a 2×2 matrix C with $\|C\|\leq\delta$ and $v=(x,y)^T\neq 0$ such that $P\circ Q(v)=\beta v$, where P has the form (5.9). Since

$$P\circ Q=\begin{pmatrix} I & 0\\ C & I\end{pmatrix}\begin{pmatrix} R_\theta & D\\ 0 & \mu R_\phi\end{pmatrix}=\begin{pmatrix} R_\theta & D\\ CR_\theta & CD+\mu R_\phi\end{pmatrix},$$

We get the equations

$$\begin{cases} R_\theta x+Dy=\beta x\\ CR_\theta x+(CD+\mu R_\phi)y=\beta y.\end{cases}$$

¿From the first equality of the above formula, we get

$$x=(\beta-R_\theta)^{-1}Dy.$$

Substituting into the second equality, we get

$$CR_\theta(\beta-R_\theta)^{-1}Dy+CDy=(\beta-\mu R_\phi)y.$$

And then

$$C(\beta-R_\theta)^{-1}Dy=(1-\frac{\mu}{\beta}R_\phi)y.$$

Take a unit vector y such that $\|Dy\|=\|D\|$. Denote

$$u=(\beta-R_\theta)^{-1}Dy, \qquad v=(1-\frac{\mu}{\beta}R_\phi)y.$$

Then $\|u\|\geq\dfrac{\mu-1}{\alpha(E,F)(\beta+1)}$ and $\|v\|\leq 1+\dfrac{\mu}{\beta}=\dfrac{\mu+\beta}{\beta}$. Take C such that $Cu=v$ and $\|C\|=\dfrac{\|v\|}{\|u\|}$.

Then

$$\|C\|\leq\frac{(\mu+\beta)(\beta+1)}{\beta(\mu-1)}\alpha(E,F)=\frac{\sqrt{\mu}+1}{\sqrt{\mu}-1}\alpha(E,F)<\frac{\sqrt{\gamma}+1}{\sqrt{\gamma}-1}\alpha(E,F).$$

¿From the above inequality, it is easy to take $\alpha=\alpha(\delta,\gamma)$ to satisfy this lemma. □

6. Proof of Proposition 2.5 for $d\leq 4$

In this section we will use the estimates in the last section to prove Proposition 2.5 for the special cases: $d=2,\ 3$ and $d=4,\ i=2$.

Proof of Proposition 2.5 for $d=2$:
Assume that $\delta=\delta(\varepsilon/2,K+\varepsilon)$ is determined by item 1 of Lemma 2.6, $\gamma=\gamma(\varepsilon,K)>0$ determined by Lemma 4.2, $\alpha=\alpha(\delta,\gamma)>0$ determined by Lemma 5.1 and $l=l(\varepsilon/2,K,\alpha,C=1,\gamma,d=2)\in\mathbb{N}$ determined by Lemma 4.4. Then we will show that l satisfies Proposition 2.5 for $d=2$.

Let $(\Sigma,f,\mathcal{E},A)$ be any 2-dimensional SPLS bounded by K such that every ε-perturbation of A exhibits an 1-eigenvalue gap.

For any $\varepsilon/2$-perturbation B of A, let the standard eigenspaces decomposition of B be

$$\mathcal{E} = E(B) \oplus F(B).$$

According to Lemma 4.2, $\gamma^n|\lambda_1(B)| \leq |\lambda_2(B)|$ and under the orthogonal decomposition

$$\mathcal{E}(x) = E(x,B) \oplus \mathcal{E}(x)/E(x,B),$$

$B^n(x)$ has the form

$$\lambda_1(B)\begin{pmatrix} 1 & * \\ 0 & \dfrac{\lambda_2(B)}{\lambda_1(B)} \end{pmatrix}.$$

Since any $\varepsilon/2$-perturbation of B also exhibits a 1-eigenvalue gap, according to Lemma 5.1,

$$\alpha(E(B), F(B)) \geq \alpha.$$

Then by Lemma 4.4 (for $C = 1$), $E(A) \oplus F(A)$ is an l-dominated splitting. This finishes the proof for this case. □

Proof of Proposition 2.5 for $d = 3$**:** We only give a proof for $i = 2$. The case $i = 1$ can be proved similarly. We will determine the uniform constant $l = l(\varepsilon, K, d = 3)$ in three steps.

First, let $\varepsilon' = \varepsilon_5(\varepsilon, K)$, $\gamma = \gamma(\varepsilon, K)$ be determined by Lemma 4.2.

Step 1: Let $\varepsilon_1' = \varepsilon_3(\varepsilon, K)$ be determined by Lemma 3.3 and $l_0 = l(\varepsilon_1', K)$ be determined by Proposition 2.5 for $d = 2$. And then let $L_0 = L(K, l_0, d = 3)$ be determined by Lemma 3.2. We claim that if $(\Sigma, f, \mathcal{E}, A)$ is any 3-dimensional complex diagonalizable SPLS bounded by K with $t(A) = (1,1,1)$ such that every ε-perturbation of A exhibits a 2-eigenvalue gap, then A admits an $(L_0, 2)$-dominated splitting

Let the standard eigenvalues decomposition of A be

$$\mathcal{E} = E_1 \oplus E_2 \oplus E_3.$$

According to Lemma 3.3, every ε_1'-perturbation of A_{23} and A_{13} exhibits a 1-eigenvalue gap and hence $E_2 \prec_{l_0} E_3$, $E_1/E_2 \prec_{l_0} E_3/E_2$. Then by Lemma 3.2, $(E_1 \oplus E_2) \prec_{L_0} E_3$. This proves the claim.

Step 2. Let $l_1 = L_0(\varepsilon/2, K + \varepsilon)$ be determined by Step 1 for constants $\varepsilon/2$ and $K+\varepsilon$, $\varepsilon_2' = \varepsilon_4(K+\varepsilon, l_1)$, $L_1 = L(K+\varepsilon, l_1)$ (we may assume that $\varepsilon_2' \leq \min\{\varepsilon/2, \varepsilon'\}$) determined by Lemma 4.1, $\delta = \delta(\varepsilon_2'/2, K + \varepsilon)$ determined by item 1 of Lemma 2.6 and $\eta = \eta(\delta)$ determined by Lemma 5.2.

We claim that if $(\Sigma, f, \mathcal{E}, A)$ is any 3-dimensional complex-diagonalizable SPLS bounded by K with $t(A) = (2, 1)$ such that every ε-perturbation of A exhibits a 2-eigenvalue gap, and for some orthonormal basis and $C \geq 1$, $A^n(x)$ has the form,

$$\begin{pmatrix} |\mu_1| R_{\theta, C} & * \\ 0 & \mu_2 \end{pmatrix} \tag{6.10}$$

with $|C^{-1} \sin\theta| \leq \eta$, where $R_{\theta, C}$ is defined by (5.8), then A has an $(L_1, 2)$-dominated splitting.

In fact, we will show that for any $\varepsilon_2'/2$-perturbation B of A with the form (6.10) and $|C^{-1}\sin\theta| \leq \eta$, then B has an $(L_1, 2)$-dominated splitting.

Let B be any $\varepsilon_2'/2$-perturbation of A with the form (6.10) and $|C^{-1}\sin\theta| \leq \eta$. According to Lemma 2.6 and 5.2 (see Lemma 2.7 for the form of perturbation), there exists $\varepsilon_2'/2$-perturbation $\widetilde{B}$ of B such that $t(\widetilde{B}) = (1,1,1)$. Since $\varepsilon_2' \leq \varepsilon/2$, every $\varepsilon/2$-perturbation of $\widetilde{B}$ exhibits a 2-eigenvalue gap. Hence $\widetilde{B}$ has an $(l_1, 2)$-dominated splitting. Then according to Lemma 4.1 and $\|\widetilde{B} - B\| \leq \varepsilon_2'/2$, B has an $(L_1, 2)$-dominated splitting.

Step 3. Let

$$\alpha = \min\{\alpha(\delta,\eta,\gamma),\ \frac{1}{2(K+\varepsilon)^{2(l_1+L_1)}+1}\},$$

where $\alpha(\delta,\eta,\gamma)$ is determined by Lemma 5.3, and

$$l_2 = l(\varepsilon_2'/2,\ K+\varepsilon,\ \alpha,\ C=1,\ \gamma,\ d=3)$$

determined by Lemma 4.4.

We claim that if $(\Sigma, f, \mathcal{E}, A)$ is any 3-dimensional complex-diagonalizable SPLS bounded by K with $t(A) = (2,1)$ such that every ε-perturbation of A exhibits a 2-eigenvalue gap, and for some orthonormal basis, $A^n(x)$ has the form (6.10), with $|C^{-1}\sin\theta| \geq \eta$, then A has an $(l_2, 2)$-dominated splitting.

Let B be any $\varepsilon_2'/2$-perturbation of A. We first show that $\alpha(E(B), F(B)) \geq \alpha$. Since $\mathrm{GL}_c(\Sigma, f, \mathcal{E})$ is dense in $\mathrm{GL}(\Sigma, f, \mathcal{E})$, we only have to prove for $B \in \mathrm{GL}_c(\Sigma, f, \mathcal{E})$.

If $t(B) = (1,1,1)$, then $E(B) \prec_{l_1} F(B)$ and hence

$$\alpha(E(B), F(B)) \geq \frac{1}{2(K+\varepsilon)^{2l_1}+1}$$

by the formula given in Lemma 4.1.

If $t(B) = (2,1)$ and for some orthonormal basis, $B^n(x)$ has the form (6.10) with $|C^{-1}\sin\theta| \leq \eta$, then according to Step 2, B has an $(L_1, 2)$-dominated splitting $E(B) \oplus F(B)$. Hence

$$\alpha(E(B), F(B)) \geq \frac{1}{2(K+\varepsilon)^{2L_1}+1}.$$

If $t(B) = (2,1)$ and for some orthonormal basis, $B^n(x)$ has the form (6.10) with $|C^{-1}\sin\theta| \geq \eta$, then $C^{-1} \geq \eta$ and $|\sin\theta| \geq \eta$. So, after a uniformly bounded change of metric, we may assume that $C = 1$. Since $\varepsilon_2' \leq \varepsilon/2$, every $\varepsilon_2'/2$-perturbation of B has a 2-eigenvalue gap, by Lemma 5.3,

$$\alpha(E(B), F(B)) \geq \alpha(\delta,\eta,\gamma).$$

This proves that $\alpha(E(B), F(B)) \geq \alpha$ for every $\varepsilon_2'/2$-perturbation B of A.

Now, according to Lemma 4.4, A admits an $(l_2, 2)$-dominated splitting.

Let

$$l = L_0 L_1 l_2.$$

Hence in any case, we have that A admits an $(l, 2)$-dominated splitting. $\square$

Proof of Proposition 2.5 for $d = 4$ and $i = 2$: The proof of this case is similar to $d = 3$. We still give the precise determination of constants for completeness.

First, let $\varepsilon' = \varepsilon_5(\varepsilon, K)$, $\gamma = \gamma(\varepsilon, K)$ be determined by Lemma 4.2.

Step 1. Let $\varepsilon_1' = \varepsilon_3(\varepsilon, K)$ be determined by Lemma 3.3 and $l_0 = l(\varepsilon_1', K)$ be determined by Proposition 2.5 for $d = 2,\ 3$. And then let $L_0 = L(K, l_0)$ be determined by Lemma 3.2. We claim that if $(\Sigma, f, \mathcal{E}, A)$ is any 4-dimensional complex-diagonalizable SPLS bounded by K such that every ε-perturbation of A exhibits a 2-eigenvalue gap and

$$t(A) \in \{(1,1,1,1),\ (2,1,1),\ (1,1,2)\},$$

then A has an $(L_0, 2)$-dominated splitting.

We omit the proof since it is similar to Step 1 for $d = 3$ and the proof for general d in §3.

Step 2. Let $l_1 = L_0(\varepsilon/2, K + \varepsilon)$ be determined by Step 1, $\varepsilon_2' = \varepsilon_4(K + \varepsilon, l_1)$, $L_1 = L(K+\varepsilon, l_1)$ (we may assume that $\varepsilon_2' < \min\{\varepsilon/2, \varepsilon'\}$) be determined by Lemma 4.1, $\delta = \delta(\varepsilon_2'/2, K + \varepsilon)$ determined by item 1 of Lemma 2.6, $\eta = \eta(\delta)$ determined by Lemma 5.2,

We claim that if $(\Sigma, f, \mathcal{E}, A)$ is any 4-dimensional complex-diagonalizable SPLS bounded by K with $t(A) = (2, 2)$ such that every ε-perturbation of A exhibits a 2-eigenvalue gap, and for some orthonormal basis and $C \geq 1,\ D \geq 1,\ A^n(x)$ has the form,

$$\begin{pmatrix} |\mu_1| R_{\theta,\,C} & * \\ 0 & |\mu_2| R_{\phi,\,D} \end{pmatrix} \tag{6.11}$$

with $|C^{-1} \sin\theta| \leq \eta$ or $|D^{-1} \sin\phi| \leq \eta$, then A has an $(L_1, 2)$-dominated splitting.

In fact, we will show that for any $\varepsilon_2'/2$-perturbation B of A with the form (6.11), $|C^{-1} \sin\theta| \leq \eta$ or $|D^{-1} \sin\phi| \leq \eta$, then B has an $(L_1, 2)$-dominated splitting.

Let B be any $\varepsilon_2'/2$-perturbation of A with the form (6.11) with $|C^{-1} \sin\theta| \leq \eta$ or $|D^{-1} \sin\phi| \leq \eta$. According to Lemma 2.6 and 5.2, there exists $\varepsilon_2'/2$-perturbation $\widetilde{B}$ of B such that $t(\widetilde{B}) = (2, 1, 1)$ or $(1, 1, 2)$. Since $\varepsilon_2' \leq \varepsilon/2$, every $\varepsilon/2$-perturbation of $\widetilde{B}$ exhibits a 2-eigenvalue gap. Hence $\widetilde{B}$ has an $(l_1, 2)$-dominated splitting. Then according to Lemma 4.1 and $\|\widetilde{B} - B\| \leq \varepsilon_2'/2$, B has an $(L_1, 2)$-dominated splitting.

Step 3. Let

$$\alpha = \min\{\alpha(\delta,\gamma),\ \frac{1}{2(K+\varepsilon)^{2(l_1+L_1)} + 1}\},$$

where $\alpha(\delta, \gamma)$ is determined by Lemma 5.5, and

$$l_2 = l(\varepsilon_2'/2,\ K + \varepsilon,\ \alpha,\ C = 1,\ \gamma,\ d = 4)$$

determined by Lemma 4.4.

We claim that if $(\Sigma, f, \mathcal{E}, A)$ is any 4-dimensional complex-diagonalizable SPLS bounded by K with $t(A) = (2, 2)$ such that every ε-perturbation of A exhibits a 2-eigenvalue gap and for some orthonormal basis, $A^n(x)$ has the form (6.11) with $|C^{-1} \sin\theta| \geq \eta$ and $|D^{-1} \sin\phi| \geq \eta$, then A has an $(l_2, 2)$-dominated splitting.

Let B be any $\varepsilon_2'/2$-perturbation of A. We first show that $\alpha(E(B), F(B)) \geq \alpha$. We only have to prove for $B \in \mathrm{GL}_c(\Sigma, f, \mathcal{E})$.

If $t(B) \in \{(1,1,1,1),(2,1,1),(1,1,2)\}$, then $E(B) \prec_{l_1} F(B)$ and hence

$$\alpha(E(B),F(B)) \geq \frac{1}{2(K+\varepsilon)^{2l_1}+1}.$$

If $t(B) = (2,2)$ and for some orthonormal basis, $B^n(x)$ has the form (6.11) with $|C^{-1}\sin\theta| \leq \eta$ or $|D^{-1}\sin\phi| \leq \eta$, then according to Step 2, $E(B) \prec_{L_1} F(B)$ and hence

$$\alpha(E(B),F(B)) \geq \frac{1}{2(K+\varepsilon)^{2L_1}+1}.$$

If the type of B is $(2,2)$ and for some orthonormal basis, $B^n(x)$ has the form (6.11) with $|C^{-1}\sin\theta| \geq \eta$ and $|D^{-1}\sin\phi| \geq \eta$, then

$$C^{-1} \geq \eta, \quad |\sin\theta| \geq \eta, \quad D^{-1} \geq \eta, \quad |\sin\phi| \geq \eta.$$

So, after a uniformly bounded change of metric, we may assume that $C = D = 1$.

If

$$\alpha(E(B),F(B)) \leq \alpha(\delta,\gamma),$$

by Lemma 5.5, there exists $\varepsilon_2'/2$-perturbation $\widetilde{B}$ (since $\mathrm{GL}_c(\Sigma,f,\mathcal{E})$ is dense in $\mathrm{GL}(\Sigma,f,\mathcal{E})$, we may assume that $\widetilde{B} \in \mathrm{GL}_c(\Sigma,f,\mathcal{E})$) of B such that

$$t(\widetilde{B}) \in \{(1,1,1,1),\ (2,1,1),\ (1,1,2)\}.$$

Since $\varepsilon_2' \leq \varepsilon/2$, every $\varepsilon/2$-perturbation of $\widetilde{B}$ exhibits a 2-eigenvalue gap, $\widetilde{B}$ has an $(l_1,2)$-dominated splitting and hence B has an $(L_1,2)$-dominated splitting. So,

$$\alpha(E(B),F(B)) \geq \frac{1}{2(K+\varepsilon)^{2L_1}+1}.$$

This proves that $\alpha(E(B),F(B)) \geq \alpha$ for every $\varepsilon_2'/2$-perturbation B of A.

Now, according to Lemma 4.4, A admits an $(l_2,2)$-dominated splitting.

Let

$$l = L_0L_1l_2.$$

Hence in any case, we have that A admits an $(l,2)$-dominated splitting. This finishes the proof for $d=4,\ i=2$. □

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