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# TORAL AUTOMORPHISMS AND CHAOTIC MAPS ON THE RIEMANN SPHERE

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Abstract. Let $L : C \to C$ be a hyperbolic automorphism. Then the hyperbolic toral automorphism $L_A : T^2 \to T^2$, induced by $L$, is a chaotic map [De pg.192]. In this paper, we characterize hyperbolic toral automorphisms by proving the converse of the above statement.

## 1. Introduction

Let $(2, 2, 2, 2)$ be ramification indices for the Riemann sphere. It is well known that the regular branched covering map corresponding to this, is the Weierstrass $\mathcal{P}$ function. Lattès [L][See also De pg.291] gives a chaotic rational function $R(z) = \frac{z^4+\frac{1}{2}g_2z^2+\frac{1}{16}g_2^2}{4z^3-g_2z}$ on $\overline{C}$ which is induced by the Weierstrass $\mathcal{P}$ function and the linear map $L(z) = 2z$ on the complex plane $C$. Recently the author classified chaotic maps of the Riemann sphere $\bar{C}$ which are induced by regular branched coverings of $\bar{C}$ and the linear map $2z$ [L1].

Let $L : C \to C$ be a hyperbolic automorphism, i.e., $L(x) = Ax$, where $A$ is a $2\times 2$ integer matrix, $|det(A)| = 1$ and hyperbolic. Then a hyperbolic toral automorphism $L_A : T^2 \to T^2$, which is induced by $L$, is a chaotic map [De pg.192].

Let $L_A : T^2 \to T^2$ be a hyperbolic toral automorphism and let $\mathcal{P} : T^2 \to \bar{C}$ be the Weierstrass $\mathcal{P}$ function. Then we have a commutative diagram which induces a homeomorphism of $C$ onto itself. Then we can construct (countably many) chaotic homeomorphisms, which is not holomorphic, induced by hyperbolic toral automorphism and the Weierstrass $\mathcal{P}$ function [L2].

Now let $A$ be a $2\times 2$ integer matrix with $|det(A)| = 1$. If $A$ is non-hyperbolic then we have 3 cases for characteristic solutions $\lambda$ of $A$ : (1) $\lambda$'s are complex numbers, (2) $\lambda = \pm 1$ or (3) $\lambda = 1$ or $\lambda = -1$ with multiple root. We characterize hyperbolic toral automorphisms by proving that if $A$ is non-hyperbolic then $L_A$ is not a chaotic map.

## 2. Hyperbolic Toral Automorphisms and Branched Covering Maps

Let $f : M \to M$ be a map of metric space $X$. A map $f : M \to M$ is *chaotic* iff $f$ has sensitive dependence on initial conditions, $f$ is topologically transitive and

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periodic points are dense in $X$. Remark that $f$ is *topologically transitive* iff for any pair of open sets $U, V \subset M$ there exists $k > 0$ such that $f^k(U) \cap V \neq \emptyset$. We refer to the reader [De] for detailed definition and examples of chaotic map.

The following simple characterization of chaotic map which is proved by Touhey [T], is very useful to prove whether a map is chaotic or not.

**Proposition 2.1.** [T] *A map $f : M \to M$ is chaotic iff for all non-empty open sets $U$ and $V$ of $M$, $f$ has a periodic orbit $\Gamma$ such that $\Gamma \cap U \neq \emptyset$ and $\Gamma \cap V \neq \emptyset$.* □

Let $\Lambda$ be the lattice induced by $w_1, w_2 \in C$ with $w_1/w_2 \notin R$. Let $L : C \to C$ be a linear map whose matrix representation is an integer matrix $A$. Then $\tilde{L}$ is clearly well-defined on $T^2$ which is induced by the square lattice. We call $\tilde{L}$ a *toral automorphism*, denoted by $L_A$.

**Definition 2.1.** *Let $L(x) = Ax$ where $A$ is a $2\times 2$ integer matrix with $det(A) = \pm 1$. If $A$ is hyperbolic, i.e., $A$ has no eigenvalues on unit circle then we call $L_A : T^2 \to T^2$ a* hyperbolic toral automorphism.

**Proposition 2.2.** [De pg.192] *Let $L_A$ be a hyperbolic toral automorphism of $T^2$. Then $L_A$ is a chaotic map.* □

Let $p : T^2 \to \bar{C}$ be a regular branched covering map from $T^2$ onto the Riemann sphere $\bar{C}$. Then the ramification indices corresponding to $p : T^2 \to \bar{C}$ are $(2,2,2,2)$, $(2,4,4)$, $(2,3,6)$ or $(3,3,3)$. Conversely, if $(\bar{C}, \nu)$ be an orbifold whose ramification indices are $(2,2,2,2)$, $(2,4,4)$, $(2,3,6)$ or $(3,3,3)$ then there exists regular branched covering map $p : T^2 \to \bar{C}$ whose ramification indices is $(2,2,2,2)$, $(2,4,4)$, $(2,3,6)$ or $(3,3,3)$ by Riemann-Hurwitz formula. See [Mi,pg.229-233], [Na] and [DH] for detailed definitions and properties of ramification indices and branched coverings.

Let $\Lambda$ be the square lattice. Now let $(2,2,2,2)$ be the ramification indices. Then the branched covering corresponding to the ramification indices $(2,2,2,2)$ is just the Weierstrass $\mathcal{P}$ function Moreover the geometry of Weierstrass $\mathcal{P}$ function is just quotient map from $T^2$ onto $\bar{C}$ such that $z$ and $-z$ have the same image.[De pg.292]

The following are the classification of branched covering maps and chaotic maps of the Riemann sphere $\bar{C}$ which are induced by regular branched coverings of $\bar{C}$ and the linear map $2z$ [L] for $(2,2,2,2)$ and [L1] for $(2,4,4)$, $(2,3,6)$ and $(3,3,3)$.

| Ramification Indices | Branched Coverings | Induced Chaotic Maps of $\overline{C}$ |
|---|---|---|
| (2,2,2,2) | $\mathcal{P}$ | $R(z) = \frac{z^4 + \frac{1}{2}g_2 z^2 + \frac{1}{16}g_2^2}{4z^3 - g_2 z}$ |
| (2,4,4) | $\mathcal{Q}(z) = \mathcal{P}(z)^2$ | $R(z) = \frac{(z + \frac{g_2}{4})^4}{16z(z-g_2)^2}$ |
| (2,3,6) | $\mathcal{Q}(z) = (\mathcal{P}(z))^3 - \frac{g_3}{4}$ | $R(z) = \frac{(z+\frac{g_3}{4})(z+\frac{9g_3}{4})^3}{4z} - \frac{g_3}{4}$ |
| (3,3,3) | $\mathcal{Q}(z) = \mathcal{P}'(z)$ | $R(z) = \left(4\left(\frac{\frac{(z^2+g_3)}{4} \quad \frac{(z^2+g_3)}{4} + 2g_3}{z^6}^{3}\right) - g_3\right)^{\frac{1}{2}}$ |

We can also construct chaotic homeomorphism induced by hyperbolic toral automorphisms and the Weierstrass $\mathcal{P}$ function, which is branched covering corresponding to the ramified indices $(2,2,2,2)$.

**Theorem 2.1.** [L2] *Let $(2,2,2,2)$ be the ramification indices and $\mathcal{P}(z)$ the Weierstrass $\mathcal{P}$ function, which is branched covering of the indices $(2,2,2,2)$. Let $L_A : T^2 \to T^2$ be a hyperbolic toral automorphism induced by the square lattice $\Lambda$. Then there exist (countably many) chaotic homeomorphisms of $C$ onto itself which is not analytic such that the diagram in Lemma 3.1 commutes.* □

But Theorem 2.1 is not true in general if the ramification indices is $(2,4,4)$, $(2,3,6)$ and $(3,3,3)$ [L2].

## 3. A Characterization of Chaotic Linear Toral Automorphisms

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an integer matrix with $det(A) = \pm 1$ and let $\lambda$ be the characteristic solutions of $A$. Then $\lambda = \frac{(a+d)\pm\sqrt{(a+d)^2-4(ad-bc)}}{2}$. We now suppose that $L_A$ is not hyperbolic, i.e., $A$ does not have real eigen values which are not in the unit circle. Then we have 3 different cases:

(1) characteristic solutions are complex numbers,
(2) characteristic solutions are $\pm 1$, or
(3) characteristic solutions are 1 or $-1$ with multiple root.

In this section, we characterize hyperbolic toral automorphisms by proving that the map $L_A$ can not be chaotic in any of the above 3 cases [Theorem 3.1, 3.2 and 3.4].

### 3.1. Case (1): Characteristic solutions are complex numbers.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an integer matrix with $det(A) = \pm 1$. Recall that if characteristic solutions $\lambda$ are complex numbers then $ad - bc = 1$ and $(a+d)^2 < 4$. Therefore $a + d = 0,\ 1\, or\ -1$. Then $a + d$ is 0, 1 and -1 iff $\lambda$ is $\pm i$, $\frac{1}{2} \pm \frac{\sqrt{3}i}{2}$ and $-\frac{1}{2} \pm \frac{\sqrt{3}i}{2}$ respectively.

**Proposition 3.1.** *Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an integer matrix with $det(A) = \pm 1$. If $A$ have complex characteristic solutions $\pm i$ then $L_A : T^2 \to T^2$ is not a chaotic map.*

**Proof**. If $A$ has a complex characteristic solutions $\pm i$ then $A$ has the form $\begin{pmatrix} n & b \\ c & -n \end{pmatrix}$. Now we can easily check that $A$ is periodic with period 4 since $det(A) = 1$. Consequently $L_A : T^2 \to T^2$ can not be chaotic. □

**Proposition 3.2.** *Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an integer matrix with $det(A) = \pm 1$. If $A$ have complex characteristic solutions $\frac{1}{2} \pm \frac{\sqrt{3}i}{2}$ or $-\frac{1}{2} \pm \frac{\sqrt{3}i}{2}$, then $L_A : T^2 \to T^2$ is not a chaotic map.*

**Proof.** We can prove the proposition by matrix multiplication. Note that we have 2 cases, $a+d = 1$ or $a+d = -1$ by the formula of $\lambda$. Consider the case $a+d = 1$. Then 2,1 component of $A^3$ is $a(ac + cd) + c(bc + d^2)$. Recall that $det(A) = 1$ since $A$ have complex characteristic solutions. Then $a(ac + cd) + c(bc + d^2) = 0$ by substituting $d = 1 - a$ and $bc = ad - 1$. We also have 1,2 component of $A^3$,

$b(a^2+bc)+d(ab+bd)=0$ by substituting $d=1-a$ and $bc=ad-1$. Similarly, we have 1,2 and 2,1 components of $A^3$ are 0 in case $a+d=-1$ by direct computation.

Note that $det(A^3)=1$, therefore $A^3=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Consequently $A$ is periodic with period 3 or 6 respectively. Consequently $L_A : T^2 \to T^2$ can not be chaotic.□

We now state one result from Proposition 3.1 and Proposition 3.2 in Case (1).

**Theorem 3.1.** *Let $A$ be a $2\times 2$ integer matrix with $det(A)=\pm 1$. If $A$ have complex characteristic solutions then the induced toral automorphism $L_A : T^2 \to T^2$ is not chaotic.* □

3.2. **Case (2): Characteristic solutions are $\pm 1$.** Let $A$ be a $2\times 2$ integer matrix with $det(A)=\pm 1$ and let $\lambda=\pm 1$ be characteristic solutions of $A$. If $det(A)=1$ then $(a+d)^2-4>0$ from the formula of $\lambda$. Then $\lambda$ can not be $\pm 1$. We now suppose that $det(A)=-1$. Then $\lambda=\frac{(a+d)\pm\sqrt{(a+d)^2+4}}{2}$. Consequently $\lambda=\pm 1$ iff $a+d=0$ and $det(A)=-1$.

**Theorem 3.2.** *Let $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an integer matrix with $det(A)=\pm 1$ and let $\lambda=\pm 1$ be characteristic solutions of $A$. Then $L_A : T^2 \to T^2$ is not chaotic.*

**Proof.** Note that $A$ has the form $A=\begin{pmatrix} n & b \\ c & -n \end{pmatrix}$. Since $det(A)=-1$, $n^2+bc=1$. Hence $A^2=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and therefore $A$ is periodic with period 2. □

3.3. **Case (3): Characteristic solutions are 1 or $-1$ with multiple root.** We will show that we can construct 3 disjoint simple closed curves which are fixed or invariant such that one of the simple closed curves maps onto itself in Case (3) . Then we will show that the induced map $L_A$ in Case (3) does not satisfy topological transitivity and therefore $L_A$ can not be chaotic.

3.3.1. *Characteristic solution is 1 with multiple root.* Let $A$ a matrix whose characteristic solution is 1 with multiple root. Then the matrix $A$ satisfies $a+d=2$ and $det(A)=1$ from the characteristic solutions formula. Note that if $\lambda=1$ with multiple root and its eigen space is $R^2$ then $Az=z$. Therefore $A=I$.

Now we can find the fixed point sets by solving $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} x+m \\ y+n \end{pmatrix}$ (equivalently $\begin{pmatrix} a-1 & b \\ c & d-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} m \\ n \end{pmatrix}$) where $m,n$ are integers. Then $(a-1)x+by=k(cx+(d-1)y)$ or $k((a-1)x+by=cx+(d-1)y)$, where $k$ is a rational number.

**Case 1:** $k=0$.
Since $k=0$ we have 2 cases; $(a-1)x+by=k(cx+(d-1)y)$ or $k((a-1)x+by=cx+(d-1)y)$. Then the matrix is $\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ respectively since $a+d=2$ and $det(A)=1$. We denote those matrices $A$ and $B$ respectively.

We consider the matrix $A$. Let $\begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$ where $m_1$ and $m_2$ are integers. Then the solution of the above equations, $x = \frac{m_1}{n}$, is the fixed point sets. Therefore if $n \geq 3$ then we have more than 2 disjoint simple closed curves as the fixed point set.

Now we consider the case $n = 1$ and $n = 2$.
In case $n = 1$ we can easily check that each simple closed curves $S_2$ and $S_3$ induced by $x = \frac{1}{2}$ and $x = \frac{1}{3}$ respectively is invariant under $L_A$. Therefore we have 3 simple closed curves such that the simple closed curve $S_1$, induced by $x = 0$ is fixed and $S_2$ and $S_3$ are invariant.

In case $n = 2$ the simple closed curves $S_1$ and $S_2$ induced by $x = 0$ and $x = \frac{1}{2}$ respectively are fixed point set. Now we also can check that the simple closed curve $S_3$ induced by $x = \frac{1}{3}$ is invariant under $L_A$.

Similarly we can find 3 disjoint simple closed curves each of which is fixed or invariant under the induced map $L_B$ when the matrix is $B$.

**Case 2:** $|k| = 1$.

**1.** $k = 1$.
$\begin{pmatrix} a-1 & b \\ c & d-1 \end{pmatrix}$ has the form $\begin{pmatrix} -p & p \\ -p & p \end{pmatrix}$ since $a + d = 2$ and $(a-1)x + by = cx + (d-1)y$. Then $-px + py = m$, and therefore $y = x + \frac{m}{p}$ is the fixed point sets. Consequently if $p \geq 3$ then $L_A$ has more than 2 simple closed curves as the fixed point set in the identification space $T^2$. We now consider when $p = 1$ and $p = 2$. Recall that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has the form $\begin{pmatrix} -p+1 & p \\ -p & p+1 \end{pmatrix}$ since $k = 1$.

When $p = 1$ we have the simple closed curve $S_1$induced by $y = x$. And each of the simple closed curves $S_2$ and $S_3$, induced by $y = x + 1/2$ and $y = x + 1/3$ respectively, is invariant under $L_A$.

When $p = 2$, we have 2 simple closed curves $S_1$ and $S_2$ induced by $y = x$ and $y = x + 1/2$ respectively as the fixed point sets. We now consider the simple closed curve $S_3$ induced by $y = x + \frac{1}{3}$. Then we can easily check that $S_3$ is invariant under $L_A$. Consequently we have 3 disjoint simple closed curves each of which is fixed or invariant.

**2.** $k = -1$. We have the same result as the case $k = 1$.

**Case 3:** $|k| \neq 1$ **and** $k \neq 0$.
It suffice to consider when $(a-1)x + by = k(cx + (d-1)y$.
Now $\begin{pmatrix} a-1 & b \\ c & d-1 \end{pmatrix}\begin{pmatrix} x \\ \frac{c}{1-d}x + \frac{n}{1-d} \end{pmatrix} = \begin{pmatrix} \frac{bn}{1-d} \\ -n \end{pmatrix}$. Therefore if $\frac{bn}{1-d}$ is an integer then the simple closed curve induced by $y = \frac{c}{1-d}x + \frac{n}{1-d}$ is the fixed point set, otherwise the simple closed curve is invariant for $1 \leq n < |1-d|$.

Now we consider the case when the number of simple closed curves induced by the above one or two. Note that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ \frac{c}{1-d}x + \frac{1}{q} \end{pmatrix} = \begin{pmatrix} x + \frac{b}{q} \\ \frac{c}{1-d}(x + \frac{b}{q}) + \frac{1}{q} \end{pmatrix}$ for $q \geq 2$ and $q \in Z$. Therefore the simple closed curve induced by $\frac{c}{1-d}x + \frac{1}{q}$ for $q \geq 2$ and $q \in Z$ is invariant set under $L_A$. Moreover the simple closed curve induced by $\frac{c}{1-d}x + \frac{m}{q}$ for $q \geq 2$ and $q \in Z$ and $m < q$ is also invariant set under $L_A$ by the same argument as the above.

**Proposition 3.3.** *Let* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be an integer matrix with* $det(A) = \pm 1$ *and let the characteristic solution be 1 with multiple root. Then the fixed point set of the induced toral automorphism* $L_A : T^2 \to T^2$ *is a simple closed curve or a disjoint union of simple closed curves, which are parallel, in the identification space* $T^2$*. Moreover if the fixed point sets is a simple closed curve or disjoint union of 2 simple closed curves then we can find more than 1 simple closed curves, parallel to the fixed simple closed curves, each of which are invariant under* $L_A$*.* □

3.3.2. *Characteristic solution is* $-1$ *with multiple root.* Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an integer matrix with $det(A) = \pm 1$ and let the characteristic solution be $-1$ with multiple root. We will show that there exist more than 2 simple closed curves which are invariant. Moreover one of the invariant simple closed curve $S_1$ maps onto itself by $L_A$, i.e., $L_A(S_1) = S_1$.

Note that $-A$ has eigen value 1 with multiple root. Therefore the induced toral automorphism $L_{(-A)} : T^2 \to T^2$ has fixed point sets which is a simple closed curve or finite disjoint union of simple closed curves by Proposition 3.3. Then the fixed point sets of $L_{(-A)}$ are invariant set under the map $L_A$. In fact, let $S_i$ be the fixed point set. Then $L_A(S_i) = S_i$ or $L_A(S_i) = S_j$ and $L_A(S_j) = S_i$. since $L_A(z) = -z$ for elements of the fixed point sets of $L_{(-A)}$.

**Corollary 3.1.** *Let* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be an integer matrix with* $det(A) = \pm 1$ *and let the characteristic solution be* $-1$ *with multiple root. Then there exist at least 3 finite disjoint union of simple closed curves which are invariant. In particular the simple closed curve induced by the line passing through the origin maps onto itself by* $L_A$*. Moreover those simple closed curves are parallel in the identification space* $T^2$*.*□

**Theorem 3.3.** *Let* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be an integer matrix with* $det(A) = \pm 1$ *and let the characteristic solution be 1 with multiple root. If the fixed point sets of* $L_A : T^2 \to T^2$ *in Proposition 4.3 is more than 2 disjoint simple closed curves then* $L_A$ *is not chaotic.*

**Proof.** It suffices to consider when the fixed point set of $L_A$ is 3 disjoint simple closed curves $S_1$, $S_2$ and $S_3$. Since $S_1$, $S_2$ and $S_3$ are parallel, $T^2 - \cup_{i=1}^{i=3} S_i$ are 3 components, denoted by $D_1$, $D_2$ and $D_3$, where $D_1$ is the component between $S_1$ and $S_2$, $D_2$ is the component between $S_2$ and $S_3$,and $D_3$ is the component between $S_3$ and $S_1$. Let $D* = D_1 \cup D_3 \cup S_1$. Then $D*$ and $D_2$ are disjoint components whose common boundary is $S_2 \cup S_3$.

Now let $U$ be a small open set in $D_2$ and let $V \subset D*$ be a small open neighborhood of $x \in S_1$ such that $U$ and $V$ do not intersects $S_2$ or $S_3$. Consider $L_A^n(V)$ for $n \in N$. Note that $S_i$'s are the fixed point sets. Therefore if there exists $n$ such that $L_A^n(V) \cap U \neq \emptyset$ then $L_A^n(V)$ must intersect $S_2$ or $S_3$. But this is impossible. In fact, if $y \in L_A^n(V)$ with $y \in S_2 \cup S_3$ then $y = L_A^{-n}(y) \in V$. This contradicts for the choice of $V$. Consequently $L_A : T^2 \to T^2$ is not topologically transitive and therefore$L_A$ is not chaotic. □

We also have the same result as Theorem 3.3 when the invariant set is disjoint union of more than 2 simple closed curves and one of them maps onto itself, whose proof is basically same as Theorem 3.3.

**Corollary 3.2.** *Let* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be an integer matrix with* $det(A) = \pm 1$ *and let the characteristic solution be 1 or -1 with multiple root. If the invariant set(including fixed point set) of* $L_A : T^2 \to T^2$ *in Proposition 3.3 and Corollary 3.1 is more than 2 disjoint simple closed curves such that one of them maps onto itself then* $L_A$ *is not chaotic.* □

We now state main results in Case (3) from Theorem 3.3 and Corollary 3.2.

**Theorem 3.4.** *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be an integer matrix with* $det(A) = \pm 1$ *and let* $\lambda = 1$ *or* $\lambda = -1$ *be characteristic solutions of* $A$ *with multiple root. Then* $L_A : T^2 \to T^2$ *is not chaotic.*□

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