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REVERSALS ON SFT'S

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Abstract. Reversals of topological dynamical systems are homeomorphisms that are skew-commuting with the underlying actions. In this article we consider reversals for subshits of finite type and their properties. We represent such reversals in terms of transition matrices and permutation matrices. Also found are necessary and/or sufficient conditions for an SFT to admit reversals of given finite order. Reversals of order 2 are treated first and the results are generalized to reversals of any finite order.

1. Introduction and preliminaries

It seems that reversible dynamical systems have been studied in connection with the classical Hamiltonian systems. Recently reversible systems are considered in measurable dynamics, and it turns out that many properties are distinguished from those in topological dynamics. For example, it is well known that measurable reversible systems are isomorphic if the underlying dynamical actions are Bernoulli of the same entropy. In [7] it is shown that if the underlying actions are Kolmogorov and isomorphic, there are examples of non-isomorphic reversible systems.

Since there is a dynamical system (X, T) which is not conjugate to its time reversal (X, T^{-1}), not every dynamical system is reversible. On the other hand, any subshift of finite type whose transition matrix is symmetric is reversible. We want to answer the following question: If a subshift of finite type is reversible, how much the transition matrix differ from symmetric matrices. In general, it is desired to find necessary and/or sufficient conditions for subshifts of finite type to be reversible.

Definition 1.1. Let (X, T) be a topological dynamical system, where X is a topological space and $T : X \to X$ a homeomorphism. A homeomorphism $\phi : X \to X$ is called a *reversal* for (X, T) if

$$T \circ \phi = \phi \circ T^{-1}.$$

The triplet (X, T, ϕ) is called a *reversal system.* We say (X, T) is *reversible.* If $\phi^n = id$ for some $n \geq 1$, then ϕ is said to be a *reversal of order n.* In particular, if $\phi^2 = id$, then ϕ is called a *flip map* or simply a *flip.*

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Let (X, T, ϕ) be a reversal system. By the definition, ϕ is a topological conjugacy between (X, T) and (X, T^{-1}). Also $T \circ \phi^2 = \phi^2 \circ T$. Hence ϕ^2 is an automorphism of (X, T). If n is odd, then

$$T \circ \phi^n = \phi^{n-1} \circ T \circ \phi = \phi^{n-1} \circ \phi \circ T^{-1} = \phi^n \circ T^{-1}$$

so that ϕ^n is a reversal for (X, T). It follows that if ϕ is of order $n \geq 1$, then n must be even, unless $T^2 = id$. If $\psi : (X, T) \to (X, T)$ is an automorphism, then $\psi \circ \phi$ and $\phi \circ \psi$ both are reversals for (X, T).

A reversal system (X, T, ϕ) is said to be (*topologically*) *conjugate* to (Y, S, ψ) if there is a homeomorphism $\theta : X \to Y$ such that

$$\theta \circ T = S \circ \theta \qquad \text{and} \qquad \theta \circ \phi = \psi \circ \theta.$$

In this case, we write $\theta : (X, T, \phi) \cong (Y, S, \psi)$, and θ is called a (*topological*) *conjugacy* from (X, T, ϕ) to (Y, S, ψ).

Let (Y, S) be a topological dynamical system and $\theta : (X, T) \to (Y, S)$ be a topological conjugacy. Defining $\psi = \theta \circ \phi \circ \theta^{-1}$, we obtain a reversal system (Y, S, ψ) which is conjugate to (X, T, ϕ). It is easy to see that $(X, T^m, T^n\phi)$ is a reversal system for any $m, n \in \mathbb{Z}$. Also T is a conjugacy from $(X, T^m, T^n\phi)$ to $(X, T^m, T^{n+2}\phi)$ and ϕ is a conjugacy from $(X, T^m, T^n\phi)$ to $(X, T^{-m}, T^{-n}\phi)$.

2. Representation of reversals in SFTs

Throughout the work, let $\mathcal{A}$ denote an alphabet, i.e., a finite set of symbols, and be equipped with discrete topology. For $x \in \mathcal{A}^{\mathbb{Z}}$ and $i \in \mathbb{Z}$ the i-th coordinate of x is denoted by x_i. For $i, j \in \mathbb{Z}$ with $i < j$, the block $x_i x_{i+1} \dots x_j$ is denoted $x_{[i,j]}$. For $x \in \mathcal{A}^{\mathbb{Z}}$, we define σx and ρx by

$$(\sigma x)_i = x_{i+1} \qquad \text{and} \qquad (\rho x)_i = x_{-i} \qquad \text{for } i \in \mathbb{Z}.$$

Then σ and ρ are homeomorphisms of $\mathcal{A}^{\mathbb{Z}}$ onto itself, and satisfy

$$\sigma \circ \rho = \rho \circ \sigma^{-1} \qquad \text{and} \qquad \rho^2 = id,$$

that is, ρ is a flip for the topological dynamical system $(\mathcal{A}^{\mathbb{Z}}, \sigma)$. This dynamical system is called the *full $\mathcal{A}$-shift*. The map σ is called the *shift map*, and ρ the *mirror map*.

Let A be an $\mathcal{A} \times \mathcal{A}$, 0-1 matrix. Denote by X_A the closed σ-invariant subset of $\mathcal{A}^{\mathbb{Z}}$ defined by A, that is,

$$\mathsf{X}_A = \{x = (x_i) \in \mathcal{A}^{\mathbb{Z}} | A_{x_i x_{i+1}} = 1 \text{ for } i \in \mathbb{Z}\}.$$

The topological dynamical system consisting of this space and the restriction of the shift, denoted σ_A, is called the *shift of finite type*(SFT) or the *topological Markov shift* defined by A.

Let A be an $\mathcal{A} \times \mathcal{A}$, 0-1 matrix. If $A = A^{\mathsf{T}}$, then X_A is ρ-invariant, and hence the mirror map ρ restricted to X_A is a flip for X_A. More generally, suppose that P is an $\mathcal{A} \times \mathcal{A}$, 0-1 matrix such that

$$AP = PA^{\mathsf{T}} \qquad \text{and} \qquad P^n = I, \quad n \geq 1. \tag{2.1}$$

Then there is a reversal, denoted $\phi_{A,P}$, for X_A that is defined as follows. Since $P^n = I$, it is a permutation matrix, that is $P^{-1} = P^{n-1} = P^\mathsf{T}$. For each $a \in \mathcal{A}$, let $\tau(a) = a^*$ be given so that $P_{aa^*} = 1$. Then $\tau : \mathcal{A} \to \mathcal{A}$ is a bijection of order n. One can easily see that

$$A_{ab} = 1 \quad \text{if and only if} \quad A_{b^*a^*} = 1 \qquad \text{for } a, b \in \mathcal{A}. \tag{2.2}$$

For $x \in \mathsf{X}_A$ define $\phi_{A,P}x \in \mathsf{X}_A$ by

$$(\phi_{A,P}x)_i = (x_{-i})^* \qquad \text{for } i \in \mathbb{Z}.$$

It follows from (2.2) that $\phi_{A,P}$ is a reversal for X_A. Also $\phi_{A,P}$ is of order n, since $\tau^n = I$. Whenever we refer to $(\mathsf{X}_A, \sigma_A, \phi_{A,P})$ as a reversal system, we always assume that A and P are 0-1 matrices satisfying (2.1) and that X_A is the SFT whose transition matrix is A.

The following theorem states that every SFT with a reversal of finite order can be represented in this way. We omit the proof since one can adopt and extend the proof of the case $n = 2$ which was given in [5].

Theorem 2.1 (Representation Theorem). *Let (X, T) be an SFT. Suppose ϕ is a reversal for (X, T) of order $n \geq 1$. Then there are 0-1 matrices A and P satisfying (2.1) such that (X, T, ϕ) is conjugate to $(\mathsf{X}_A, \sigma_A, \phi_{A,P})$.*

The case $n = 2$ is of particular interest.

Corollary 2.2. *Let (X, T) be an SFT. Suppose ϕ is a flip for (X, T). Then there are 0-1 matrices A and P such that (X, T, ϕ) is conjugate to $(\mathsf{X}_A, \sigma_A, \phi_{A,P})$, and satisfying*

$$AP = PA^\mathsf{T} \qquad \text{and} \qquad P^2 = I. \tag{2.3}$$

If the matrix A is symmetric and P is the identity matrix in the above corollary, the flip map is very simple. However, the conjugacy of the underlying SFTs does not necessarily implies the conjugacy of the flip systems as one can see in the following example.

Example 2.3. One can easily check that two symmetric matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

are strongly shift equivalent and so $(\mathsf{X}_A, \sigma_A) \cong (\mathsf{X}_B, \sigma_B)$. We see that $\phi_{B,I}(x) = \sigma_B(x)$ for each 4-periodic point x in X_B, and that there are no such 4-periodic points in X_A. Thus $(\mathsf{X}_A, \sigma_A, \phi_{A,I})$ is not conjugate to $(\mathsf{X}_B, \sigma_B, \phi_{B,I})$.

We provide an example of a reversal that is not of finite order.

Example 2.4. Let $X = \{0, 1, 2, 3\}^\mathbb{Z}$. Define $\Gamma : \mathcal{B}_2(X) \to \mathcal{B}_1(X)$ by

$$\begin{array}{lllllll} \Gamma : & 00, 01, 20, 21 & \mapsto & 0 & 02, 03, 22, 23 & \mapsto & 1 \\ & 10, 11, 30, 31 & \mapsto & 2 & 12, 13, 32, 33 & \mapsto & 3 \end{array}$$

Check that $\gamma = \Gamma_\infty^{[0,1]} : X \to X$ is a conjugacy and $\gamma^2 = \sigma$. Put $\phi = \rho \circ \gamma$. Since ρ is a reversal for X, so is ϕ. Fix $m \geq 1$ and consider the periodic point

$$x = (10^m)^\infty = \cdots 10^m.10^m 10^m 1 \cdots \in X.$$

Then $\gamma(x) = (20^m)^\infty$, so that $\phi(x) = \rho(\gamma(x)) = \gamma(x)$ and hence

$$\phi^2(x) = (\rho \circ \gamma)(\gamma(x)) = \rho(\sigma(x)) = \sigma^{-1}(x).$$

Thus

$$\phi^{2m}(x) = (\phi^2)^m(x) = \sigma^{-m}(x) = \sigma(x) \neq x.$$

Therefore $\phi^{2m} \neq id$ for any $m \geq 1$.

3. Existence of a reversal

In this section, we search for conditions for an SFT to be reversible.

Let A be a square matrix with nonnegative integer entries. Let G_A denote the graph associated to A, i.e., A is the adjacency matrix of the graph G_A. The SFT induced by G_A will be denoted X_A and accompanied by the shift map, denoted σ_A, to represent a topological dynamical system.

Theorem 3.1. *Let $A = RS$ for some symmetric matrices R and S with nonnegative integer entries. Then the subshift X_A admits a flip.*

Proof. Let G_A and G_{A^T} denote the graphs associated to A and A^T respectively. There is a natural bijection between states of G_A and states of G_{A^T}. For each state i in G_A, i^* denotes the corresponding state in G_{A^T}. Also, for each edge a in G_A from a state i to j, we denote by a^* the unique reversed edge in G_{A^T} from j^* to i^*. The mirror map $\rho : X_A \to X_{A^T}$ is defined by $\rho(x)_i = x^*_{-i}$ for $x = \{x_i\} \in X_A$. It is obvious that ρ is continuous.

We construct a graph $G_{R,S}$. Start with the disjoint union of G_A and G_{A^T}. For each state i in G_A and each state j^* in G_{A^T}, add R_{ij} edges from i to j^* (called R-edges), and S_{ji} edges from j^* to i (called S-edges), completing the graph. An R, S-path is defined as an R-edge followed by an S-edge forming a path in $G_{R,S}$; an S, R-path is defined similarly.

Let i and j be states in G_A. Since $A = RS$, there is a bijection from the set of A-edges from i to j and the set of R, S-paths from i to j. We denote a fixed choice of bijection by $a \mapsto \mathbf{r}(a)\mathbf{s}(a)$, and its inverse by $rs \mapsto \mathbf{a}(rs)$. Let i be a state in G_A, and j^* a state in G_{A^T}. If $i \neq j$, then since R is symmetric, there is a bijection between R-edges from i to j^* and R-edges from j to $i*$. For an R-edge r from i to j^*, we denote by r' the image of r under this bijection. If $i = j$, then we simply define $r' = r$. We note that $r = r''$. For an S-edge s, s' is similarly defined. Then it is trivial to see that the map $a^* \mapsto \mathbf{s}(a)'\mathbf{r}(a)'$ is a bijection from the set of edges in G_{A^T} to the set of S, R-paths. Now we define a sliding block code $\gamma : X_{A^T} \to X_A$ by $\gamma(x^*)_i = \mathbf{a}(\mathbf{r}(x_i)'\mathbf{s}(x_{i+1})')$ for $x^* = \{x^*_i\} \in X_{A^T}$.

Finally we show that $\phi \equiv \gamma \cdot \rho$ is a flip on X_A. It is easy to see that

$$\phi \cdot \sigma_A^{-1} = \gamma \cdot \rho \cdot \sigma_A^{-1} = \gamma \cdot \sigma_{A^T} \cdot \rho = \sigma_A \cdot \gamma \cdot \rho = \sigma_A \cdot \phi. \tag{3.1}$$

□

Corollary 3.2. *Let A be a square matrix with nonnegative integer entries. Then the subshift X_A admits a flip if and only if A is strongly shift equivalent to a matrix that is a product of two symmetric ones with nonnegative integer entries.*

Example 3.3. Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

We set

$$U = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } V = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Then we note that $A = UV$ and $B = VU$ so that $\mathsf{X}_A \cong \mathsf{X}_B$. Since B is symmetric, X_B admits a flip and so does X_A. It is not difficult to show, however, that no pair of symmetric matrices R, S satisfies $A = RS$.

Even if an SFT admits a flip, its adjacency matrix may not a product of two symmetric matrices as one can see in the above example. Thus the adjacency matrix is strong shift equivalent to its transpose not necessarily through symmetric matrices. However, the following theorem asserts that some power of such a adjacency matrix is elementarily shift equivalent to its transpose through symmetric matrices.

Theorem 3.4. *Let A be a square matrix with nonneagetive integer entries, and let X_A admits a flip. Then there exist symmetric matrices K, L with nonnegative integer entries, and an integer l satisfying the relations*

$$AK = KA^T, \qquad LA = A^T L, \tag{3.2}$$

$$A^l = KL, \qquad (A^T)^l = LK. \tag{3.3}$$

Proof. In view of the Representation Theorem, we may assume that A is strongly shift equivalent to B such that

$$BM = MB^T, \qquad NB = B^T N, \tag{3.4}$$

$$B^n = MN, \qquad (B^T)^n = NM \tag{3.5}$$

for some symmetric M, N, and even integer n. Since A and B are strongly shift equivalent, we have R and S for which $A = RS$ and $B = SR$. We set $K = RMR^T$ and $L = S^T NS$. Then it is straightforward to check that K and L satisfy the required properties in the theorem with $l = n + 2$. □

We attempt to generalize Theorem 3.4 in order to find conditions for an SFT to have a reversal of any finite order. First we need to extend the concept of symmetric matrices.

Definition 3.5. Let $K = \{1, \cdots, k\}$, $k \geq 1$, and let P be a $k \times k$ permutation matrix. Let $\tau : K \to K$ denote the bijection such that $P_{i\tau(i)} = 1$ for each $i \in K$. A $k \times k$ matrix A is said to be *P-symmetric* if $A_{ij} = A_{\tau(j)i}$ for all $i, j \in K$, or equivalently, if $A^{\mathsf{T}} = PA = AP$.

Theorem 3.6. *Let $A = RS$ where R and S^{T} are nonnegative integer matrices that are P-symmetric for some permutation matrix P of order $n \geq 1$. Then there is a reversal for X_A of order $2n$.*

Corollary 3.7. *Let A be a square matrix with nonnegative integer entries. Then the following are equivalent.*

i) *There is a reversal for X_A of order $2n$, $n \geq 1$.*
ii) *A is strong shift equivalent to a matrix of the form RS where R and S^{T} are P-symmetric for some permutation matrix P of order n.*

Proof. It follows from Theorem 3.6 that (ii) implies (i). Next, suppose that there is a permutation matrix Q of order $2n$ such that $AQ = QA^{\mathsf{T}}$. Let $P = Q^2$ so that P is a permutation of order n. Then $Q = PQ^{\mathsf{T}}$ and $QA = Q^2A^{\mathsf{T}}Q^{\mathsf{T}} = P(QA)^{\mathsf{T}}$. Thus $A = Q^{\mathsf{T}}(QA)$ where Q^{T} and $(QA)^{\mathsf{T}}$ are P-symmetric. By the Representation Theorem (see Theorem 2.1), (i) implies (ii). □

One can ask the following question. If an SFT admits a reversal of some order, does it admit one of other order? Proposition 3.8 states that an SFT with flips admits reversals of higher orders. One can prove the following theorem by using the "marker argument."

Proposition 3.8. *Let X be a mixing shift of finite type. If X admits a flip, then there is a reversal ϕ for X such that $\phi^2 \neq id$.*

One may ask if there is an SFT which admits a reversal of high order but no reversals of low order. We can construct an SFT which has a reversal of order 4 but no flips though the SFT is reducible.

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