Trends in Mathematics
Information Center for Mathematical Sciences
Volume 7, Number 2, December, 2004, Pages 105–109

DISPERSE SYSTEMS OF CHARACTERISTIC 0^+

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Abstract. We introduce the concepts of characteristic 0^+ and stabilities in disperse systems. Also we obtain a necessary and sufficient conditions of the notion of characteristic 0^+.

1. INTRODUCTION

The aim of this paper is to generalize the theory of characteristic 0^+ to the case of multi-valued dynamical systems. Such systems appear to be rather suitable for describing the global behavior of processes in optimal control and economic dynamics. Also the systems are used to describe multi-valued differential equations. The properties of set-valued dynamical systems has been investigated in several papers. See [3,4,5]. The results of the theory of characteristic 0^+ are well-known in [1].

Now we recall some notions of disperse systems.

Let (X, d) and (Y, ρ) be locally compact metric spaces and let $F(Y)$ the set of all nonempty compact subsets of Y endowed with the Hausdorff topology. A set-valued function $f : X \to F(Y)$ is said to be *upper* [*lower*]*semicontinuous* at $x \in X$ if for each $\varepsilon > 0$, there is a $\delta > 0$ such that $d(x, y) < \delta$ implies $f(y) \subseteq B(f(x), \varepsilon)[f(x) \subseteq B(f(y), \varepsilon)]$. A well-known result is that f is upper semicontinuous at $x \in X$ if and only if for every neighborhood U of $f(x)$, there exists a neighborhood V of x such that $f(y) \subseteq U$ for all $y \in V$. And it is also clear that f is lower semicontinuous at $x \in X$ if and only if for every open subset U of M with $U \cap f(x) \neq \emptyset$, there is a neighborhood V of x such that $U \cap f(y) \neq \emptyset$ for all $y \in V$.

f is said to be *upper* [*lower*]*semicontinuous* if f is upper [lower] semicontinuous at every point of X. f is said to be *continuous* if f is upper semicontinuous and lower semicontinuous. Let X be a locally compact metric space. A *disperse system* on X is a triple $(X, \mathbb{R}, f)$ where $f : X \times \mathbb{R} \to F(X)$ is a map satisfying the following conditions :

(i) $f(x, 0) = x$ for all $x \in X$,
(i) if $t_1 t_2 > 0$, then $f(f(x, t_1), t_2) = f(x, t_1 + t_2)$,
(ii) if $y \in f(x, t)$, then $x \in f(y, -t)$
(ii) f is continuous.

2000 *Mathematics Subject Classification.* primary 37B20; secondary 37B25.
Key words and phrases ; characteristic 0^+, stability, disperse systems.

For $x \in X$, let $K^+(x) = \overline{f(\{x\} \times \mathbb{R}^+)}$ and $L^+(x) = \bigcap_{t\geq 0} \overline{f(\{x\} \times [t,\infty))}$. Throughout this paper, we assume that $K^+(x)$ is compact for all $x \in X$. The purpose of this paper is to study necessary and sufficient conditions of concept of characteristic 0^+.

Throughout this paper, let (X, d) be a locally compact metric space and $f : X \to F(X)$ a disperse system.

2. Disperse systems of characteristic 0^+

For $x \in X$, let $D^+(x) = \bigcap_{U\in N_x} \overline{f(U \times \mathbb{R}^+)}$ where N_x is the set of all neighborhoods of x. We say that $x \in X$ is a *point of characteristic* 0^+ if $D^+(x) = K^+(x)$.

Theorem 2.1. *Let $x \in X$. Then $K^+ : X \to F(X)$ is lower semicontinuous at x.*

Proof. It is obvious. □

Corollary 2.2. *Let $x \in X$. Then we have that K^+ is continuous at x if and only if K^+ is upper semicontinuous at x.*

Lemma 2.3. *Let $f(x,t)$ be a connected subset of X for each $(x,t) \in X \times \mathbb{R}^+$. If W is connected in $X \times \mathbb{R}^+$, then $f(W)$ is a connected subset of X.*

Proof. Suppose that $f(W)$ is a disjoint union of nonempty open subsets A and B in $f(W)$. Let $U = \{p \in W \mid f(p) \subseteq A\}$ and $V = \{p \in W \mid f(p) \subseteq B\}$. For every element p of W, since $f(p)$ is connected subset of $f(W)$, either $f(p) \subseteq A$ or $f(p) \subseteq B$. Then either $p \in U$ or $p \in V$. Hence we know that $U \neq \emptyset$, $V \neq \emptyset$, $U \cap V = \emptyset$ and $U \cup V = W$. By the hyppothesis, U and V are both open in W. Thus W is disconnected. This is a contradiction. □

Corollary 2.4. *Let $f(x,t)$ be a connected subset of X for each $(x,t) \in X \times \mathbb{R}^+$. Then for every $T > 0$, $f(\{x\} \times [T,\infty))$ is a connected subset of X.*

Theorem 2.5. *Let $x \in X$ and $f(x,t)$ be a connected subset of X for each $(x,t) \in X \times \mathbb{R}^+$. Then K^+ is continuous at x if and only if x is a point of characteristic 0^+.*

Proof. Let $y \in D^+(x) - K^+(x)$. There exists a neighborhood U of $K^+(x)$ such that $y \notin \overline{U}$. Since K^+ is upper semicontinuous at x, there is a neighborhood V of x such that $K^+(z) \subseteq U$ for all $z \in V$. We have $y \in D^+(x) \subseteq \overline{f(V \times \mathbb{R}^+)} \subseteq \overline{U}$. This is a contradiction. Thus $D^+(x) = K^+(x)$ and so x is a point of characteristic 0^+.

Conversely let x be a point of characteristic 0^+. Given any neighborhood U of $K^+(x)$, there exists a neighborhood V of $K^+(x)$ such that $\overline{V} \subseteq U$ and that $\overline{V}$ is compact. Now let us show that there exists a neighborhood W of x such that $f(W \times \mathbb{R}^+) \subseteq V$. Assume that $f(W \times \mathbb{R}^+) \nsubseteq V$ for all neighborhood W of x. Since $x \in K^+(x) \subseteq V$, there exists an $\varepsilon > 0$ with $B(x,\varepsilon) \subseteq V$. For each positive integer i, since $f(B(x, \frac{\varepsilon}{i}) \times \mathbb{R}^+) \nsubseteq V$, there exist $x_i \in B(x, \frac{\varepsilon}{i})$ and $t_i \in \mathbb{R}^+$ such that

$f(x_i, t_i) \nsubseteq V$. Since $f(\{x_i\} \times [0, t_i])$ is connected, we have $f(\{x_i\} \times [0, t_i]) \cap \partial V \neq \emptyset$. Let $y_i \in f(\{x_i\} \times [0, t_i]) \cap \partial V$. Since ∂V is compact, we may assume that $y_i \to x$ and $y_i \in f(\{x_i\} \times [0, t_i]) \subseteq f(\{x_i\} \times \mathbb{R}^+)$. we have $y \in D^+(x) = K^+(x) \subseteq V$. This is a contradiction. Thus there exists a neighborhood W such that $f(W \times \mathbb{R}^+) \subseteq V$. If $y \in W$, then we have

$$\overline{K^+(y)} \subseteq f(W \times \mathbb{R}^+) \subseteq \overline{V} \subseteq U.$$

Thus K^+ is upper semicontinuous at x. By Corollary 2.2, K^+ is continuous at x. □

The subset M of X is said to be *stable* if for each neighborhood U of M, there exists a neighborhood V of M such that $f(V \times \mathbb{R}^+) \subseteq U$.

Theorem 2.6. *Let $x \in X$. If K^+ is upper semicontinuous on $K^+(x)$, then $K^+(x)$ is stable.*

Proof. Let U be a neighborhood of $K^+(x)$. For each $y \in K^+(x)$, U is a neighborhood of $K^+(y)$. Since K^+ is upper semicontinuo at y, there exists a neighborhood V_y of y such that $K^+(z) \subseteq U$ for all $z \in V_y$. Since

$$f(\{z\} \times \mathbb{R}^+) \subseteq \overline{f(\{z\} \times \mathbb{R}^+)} = K^+(z) \subseteq U$$

for all $z \in V_y$, we have $f(V_y \times \mathbb{R}^+) \subseteq U$. Let $V = \bigcup_{y \in K^+(x)} V_y$. Then V is a neighborhood of $K^+(x)$ and we have

$$f(V \times \mathbb{R}^+) = f(\cup_{y \in K^+(x)}(V_y \times \mathbb{R}^+)) = \bigcup_{y \in K^+(x)} f(V_y \times \mathbb{R}^+) \subseteq U.$$

Hence $K^+(x)$ is stable. □

Theorem 2.7. *K^+ is continuous if and only if $K^+(x)$ is stable for all $x \in X$.*

Proof. Let $x \in X$ and let $K^+(x)$ be a stable set. Given any neighborhood U of $K^+(x)$, there exists a neighborhood V of K^+ such that $\overline{V} \subseteq U$. Since $K^+(x)$ is stable, there exists a neighborhood W of K^+ such that $f(W \times \mathbb{R}^+) \subseteq V$. So W is a neighborhood of x. For any $y \in W$, we have

$$K^+(y) \subseteq \overline{f(W \times \mathbb{R}^+)} \subseteq \overline{V} \subseteq U.$$

Thus K^+ is upper semicontinuous at x. By Corollary 2.2, K^+ is continuous at x.

Conversely it is obvious by Theorem 2.6. □

3. The stabilities of K^+ and L^+.

Definition 3.1. A subset M of X is called *minimal* if it is nonempty, closed and invariant, and no proper subset of M has these properties.

Remark 3.2. A nonempty subset M of X is minimal if and only if $\overline{O(x)} = M$ for any $x \in M$.

Theorem 3.3. *If $K^+(x)$ is stable for all $x \in X$, then $L^+(x)$ is stable.*

Proof. Since $L^+(x)$ is positively invariant and closed, $L^+(x)$ contains a minimal set M. Let $y \in L^+(x) - M$. There exists a neighborhood U of M such that $y \notin \overline{U}$. Let $z \in M$. We have $K^+(z) = M \subseteq U$. Since $K^+(z)$ is stable, there is a neighborhood V of $K^+(z)$ such that $f(V \times \mathbb{R}^+) \subseteq U$. There exists a $t \in \mathbb{R}^+$ with $f(x,t) \subseteq V$. Since $f(\{x\} \times [t,\infty)) = f(f(x,t) \times \mathbb{R}^+) \subseteq f(V \times \mathbb{R}^+) \subseteq U$, we have

$$y \in L^+(x) \subseteq \overline{f(\{x\} \times [t,\infty))} \subseteq \overline{U}.$$

This is a contradiction. Thus $M = L^+(x)$. Since $L^+(x)$ is minimal, $K^+(x) = L^+(x)$ for all $y \in L^+(x)$ by Remark 3.2. From the assumption, $L^+(x)$ is stable. □

Lemma 3.4. *For every neighborhood U of $L^+(x)$, there exists an $t \in \mathbb{R}^+$ such that $f(\{x\} \times [t,\infty)) \subseteq U$.*

Proof. Since X is locally compact, there exists a neighborhood V of $L^+(x)$ such that $\overline{V} \subseteq U$ and that $\overline{V}$ is compact. Let us show that there is a $t \in \mathbb{R}^+$ with $f(\{x\} \times [t,\infty)) \subseteq \overline{V}$. Assume that for each $t \in \mathbb{R}^+$, there is an $s \in [t,\infty)$ such that $f(x,s) \nsubseteq \overline{V}$. Let $y \in L^+(x)$. There are sequences $\{t_n\}, \{y_n\}$ with $y_n \in f(x,t_n)$ such that $t_n \to \infty$ and $y_n \to y$. Since V is a neighborhood of y, we may assume that $y_n \in V$ for all n. There exists an $s_n \in [t_n,\infty)$ such that $f(x,s_n) \nsubseteq \overline{V}$. Since $f(\{x\} \times [t_n,s_n])$ is connected, we have $f(\{x\} \times [t_n,s_n]) \cap \partial V \neq \emptyset$. Then there is $z_n \in f(x,r_n) \cap \partial V$ for some $r_n \in [t_n,s_n]$. Since ∂V is compact, (z_n) has a convergent subsequence. Without loss of generality, we can assume that $z_n \to z$ for some element $z \in \partial V$. Since $r_n \to \infty$, we have $z \in L^+(x)$. This is a contradiction. Thus there exists a $t \in \mathbb{R}^+$ such that $f(\{x\} \times [t_n,\infty)) \subseteq \overline{V} \subseteq U$. □

Theorem 3.5. *If $L^+(x)$ is stable, then $K^+(x)$ is stable.*

Proof. Suppose that $K^+(x)$ is not stable. Then exists a neighborhood U of $K^+(x)$ such that $f(V \times \mathbb{R}^+) \nsubseteq U$ for all neighborhoods V of $K^+(x)$. And then there exists a neighborhood W of $K^+(x)$ such that $\overline{W} \subseteq U$ and that $\overline{W}$ is compact. There exists an $\varepsilon > 0$ with $B(K^+(x),\varepsilon) \subseteq W$. For each n, since $f(B(K^+(x),\frac{\varepsilon}{n}) \times \mathbb{R}^+) \nsubseteq W$, there exists an $x_n \in B(K^+(x),\frac{\varepsilon}{n})$ such that $f(\{x_n\} \times \mathbb{R}^+) \nsubseteq W$. By Lemma 2.3, $f(\{x_n\} \times \mathbb{R}^+)$ is connected. Since $x_n \in B(K^+(x),\frac{\varepsilon}{n}) \subseteq W$, we have $f(\{x_n\} \times \mathbb{R}^+) \cap \partial W \neq \emptyset$. Then there exist a positive real number t_n and an element y_n of $f(x_n,t_n) \cap \partial W$. Since ∂W is compact, we may assume that $y_n \to y$ for some

$y \in \partial W$. Since $x_n \in \overline{W}$ and $\overline{W}$ is compact, we may assume that $x_n \to z$ for some $z \in \overline{W}$. It is clear that $z \in K^+(x)$.

If (t_n) is bounded, then we may assume that there exists positive real number t such that $t_n \to t$. Since $(x_n, t_n) \to (z, t)$ and $y_n \in f(x_n, t_n)$, there exists an $z_n \in f(z, t)$ such that $d(y_n, z_n) \to 0$ by the upper semicontinuity of f at (z, t). Since $y_n \to y$, we have $z_n \to y$. Thus

$$y \in \overline{f(z,t)} = f(z,t) \subseteq K^+(x) \subseteq W.$$

This is a contradiction.

If (t_n) is unbounded, we may assume that $t_n \to \infty$. Since $L^+(x)$ is stable, there exists a neighborhood V of $L^+(x)$ such that $f(V \times \mathbb{R}^+) \subseteq W$. Then there exists a positive real number t such that $f(\{x\} \times [0, \infty)) \subseteq V$. Then we have

$$z \in K^+(x) = f(\{x\} \times \mathbb{R}^+) \cup L^+(x).$$

If $z \in f(\{x\} \times \mathbb{R}^+)$, then there exists an $s \in \mathbb{R}^+$ such that $z \in f(x, s)$. Thus we have

$$f(z,t) \subseteq f(f(x,s),t) = f(x, s+t) \subseteq f(\{x\} \times [t, \infty)) \subseteq V.$$

Since f is upper semicontinuous at (z, t) there exists a neighborhood A of z such that $f(A \times \{t\}) \subseteq V$. There is a natural number n with $x_n \in A$ and $t_n \geq t$. Thus we have

$$\begin{aligned} f(x_n, t_n) &\subseteq f(\{x_n\} \times [t, \infty)) \\ &= f(f(x_n, t) \times \mathbb{R}^+) \\ &\subseteq f(f(A \times \{t\}) \times \mathbb{R}^+) \\ &\subseteq f(V \times \mathbb{R}^+) \subseteq W. \end{aligned}$$

This is a contradiction.

If $z \in L^+(x) \subseteq V$, then there exists an n with $x_n \in V$. Thus we have

$$f(\{x_n\} \times \mathbb{R}^+) \subseteq f(\{v\} \times \mathbb{R}^+) \subseteq W.$$

This is a contraction. Hence $K^+(x)$ is stable.

□

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