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NONZERO LYAPUNOV EXPONENTS ON PERIODIC ORBITS UNDER PERTURBATIONS

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ABSTRACT. Consider a C^2 vector field S with a hyperbolic periodic orbit Γ that is not a singularity. Denote by $\mathcal{X}^2(\Gamma)$ the set of C^2 vector fields preserving the periodic orbit Γ together with its period. We construct an open neighborhood $\mathcal{U}(S) \subset \mathcal{X}^2(\Gamma)$ such that each vector field X in $\mathcal{U}(S)$ preserves all nonzero Lyapunov exponents with respect to S on Γ.

1. INTRODUCTION

Lyapunov exponents, which measure the asymptotic exponential rate at which infinitesimally nearby points approach or move away from each other as time increases to infinity, are in general very sensitive in the sense that neither value nor sign of the Lyapunov exponents is preserved under smooth perturbations, see [1][2][4]. However, if we fix an differential ergodic system and perturb the system by Liao small perturbation, which means to perturb a certain linear systems of differential equations arising from a global linearization along a transitive orbit for the ergodic system , then a result from [5] asserts that both value and sign of Lyapunov exponents of the ergodic system are preserved. We point out in the present paper by combining certain technique from [3] with main result in [5] that Liao perturbation keeps nonzero Lyapunov exponents unchanged not only for perturbling a certain linear skew flows over a fixed ergodic system but also for perturbing a certain vector fields on state manifold.

Let us denote by M^n a compact smooth n-dimensional Riemannian manifold and by S a C^2 differential system, or other words, a C^2 vector field on M^n. We consider a hyperbolic periodic orbit Γ of S. Denote by $\mathcal{X}^2(M^n, \Gamma)$, or in brief $\mathcal{X}^2(\Gamma)$, the set of C^2 vector fields on M^n that preserve the periodic orbit Γ and its period. We will construct an open neighborhood $\mathcal{U}(S) \subset \mathcal{X}^2(M^n, \Gamma)$ such that each vector field X in $\mathcal{U}(S)$ preserves both value and sign of Lyapunov exponents with respect to S on Γ.

The given vector field S induces as usual a one-parameter transformation group $\phi_t : M^n \to M^n$, $t \in R$ on the state manifold and therefore a one-parameter transformation group $\Phi_t = d\phi_t : TM^n \to TM^n$, $t \in R$ on the tangent bundle. A probability

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ν on M^n is $\phi-$invariant if it is ϕ_t-invariant for any $t \in R$. A $\phi-$invariant probability is called $\phi-$ ergodic if every Borel set ϕ_t invariant for any $t \in R$ has zero or full probability. Denote by $C^0(M^n, R)$ the set of all continuous functions on M^n. By a basin of an ergodic probability ν we mean the subset defined by

$$Q_\nu(\phi) := \{x \in M^n;\ \lim \frac{1}{t}\int_0^t f(\phi_s(x))ds\ = \int_{M^n} f d\nu,\ \ \forall f \in C^0(M^n, R)\}.$$

Let M denote the subset of all ordinary points, namely, $M = \{x \in M^n; S(x) \neq 0\}$. We define the conjugate bundle(conjugate to S) $\mathcal{D}(S) = \cup_{x\in M}\mathcal{D}(x)$ by indicating its fiber

$$\mathcal{D}(x) := \{u \in T_xM^n;\ < u, S(x) >= 0\}.$$

For a given point $x \in M$ and a time $t \in R$ and a vector $v \in \mathcal{D}(x)$ there exists uniquely a vector $v(t) \in \mathcal{D}(\phi_t(x))$ suct that $\Phi_t(v) = v(t) + S(\phi_t(x))$. We define a linear map $\tilde{\Phi}_t(x) : \mathcal{D}(x) \to \mathcal{D}(\phi_t(x))$ by $\tilde{\Phi}_t(x)(v) = v(t)$.

Definition 1.1. *Fix a C^2 vector field (M^n, S) and a $\phi-$invariant ergodic probability ν which is not atomic on a singularity. Take a C^2 vector field (M^n, X), with the induced flows $\psi_t : M^n \to M^n$ on state manifold and $\Psi_t : TM^n \to TM^n$ on tangent bundle and $\tilde{\Psi}_t : \mathcal{D}(X) \to \mathcal{D}(X)$ on conjugate bundle. The vector field (M^n, X) is called a $\nu-$essential perturbation to (M^n, S) if there exists open sets $U \subset V \subset M^n$, and an interval $[0, T] \subset R$ and a linear cocycle $r(x, t) : \mathcal{D}(S)(x) \to \mathcal{D}(S)(\phi_t(x))$, $x \in U$, $t \in [0, T]$(i.e., $r(x, t + s) = r(r(x, s), t) \circ r(x, s)$, whenever $s,\ t,\ s + t \in [0, T]$) so that the following properties hold.*

(1). $V \cap \{x \in M^n |\ S(x) = 0$ or $X(x) = 0\} \neq \emptyset$, $\nu(Q_\nu(\phi) \cap U) > 0$, $\nu(B_\nu(\phi) \cap (M^n \setminus U)) > 0$;

(2). $X(x) = S(x),\ \ x \in Q_\nu(\phi) \cap (M^n \setminus V)$;

(3). $\Psi_t(x) = \Phi_t(x),\ \ x \in M^n \setminus V,\ t \in R$;

(4). $\tilde{\Psi}_t(x) = \tilde{\Phi}_t(x) + r(x, t),\ \ x \in U,\ t \in [0, T]$.

(5). $X \neq S$ and $\|X - S\|_i$ small, where $\|.\|$ denotes C^i normal for $i = 1, 2$.

We will show in Section 3 the existence of ν-essential perturbations and the existence of an open neighborhood of S consisting of such perturbations for an atomic probability on an periodic orbit.

Now we describe our main theorem in the present paper.

Theorem 1.2. *Let (M^n, S) be a C^2 vector field with the induced flows $\phi_t,\ \Phi_t,\ \tilde{\Phi}_t$. Let Γ denote a hyperbolic periodic orbit of ϕ_t, which supports an atomic probability ν. Let*

$$\Sigma(\Gamma, S) = \{\lambda_1 < \lambda_2 < ... < \lambda_{n-1}|\ \lambda_i \neq 0,\ i = 1, ..., n-1\}$$

denote the spectrum of nonzero Lyapunov exponents on (Γ, S). Then there is an open C^2 neighborhood $\mathcal{U}(S) \subset \mathcal{X}^2(M^n, \Gamma)$ of S consisting of ν- essential perturbations so that for each $X \in \mathcal{U}(S)$ with the induced flows $\psi_t,\ \Psi_t,\ \tilde{\Psi}_t$, the spectrum $\Sigma(\Gamma, X)$ of all Lyapunov exponents except one zero exponent determined by X coincides with $\Sigma(\Gamma, S)$,

$$\Sigma(\Gamma, S) = \Sigma(\Gamma, X) = \{\lambda_1 < \lambda_2 < ... < \lambda_{n-1}|\ \lambda_i \neq 0,\ i = 1, ..., n-1\}.$$

In 1979 in [3] Liao studied the perturbations that preserve one periodic orbit together with its period and got a theoretical formula for Lyapunov exponents for periodic orbit under the perturbing. We point out further in Theorem 1.2 in the present paper based on some technique in [3] and main theorem in [5] that the nonzero Lyapunov exponents of a periodic orbit are constant under small perturbations that preserve the periodic orbit together with the period. We will introduce several bundles and flows on the bundles in Section 2 and consider standard maps of Liao[3] and $\nu-$essential perturbation in Section 3. In Section 4 we will complete the proof of Theorem 1.2.

2. Bundles and flows

We start from a C^2 vector field S on a compact smooth n-dimensional Riemannian manifold M^n, and its induced one-parameter transformation groups $\phi_t\colon M^n \to M^n$, $t \in R$ on the state manifold and $\Phi_t = d\phi_t\colon TM^n \to TM^n$, $t \in R$ on the tangent bundle.

Fix some integer ℓ, $1 \le \ell \le n$. Construct a bundle $\mathcal{U}_\ell = U_{x\in M^n}\mathcal{U}_\ell(x)$ of ℓ-frames, where the fiber over x is

$$\begin{aligned}\mathcal{U}_\ell(x) \quad &= \{(u_1, u_2, \dots, u_\ell) \in T_xM^n \times T_xM^n \times \cdots \times T_xM^n \mid u_1, u_2, \dots, u_\ell \\ &\qquad\qquad\qquad \textit{are linearly independent}\}.\end{aligned}$$

Let $p_\ell\colon\ \mathcal{U}_\ell \to M^n$ denote the bundle projection. Write by $proj_k\colon \mathcal{U}_\ell \to TM^n$ for the map which sends $\alpha \in \mathcal{U}_\ell$ to the k-th vector in α. The vector field S induces then a one-parameter transformation group, which we denote (with the same notation as the tangent map for the sake of simplicity) by Φ_t, $t \in R$, namely,

$$\Phi_t(u_1, u_2, \dots, u_\ell) = (d\phi_t(u_1),\, d\phi_t(u_2), \dots, d\phi_t(u_\ell)).$$

For $\alpha = (u_1, u_2, \dots, u_\ell) \in \mathcal{U}_\ell$ and a nondegenerate $\ell \times \ell$ matrix $B = (b_{ij})$ we write

$$\alpha \circ B = \left(\sum_{i=1}^{\ell} b_{i1}u_i, \sum_{i=1}^{\ell} b_{i2}u_i, \dots, \sum_{i=1}^{\ell} b_{i\ell}u_i\right).$$

Then $\Phi_t(\alpha \circ B) = \Phi_t(\alpha) \circ B$. By Gram-Schmidt orthogonalizing process there exists a unique triangular matrix $\Gamma(\alpha)$ with diagonal elements 1 and with elements 0 below its diagonal such that $\alpha \circ \Gamma(\alpha)$ is orthogonal.

Construct a bundle $\mathcal{F}_\ell = U_{x\in M^n}\mathcal{F}_\ell(x)$ of ℓ-orthogonal frames, where the fiber over x is

$$\mathcal{F}_\ell(x) = \{(u_1, u_2, \dots, u_\ell) \in \mathcal{U}_\ell(x) \mid \langle u_i, u_j\rangle = 0,\ 1 \le i \ne j \le \ell\}.$$

The bundle projection is given by $q_\ell = p_\ell|\mathcal{F}_\ell$. The vector field S then induces a one-parameter transformation group

$$\chi_t\colon \mathcal{F}_\ell \to \mathcal{F}_\ell, \qquad \alpha \mapsto \Phi_t(\alpha) \circ \Gamma(\Phi_t(\alpha)).$$

If we define $\pi\colon\ \mathcal{U}_\ell \to \mathcal{F}_\ell$ by $\alpha \mapsto \alpha \circ \Gamma(\alpha)$ then $\chi_t(\alpha) = \pi \circ \Phi_t(\alpha)$.

Construct a bundle $\mathcal{F}_\ell^\# = U_{x\in M^n}\mathcal{F}_\ell^\#(x)$ of orthonormal ℓ-frames, where the fiber over x is

$$\mathcal{F}_\ell^\#(x) = \{(u_1, u_2, \ldots, u_\ell) \in \mathcal{F}_\ell(x) \mid \|u_i\| = 1,\ i = 1, 2, \ldots, \ell\}.$$

Then $\mathcal{F}_\ell^\#$ is a compact metrizable space. Let $\pi^\# : \mathcal{F}_\ell \to \mathcal{F}_\ell^\#$ be given by

$$\pi^\#(u_1, u_2, \ldots, u_\ell) = \left(\frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}, \ldots, \frac{u_\ell}{\|u_\ell\|}\right).$$

Setting $\chi_t^\# = \pi^\# \circ (\chi_t|\mathcal{F}_\ell^\#)$ for $t \in R$, we get a one-parameter transformation group $\chi_t^\# : \mathcal{F}_\ell^\# \to \mathcal{F}_\ell^\#$, $t \in R$. Let $q_\ell^\# = q_\ell|\mathcal{F}_\ell^\#$, then $q_\ell^\#$ is a bundle projection. The following commutable properties hold clearly:

$$q_\ell \circ \chi_t = \phi_t \circ q_\ell, \quad q_\ell^\# \circ \chi_t^\# = \phi_t \circ q_\ell^\#, \quad \chi_t^\# \circ \pi^\# = \pi^\# \circ \chi_t.$$

Based on the subspace $M = \{x \in M^n |\ S(x) \neq 0\}$ of M^n we define a conjugate bundle $\mathcal{E} = \cup_{x\in M}\mathcal{E}(x)$ of orthonormal $n-1$ frames by fiber

$$\mathcal{E}(x) = \{(u_2, ..., u_n) \in \mathcal{F}_{n-1}^\#(x)|\ < u_i, S(x) >= 0,\ i = 2, ..., n\}.$$

Define a map

$$\gamma :\ \mathcal{E} \to \mathcal{F}_n^\#,\quad \alpha \to (\frac{S(x)}{\|S(x)\|}, \alpha)$$

and a one parameter transformation group $\Theta_t :\ \mathcal{E} \to \mathcal{E}$,

$$\Theta_t(\alpha) = (proj_2\chi_t^\#(\frac{S(x)}{\|S(x)\|}, \alpha), ..., proj_n\chi_t^\#(\frac{S(x)}{\|S(x)\|}, \alpha)),\ \alpha \in \mathcal{E}(x).$$

Let $\tau :\ \mathcal{E} \to M$ denote the bundle projection, then

$$\gamma \circ \Theta_t(\alpha) = \chi_t^\#(\gamma(\alpha)),\quad \tau \circ \Theta_t(\alpha) = \phi_t \circ \tau(\alpha),\ t \in R.$$

3. Standard maps and essential perturbations

For $\beta \in \mathcal{F}_n^\#$ and $\alpha \in \mathcal{E}$, we define standard maps

$$\mathcal{P}_\beta :\ R^{n+1} \to M^n,\quad \tilde{\mathcal{P}}_\alpha :\ R^n \to M^n$$

by

$$\mathcal{P}_\beta(t, y_1, ..., y_n) = \exp(\Sigma_{k=1}^n y_k\, proj_k\chi_t^\#(\beta)),$$
$$\tilde{\mathcal{P}}_\alpha(t, y_2, ..., y_n) = \mathcal{P}_{\gamma(\alpha)}(t, 0, y_2, ..., y_n),$$

where $\tau :\ \mathcal{E} \to M$ denotes the bundle projection, and $\gamma(\alpha) = (\frac{S(\tau(\alpha))}{\|S(\tau(\alpha))\|}, \alpha)$ as in Section 2. The map

$$\mathcal{P}_{\beta,t} :\ R^n \to M^n$$

defined by $\mathcal{P}_{\beta,t}(y) = \mathcal{P}_\beta(t, y)$ is clearly C^∞. ¿From the compactness of M^n there is a constant $\zeta_0 > 0$ such that for each $(t, \beta) \in R \times \mathcal{F}_n^\#$ the map $\mathcal{P}_{\beta,t}$ sends $B_0 = \{y \in R^n |\ \|y\| < \zeta_0\}$ homeomorphically into M^n. Take on B_0 a unique vector field

$$\hat{S}_\beta(t, y) = (\hat{S}_\beta^1(t, y), ..., \hat{S}_\beta^n(t, y)),\quad y \in B_0$$

such that

$$d\mathcal{P}_{\beta,t}(\hat{S}_\beta(t,y)) = d\mathcal{P}_\beta(\frac{\partial}{\partial t}|_{(t,y)}).$$

For given $X \in \mathcal{X}^2(M^n)$, a perturbation to S, and for fixed (t,y), there is on B_0 a unique vector field

$$\bar{X}_\beta(t,y) = (\bar{X}^1_\beta(t,y), ..., \bar{X}^n_\beta(t,y)) \in R^n, \quad y \in B_0$$

such that $d\mathcal{P}_{\beta,t}(\bar{X}_\beta(t,y)) = X(\mathcal{P}_\beta(t,y))$.

For given $\alpha \in \mathcal{E}$, set

$$\mathcal{X}(\alpha) = \{X \in \mathcal{X}^2(M^n) | \bar{X}^1_{\gamma(\sigma)}(0,0) > 0 \ \ \forall \sigma \in \mathcal{E}_{\phi_t(\tau(\alpha))}, \ t \in R\}.$$

Set

$$\tilde{L}_{X,\alpha} = \{(t,z) \in R \times R^{n-1} | \ \|z\| < \zeta_0, \ \bar{X}^1_{\gamma(\alpha)}(t,0,z) > 0, \ \hat{S}^1_{\gamma(\alpha)}(t,0,z) > 0\}.$$

We define $\tilde{X}_\alpha(t,z) \in R^{n-1}$ by

$$(0, \tilde{X}_\alpha(t,z)) = \frac{\hat{S}^1_{\gamma(\alpha)}(t,0,z)}{\bar{X}^1_{\gamma(\alpha)}(t,0,z)} \bar{X}_{\gamma(\alpha)}(t,0,z) - \hat{S}_{\gamma(\alpha)}(t,0,z).$$

We call then for $X \in \mathcal{X}(\alpha)$

$$(\tilde{X}_\alpha) \qquad \frac{dz}{dt} = \tilde{X}_\alpha(t,z), \quad (t,z) \in \tilde{L}_{X,\alpha}$$

Liao standard system of differential equations. Set

$$\tilde{R}_\alpha(t) = (\frac{\partial \tilde{S}_\alpha(t,z)}{\partial z})_{z=0}.$$

Then $(\tilde{X}_\alpha)$ becomes(see [3])

$$\frac{dz}{dt} = z\tilde{R}_\alpha(t)^{tr} + f(t,z) \qquad (t,z) \in \tilde{L}_{X,\alpha},$$

where $f(t,z)$ is continuous on $\tilde{L}_{X,\alpha}$ and Lipschitz in z.

The following proposition is from directly Main Theorem in [5]

Theorem 3.3. *Let (M^n, S) be a C^2 vector field that induces several one parameter transformation groups $\phi_t : M^n \to M^n$, $\Phi_t : TM^n \to TM^n$, $\tilde{\Phi}_t : \mathcal{D} \to \mathcal{D}, t \in R$ as in Section 1.* Suppose that a ϕ-invariant and ergodic probability has $n-1$ simple nonzero Lyapunov exponents

$$\lambda_1 < \lambda_2 < \ldots < \lambda_{n-1}, \tag{3.1}$$

together with 1 zero Lyapunov exponents. Then for ν a. a. $x \in M^n$ and $\alpha = \alpha(orbit(x,\phi)) \in \mathcal{E}(x)$ the reduced linear system of differential equations

$$\frac{dy}{dt} = yA_{n-1\times n-1}(t), \quad y \in R^{n-1}, \quad t \in R, \tag{3.2}$$

where $A_{n-1\times n-1}(t) = \tilde{R}_\alpha(t)^{tr}$, is well defined and has the following properties:

(1) The matrix $A_{n-1\times n-1}(t)$ is uniformly bounded and continuous with respect to t.

(2) There exist $u_1, u_2, \ldots, u_\ell \in R^\ell$ such that

$$\lim_{t\to\infty} \frac{1}{t} \log\big\|y(t,u_i)\big\| = \lambda_i,$$

where $y(t,v)$ denotes a unique solution of the initial value problem

$$\frac{dy}{dt} = yA_{n-1\times n-1}(t), \qquad y(0,v) = v. \tag{3.3}$$

(3) Consider a perturbation of the linear system

$$\frac{dy}{dt} = yA_{n-1\times n-1}(t) + f(t,y), \qquad \sup_{t\in R, y\in R^{n-1}} \|f(t,y)\| \le L < \infty, \tag{3.4}$$

where $f(t,y)$ is Lipschitz in y. Then there exist $u_1^*, u_2^*, \ldots, u_\ell^* \in R^{n-1}$ such that

$$\lim_{t\to\infty} \frac{1}{t} \log\big\|y(t,u_i^*)\big\| = \lambda_i,$$

where $y(t,v)$ denotes a unique solution of the initial value problem

$$\frac{dy}{dt} = yA_{n-1\times n-1}(t) + f(t,y), \qquad y(0,v) = v. \tag{3.5}$$

4. Proof of Theorem 1.2

Given $T > 0$ and $\zeta > 0$, $\zeta < \zeta_0$ define a cube

$$C[T,\zeta] = \{(t,z) \in [0,T] \times R^{n-1} | \ \|z\| < \zeta\}.$$

The cube $C[T,\zeta]$ is called to be allowable to a given $\alpha \in \mathcal{E}(x)$, $x \in M$, if both the following to properties holds.

(1). $C[T,\zeta] \subset \tilde{L}_{S,\alpha}$, $\zeta \le \tilde{\zeta}$,

(2). The standard map $\tilde{\mathcal{P}}_\alpha : C[T,\zeta] \to M^n$ is one-to one.

A C^2 vector field

$$Z(t,z) = (Z^1(t,z), Z^2(t,z), ..., Z^n(t,z))$$

on $R \times R^{n-1}$ is called to be allowable to a given $\alpha \in \mathcal{E}(x)$, if for a cube $C[T,\zeta]$ allowable to α both the following two properties hold.

(1). $Z(t,z) = 0$, $(t,z) \in R^n \setminus C[T,\zeta]$;

(2). $Z^1(t,z) = 0$, $\forall (t,z) \in R^n$.

By using an allowable cube $C[T,\zeta]$ and an allowable vector field $Z(t,z)$ to a given $\alpha \in \mathcal{E}(x)$ we define a C^2 vector field on M^n as

$$\pi_\alpha^*(Z)(x) = \left\{ \begin{array}{cc} d\tilde{\mathcal{P}}_\alpha(Z(t,z)), & x \in \tilde{\mathcal{P}}_\alpha(t,z), \ \ (t,z) \in C[T,\zeta] \\ 0, & x \in M^n \setminus \mathcal{P}_\alpha C[T,\zeta]. \end{array} \right\}$$

The following Proposition 4.4 and Proposition 4.5 and their proof are from [3] with a bit modification.

Proposition 4.4. *Let* $X = S + \pi_\alpha^*(Z)$. *Suppose*

$$orb(\tau(\alpha), \phi) \cap \tilde{\mathcal{P}}_\alpha(C[T, \zeta]) = \{\phi_t(\tau(\alpha))|\ 0 \le t \le T\}.$$

Then

$$\bar{X}_{\gamma(\alpha)}(t, 0, z) = \bar{S}_{\gamma(\alpha)}(t, 0, z) + Z(t, z), \quad (t, z) \in C[T, \zeta],$$

$$\tilde{X}_\alpha(t, z) = \tilde{S}_\alpha(t, z) + \frac{\hat{S}^1_{\gamma(\alpha)}(t, 0, z)}{\bar{S}^1_{\gamma(\alpha)}(t, 0, z)}(Z^2(t, z), ..., Z^n(t, z)), \quad (t, z) \in C[T, \zeta].$$

Proof. Consider C^∞ maps $\varphi_t,\ \omega_t\ : R^{n-1} \to R^n$ defined by

$$\varphi_t(z) = (t, z), \quad \omega_t(z) = (0, z).$$

Put

$$Z_t(z) := (Z^2(t, z), ..., Z^n(t, z)) \in R^{n-1}.$$

Then for $z \in R^{n-1}$,

$$\tilde{\mathcal{P}}_\alpha \varphi_t(z) = \tilde{\mathcal{P}}_\alpha(t, z) = \mathcal{P}_{\gamma(\alpha)}(t, 0, z) = \mathcal{P}_{\gamma(\alpha),t}\omega_t(z)$$

and thus

$$\tilde{\mathcal{P}}_\alpha \varphi_t = \mathcal{P}_{\gamma(\alpha),t}\omega_t.$$

Observe that $Z^1(t, z) = 0$ and

$$d\varphi_t(Z_t(z)) = (0, Z^2(t, z), ..., Z^n(t, z)) = Z(t, z) = d\omega_t(Z_t(z)),$$

we have, for $(t, z) \in C[T, \zeta]$,

$$\begin{aligned} & \pi_\alpha^*(Z)(\mathcal{P}_{\gamma(\alpha)}(t, 0, z)) \\ = & \pi_\alpha^*(Z)(\tilde{\mathcal{P}}_\alpha(t, z)) \\ = & d\tilde{\mathcal{P}}_\alpha(Z(t, z)) \\ = & d\tilde{\mathcal{P}}_\alpha \circ d\varphi_t(Z_t(z)) \\ = & d\mathcal{P}_{\gamma(\alpha),t} \circ d\omega_t(Z_t(z)) \\ = & d\mathcal{P}_{\gamma(\alpha),t}(Z(t, z)). \end{aligned}$$

From definitions of $\bar{X}_{\gamma(\alpha)}$ and $\bar{S}_{\gamma(\alpha)}$ we get

$$\bar{X}_{\gamma(\alpha)}(t, 0, z) = \bar{S}_{\gamma(\alpha)}(t, 0, z) + Z(t, z), \quad (t, z) \in C[T, \zeta].$$

Let $x(t) := \phi_t(\tau(a)) \in M^n$. Since $Z(t, z)$ is allowable to a cube $C[T, \zeta]$ and $C[T, \zeta] \subset \tilde{L}_{S,\alpha}$, then

$$\bar{X}^1_{\gamma(\alpha)}(t, 0, z) = \bar{S}^1_{\gamma(\alpha)}(t, 0, z), \quad (t, z) \in C[T, \zeta].$$

Therefore

$$\begin{aligned} \tilde{X}_\alpha(t, z) = & \tilde{S}_\alpha(t, z) + \tfrac{\hat{S}^1_{\gamma(\alpha)}(t,0,z)}{\bar{X}^1_{\gamma(\alpha)}(t,0,z)}(Z^2(t, z), ..., Z^n(t, z)) \\ = & \tilde{S}_\alpha(t, z) + \tfrac{\hat{S}^1_{\gamma(\alpha)}(t,0,z)}{\bar{S}^1_{\gamma(\alpha)}(t,0,z)}(Z^2(t, z), ..., Z^n(t, z)), \quad (t, z) \in C[T, \zeta]. \end{aligned}$$

Proposition 4.5. *Let α, $C[T,\zeta]$, $Z=(0,\tilde{Z})=(0,Z^2,...,Z^n)$ be as in Proposition 4.4. Let $P=\{\phi_t(\tau(\alpha))|\ t\in R\}$ be a periodic orbit of S with period T_0 and let $Z(t,0)=0\in R^n\ \forall t\in R$. Suppose $P\cap\tilde{\mathcal{P}}_\alpha C[T,\eta]=\{\phi_t(\tau(\alpha))|\ t\in[0,T]\}$ for some $\eta\le\zeta$. Then P is a periodic orbit of $X=S+\pi_\alpha^*(Z)$ with T_0 as period.*

Proof It is clear $T<T_0$. We claim that

$$\tilde{X}_\alpha(t,z)=\tilde{S}_\alpha(t,z)+\frac{\hat{S}^1_{\gamma(\alpha)}(t,0,z)}{\bar{S}^1_{\gamma(\alpha)}(t,0,z)}(Z^2(t,z),...,Z^n(t,z))$$

follows not only for $(t,z)\in C[T,\zeta]$ as showed in Proposition 4.4, but also for $(t,z)\in C[T_0,\eta]\subset\tilde{L}_{X,\alpha}\cap\tilde{L}_{S,\alpha}$, where η is small. In fact, take $\eta<\zeta$ small we get $C[T_0,\eta]\subset\tilde{L}_{X,\alpha}\cap\tilde{L}_{S,\alpha}$. Let

$$C:=\{(t,z)\in C[T_0,\eta]|\ t\in[T,T_0]\}.$$

Then

$$\tilde{\mathcal{P}}_\alpha(C)\cap int\tilde{\mathcal{P}}_\alpha C[T,\eta]=\emptyset.$$

For $(t,z)\in C$ we have $X(x)=S(x),\quad x=\tilde{\mathcal{P}}_a(t,z)$, which implies

$$\bar{X}^1_{\gamma(\alpha)}(t,0,z)=\bar{S}^1_{\gamma(\alpha)}(t,0,z)>0.$$

Thus for $(t,z)\in C[T_0,\eta]$ we prove the claim by the following

$$\begin{aligned}\tilde{X}_\alpha(t,z)&=\tilde{S}_\alpha(t,z)+\frac{\hat{S}^1_{\gamma(\alpha)}(t,0,z)}{\bar{X}^1_{\gamma(\alpha)}(t,0,z)}(Z^2(t,z),...,Z^n(t,z))\\&=\tilde{S}_\alpha(t,z)+\frac{\hat{S}^1_{\gamma(\alpha)}(t,0,z)}{\bar{S}^1_{\gamma(\alpha)}(t,0,z)}(Z^2(t,z),...,Z^n(t,z)).\end{aligned}$$

Since $Z^1(t,z)=0$ and $Z(t,0)=0,\ i=2,...,n$ so

$$X(\phi_t(\tau(\alpha)))=S(\phi_t(\tau(\alpha))),\ t\in R$$

and P is a periodic orbit of X with period T_0.

Proof of Main Theorem

Consider the given hyperbolic periodic orbit Γ of S with period T_0 and an atomic probability ν. Let

$$\Sigma(\Gamma,S)=\{\lambda_1<\lambda_2<...<\lambda_{n-1}|\ \lambda_i\ne0,\ i=1,...,n-1\}$$

together with one zero exponent determined by $S(\Gamma)$ be the Lyapunov spectrum. Take arbitrarily $x\in\Gamma$ and $\alpha\in\mathcal{E}(x)$. Take and fix T with $T<T_0$ and ζ with $\zeta<\zeta_0$ and construct an allowable cube $C[T,\eta]$(η will be determined later) to α and an allowable C^2 vector field $Z(t,z)=(0,\tilde{Z}(t,z)),\ \tilde{Z}(t,0)=0$ as in Section 3. Moreover, we require that both $\tilde{Z}(t,z)$ and $\frac{\partial\tilde{Z}(t,z)}{\partial z}|_{z=0}$ are bounded. Define

$$\mathcal{U}(S)=\{X=S+\pi_\alpha^*(Z(t,z));\pi_\alpha^*(Z(t,z))\ \textit{is small in}\ c^2\ \textit{sence}\}.$$

Then $\mathcal{U}$ is a C^2 neighborhood of S in $\mathcal{X}^2(\Gamma)$ by Proposition 4.5. Take arbitrarily $X\in\mathcal{U}$. Let us consider the linear system

$$\frac{dz}{dt}=z(\tilde{R}_\alpha(t)^{tr}+\frac{\partial\tilde{Z}(t,z)}{\partial z}|_{z=0}),\qquad t\in R. \tag{4.1}$$

¿From Proposition 4.5 and the Claim in its proof, there is $\eta < \zeta$, where ζ is as in Proposition 4.4, such that

$$\tilde{X}_\alpha(t,z) = \tilde{S}_\alpha(t,z) + \frac{\hat{S}^1_{\gamma(\alpha)}(t,0,z)}{\bar{S}^1_{\gamma(\alpha)}(t,0,z)}(\tilde{Z}(t,z))$$

for $(t,z) \in C[T_0,\eta]$. Let us take and fix this $\eta > 0$ in the definition of $\mathcal{U}(\mathrm{S})$. Since $\tilde{Z}(t,0) = 0$ it follows $\tilde{X}_\alpha(t,0) = \tilde{S}_\alpha(t,0)$. This shows that $z = 0$ satisfies the Liao standard system of equations

$$(\tilde{X}_\alpha) \qquad \frac{dz}{dt} = \tilde{X}_\alpha(t,z)$$

in Section 3 for $t \in [0,T_0]$. Then X preserves periodic orbit Γ together with its period. Now

$$\frac{\partial X_\alpha(t,z)}{\partial z}|_{z=0} = \tilde{R}_\alpha(t) + \frac{\partial \tilde{Z}(t,z)}{\partial z}|_{z=0}. \quad t \in [0,T_0].$$

Noting T_0 being period of Γ, the linear system of (X_α) is

$$\frac{dz}{dt} = z(\tilde{R}_\alpha(t)^{tr} + \frac{\partial \tilde{Z}(t,z)}{\partial z}|^{tr}_{z=0}) \qquad t \in R.$$

Since $Z(t,z) = (0,\tilde{Z}(t,z)) = 0$ for $(t,z) \in R^n \setminus C[T,\zeta_0]$ and X is C^2 close to S, where ζ_0 is given in Section 3, so $Sup_{(t,z)\in R\times R^{n-1}} z(\frac{\partial \tilde{Z}(t,z)}{\partial z})_{z=0} < \infty$. From Theorem 3.1, there exist $z_i,\ z_i^* \in R^{n-1}$ such that

$$\begin{aligned}\lambda_i = &\ \lim_{t\to\infty} \tfrac{1}{t} \log \|z(t,z_i)\| \\ &= \lim_{t\to\infty} \tfrac{1}{t} \log \|z(t,z_i^*)\|,\end{aligned}$$

where $z(t,z_i)$ and $z(t,z_i^*)$ are unique solutions of the initial value problems

$$\left\{\begin{array}{ccc} \frac{dz}{dt} & = z & \tilde{R}_\alpha(t)^{tr} \\ z(0,z_i) & = & z_i \end{array}\right\}$$

and

$$\left\{\begin{array}{ccc} \frac{dz}{dt} & = & z(\tilde{R}_\alpha(t)^{tr} + \frac{\partial \tilde{Z}(t,z)}{\partial z}|_{z=0} \\ z(0,z_i^*) & = & z_i^*, \end{array}\right\}$$

respectively, $i = 1,...,n-1$. Again from Theorem 3.1, the limits

$$\lim_{t\to\infty} \frac{1}{t} \log \|z(t,z_i^*)\|, \quad i = 1,...,n-1$$

are spectrum of nonzero Lyapunov exponents of X supported on Γ. This completes the proof.

Remark 4.6. *When (Γ,S) has ℓ simple nonzero exponents together with $n-\ell$ zero Lyapunov exponent, then all of the ℓ nonzero Lyapunov exponents are preserved in (Γ,X) for $X \in \mathcal{U}(S)$ in Theorem 1.2 by using a technique of reducing system of differential equations given in [5], where $1 < \ell < n$.*

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