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THE BOUNDARY OF HYPERBOLICITY FOR HÉNON-LIKE FAMILIES

YONGLUO CAO, STEFANO LUZZATTO, AND ISABEL RIOS

ABSTRACT. We consider two-dimensional Hénon-like families of diffeomorphisms $H_{a,b}$ depending on parameters a, b. We study the hyperbolicity of the horseshoe which exists for large values of the parameter a and show that this hyperbolicity is preserved uniformly until a first tangency takes place.

1. Introduction

Our aim in this paper is to study the *boundary of hyperbolicity* of certain families of two dimensional maps.

Definition 1. A family $\{f_a : \mathbb{R}^2 \to \mathbb{R}^2\}$ of C^3 plane diffeomorphisms is called a *strongly dissipative Hénon-like family* if it can be written in the form

$$f_a(x) = (1 - ax^2, 0) + \psi(x, y) \quad \text{with} \quad \|\psi\|_{C^3} \leq b$$

for some sufficiently small constant $b > 0$.

The standard *Hénon family* is a special case of the above, given by $f_{a,b}(x, y) = (1-ax^2+by, bx)$ or other essentially equivalent parametrizations such as $f_{a,b}(x, y) = a - x^2 - by, x$. The dynamics of Hénon and Hénon-like maps has been subject of intense study ever since their introduction in [7] but it is still relatively poorly understood. One of the earliest rigorous results on the subject is [6] in which it was shown that the non-wandering set $\Omega(f_{a,b})$ is uniformly hyperbolic for all $b \geq 0$ and all sufficiently large a depending on b. Their arguments involve some open conditions on the derivatives and thus extend easily to Hénon like families. On the other hand, for small $b \neq 0$ and $a \leq 2$ there exists positive probability of having "strange attractors" which contain tangencies between stable and unstable leaves [1, 11, 16, 9]. These attractors cannot be uniformly hyperbolic due to the presence of tangencies but turn out to satisfy weaker *nonuniform hyperbolicity* conditions [4, 3, 2]. The main purpose of this paper is to describe the *transition* between these two regimes by identifying and describing some of the properties of the *boundary of (uniform) hyperbolicity*. Before stating our main result we recall the relevant notions of hyperbolicity. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 diffeomorphism.

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Definition 2. A compact invariant set Ω is *uniformly hyperbolic* (with respect to f) if there exists constants $C^u, C^s > 0, \lambda^u > 0 > \lambda^s$ and a *continuous* decomposition $T\Omega = E^s \oplus E^u$ of the tangent bundle such that for every $x \in \Omega$, every non-zero vector $v^s \in E^s_x$ and $v^u \in E^u_x$ and every $n \geq 1$ we have

$$\|Df^n_x(v^s)\| \leq C^s e^{\lambda^s n} \quad \text{and} \quad \|Df^n_x(v^u)\| \geq C^u e^{\lambda^u n}. \tag{1}$$

By standard hyperbolic theory, the stable and unstable subspaces E^s_x, E^u_x are tangent everywhere to the stable and unstable manifolds. In particular uniform hyperbolicity is incompatible with the presence of any tangencies between any stable and any unstable invariant manifold associated to points of Ω. However there are several weaker notions of hyperbolicity.

Definition 3. Let μ be an f-invariant ergodic probability measure with support in Ω. We say that μ is *hyperbolic* is there exist constants $\lambda^u > 0 > \lambda^s$ and a *measurable* decomposition $T\Omega = E^s \oplus E^u$ such that for μ-almost every x and every non-zero vector $v^s \in E^s_x$ and $v^u \in E^u_x$ we have

$$\lim_{n\to\infty} \frac{1}{n} \log \|Df^n_x(v^s)\| < \lambda^s \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} \log \|Df^n_x(v^u)\| > \lambda^u. \tag{2}$$

We remark that the fact that the limits in (2) exist and are constant μ-almost everywhere follows by the extremely general Oseledec's Ergodic Theorem [12]. They are called the *Lyapunoiv exponents* associated to the measure μ. The hyperbolicity condition requires that the Lyapunov exponents be different from 0 and was first formulated by Pesin [14, 15]. Clealry (1) implies (2) for any μ. The converse however is false in general: the measurable decomposition may not extend to a continuous one on all of Ω and the exponential expansion and contraction in (2) implies only a limited version of (1) in which the constants C^s, C^u are measurable functions of x and not uniformly bounded away from 0. Condition (2) is sometimes called *nonuniform hyperbolicity.* In particular nonuniform hyperbolicity is consistent in principle with the existence of tangencies between stable and unstable manifolds. Notice however that there may be many, even infinitely many, ergodic invariant probability measures supported in Ω and no reason why some may be hyperbolic while others may not. Even if they are all hyperbolic the corresponding Lyapunov exponents may not be uniformly bounded way from 0. The situation in which all Lyapunov exponents of all ergodic invariant measures are uniformly bounded away from zero is, in some sense, as "uniformly hyperbolic" as one can get while admitting the existence of tangencies. This situation can indeed occur: a first example of a set satisfying this property was given in [5], here we give a more sophisticated argument to show that this is in fact the form of hyperbolicity which occurs at the boundary of uniform hyperbolicity for strongly dissipative Hénon-like diffeomorphisms.

Let $\mathcal{X}^3$ denote the space of all C^3 maps on $\mathbb{R}^3$ and $\tilde{\mathcal{X}}^3 \subset \mathcal{X}^3$ the subset of all C^3 diffeomorphisms. We shall be interested in a neighbourhood of the map $f_* \in \mathcal{X}^3$ where $f_*(x, y) = (1 - 2x^2, 0)$.

Theorem 1. *There exists a open neighbourhood $\mathcal{U} \subset \tilde{\mathcal{X}}^3$ of f_* and a codimension-1 submanifold $\mathcal{U}^* \subset \mathcal{U}$ which seperates $\mathcal{U}$ into two regions $\mathcal{U}^-$ and $\mathcal{U}^+$ such that the following properties are satisfied.*

(1) *For all $f \in \mathcal{U}^+$ the non-wandering set $\Omega(f)$ is uniformly hyperbolic with expansion and contraction rates bounded away from 0 uniformly in $\mathcal{U}^+$.*

(2) *For all $f \in \mathcal{U}^*$ the non-wandering set $\Omega(f)$ set contains an orbit of tangency associated to the stable and unstable manifolds of the two fixed points, and is "almost" uniformly hyperbolic in the sense that all Lyapunov exponents associated to all ergodic invariant probability measures supported on$\Omega(f)$ are bounded away from 0 uniformly in $\Omega(f)$ and in $\mathcal{U}^*$.*

2. Geometric constructions and constants

We start by describing some of the geometrical properties of maps f in a neighbourhood of f_*.

2.1. **Constants.** Notice first of all that f_* has two fixed points p_* and $q_* = (-1, 0)$. Both points are hyperbolic with horizontal and vertical eigenspaces and a contracting eigenvalue equal to 0. Notice also that $f_*(1, 0) = q_*$. We fix positive constants

$$\lambda \in (0, \log 2), \delta, \eta, b, \varepsilon, C_\varepsilon$$

according to certain conditions which will be given below, and let N be sufficiently large so that

$$\sqrt{\delta/3}\sqrt{3/\sqrt{5}}^N > 1.$$

Then we define

$$\mathcal{Q} := B_\delta(q_*) \text{ and } \mathcal{Q}' := B_{2\delta}(q_*)$$

to be open balls of radius δ and 2δ respectively centred at q_*. Then we let $\mathcal{V}$ be a neighbourhood of $(1, 0)$ such that $f_*(\mathcal{V}) \subset \mathcal{Q}'$ and, for f close to f_*, define

$$\mathcal{Q}_n(f) = \{x : f^n(x) \in \mathcal{Q} \forall\ 0 \leq i \leq n\}$$

and

$$\mathcal{V}_n(f) = f^{-1}(\mathcal{Q}_n(f)) \cap V.$$

Now define

$$\Delta_\varepsilon = \{(x, y) \in (-2, 2) \times (-3, 2b) : |x| \leq \varepsilon\}.$$

2.2. **The neighbourhood $\mathcal{U}$.** It follows from standard hyperbolic dynamics that there exists a neighbourhood $\mathcal{U}$ of f_* such that for all $f \in \mathcal{U}$ p_*, q_* admit an *analytic continuation* q_f, p_f as hyperbolic fixed points of f and the geometry of the stable and unstable manifolds $W^u(p_f)$ and $W^s(q_f)$ of the two fixed points is as in Figure 2 (by the continuous dependence of compact parts of stable and unstable manifold f, see e.g. [13]). Moreover by choosing $\mathcal{U}$ small enough we can ensure that we have, for every $f \in \mathcal{U}$,

$$q_f \in f(\mathcal{V}) \subset \mathcal{Q}'; \tag{3}$$

$$\|Df(x) - Df_*(q_*)\| \leq \eta \quad \forall\ x \in \mathcal{Q}; \tag{4}$$

$$f(\Delta_\varepsilon) \subset \mathcal{V}_N; \tag{5}$$

$\forall\, k \geq 1, \forall\, x$ with $x, f(x), \ldots, f^{k-1}(x) \notin \Delta_\varepsilon$, and vector $v = (v_1, v_2)$ with $|v_2|/|v_1| < \varepsilon$ we have

$$\|Df_x^k(v)\| \geq C_\varepsilon e^{\lambda k}\|v\|. \tag{6}$$

If we also have $f^k(x) \in \Delta_\varepsilon$ then we have

$$\|Df_x^k(v)\| \geq e^{\lambda_\varepsilon k}\|v\|. \tag{7}$$

2.3. **The non-wandering set.** ¿From now on we shall always work in the neighbourhood $\mathcal{U}$ of f_* for which the above conditions hold. We let $\mathcal{D}_f$ denote the closed topological disc bounded by compact pieces of the $W^u(p_f)$ and $W^s(q_f)$ and

$$\Delta_f = \{x \in \Delta_\varepsilon : f(x) \notin \mathcal{D}_f\} \subset \Delta_\varepsilon. \tag{8}$$

In the section 2.4 we prove the following

Proposition 1. *For all $f \in \mathcal{U}$ we have*

$$\Omega_f \subset \overline{W^u(p_f)} \cap \mathcal{D}_f.$$

By Lemma 1 and (8) it follows that

$$\Omega_f \subset \mathcal{D}_f \setminus \Delta_f.$$

Since we have a uniformly hyperbolic structure in $\mathcal{D}_f \setminus \Delta_\varepsilon$ by (6) and (7) the aim is to extend this uniformly hyperbolic structure to $\mathcal{D}_f \setminus \Delta_f$ without any loss of hyperbolicity for all $f \in \mathcal{U}$. Notice that this includes parameter values for which we are "arbitrarily close" to having a tangency. The key idea here is to take advantage of the fact that points in $f(\Delta_\varepsilon \setminus \Delta_f)$ are very close to the stable manifold of the fixed point q_f. Therefore their future orbits will spend a long time in the small neighbourhood $\mathcal{Q}$ of q in which some hyperbolic structure exists . The point then is to show that the amount of time spent in $\mathcal{Q}$ is sufficient to recover the hyperbolicity "lost" as a consequence of the return to $\Delta_\varepsilon \setminus \Delta_f$.

To simplify the notation we shall generally omit the subscript f where this doe snot lead to confusion.

2.4. **Proof of Proposition 1.** The statement in Proposition 1 follows immediatly from the following three Lemmas.

Lemma 1. $\Omega(f) \subset (-2, 2) \times (-3, 2b)$.

Lemma 2. $\Omega(f) \subset \mathcal{D} \subset (-2, 2) \times (-3, 2)$.

Lemma 3. $\Omega(f) \subset \overline{W^u(p)}$.

3. Shadowing

3.1. **Hyperbolic coordinates.** Our strategy for the proof of the key Proposition 2 invloves the use of *Hyperbolic Coordinates*, a notion which appeared in print first in [9] and was developed in [10] and [8] as an alternative framework with which to approach the classical theory of invariant manifolds. For completeness, we recall the definitions and prove the basic properties of hyperbolic coordinates in this section. These are general estimates which do not depend on the specific situation we are dealing with in the paper. We state them therefore in the general context of C^2 diffeomorphisms of a Riemannian surface M.

In the context of hyperbolic fixed points (or general uniformly hyperbolic sets) we are used to thinking of the eigenspaces (or the subspaces given by the hyperbolic decomposition) as providing the basic axes or coordinate system associated to the hyperbolicity. However this is not necessarily the most natural splitting of the space. For $z \in M$ and $k \geq 1$ we let

$$F_k(z) = \|Df_z^k\| \quad \text{and} \quad E_k(z) = \|(Df_z^k)^{-1}\|^{-1}$$

denote the maximum expansion and the maximum contraction respectively of Df_z^k. Then we think of the quantity

$$H_k(z) = E_k(z)/F_k(z)$$

as the *hyperbolicity* of Df_z^k. Notice that $H_k \leq 1$. The hyperbolicity condition $H_k = E_k/F_k < 1$ implies that the linear map Df^k maps the unit circle $\mathcal{S} \subset T_zM$ to an (non-circular) ellipse $\mathcal{S}_k = Df_z^k(\mathcal{S}) \subset T_{f^k(z)}M$ with well defined major and minor axes. The unit vectors $e^{(k)}, f^{(k)}$ which are mapped to the minor and major axis respectively of the ellipse, and are thus the *most contracted* and *most expanded* vectors respectively, are given analytically as solutions to the differential equation $d\|Df_z^k(\cos\theta, \sin\theta)\|/d\theta = 0$ which can be solved to give the explicit formula

$$\tan 2\theta = \frac{2(\partial_x\Phi_1^k\partial_y\Phi_1^k + \partial_x\Phi_2^k\partial_y\Phi_2^k)}{(\partial_x\Phi_1^k)^2 + (\partial_x\Phi_2^k)^2 - (\partial_y\Phi_1^k)^2 - (\partial_y\Phi_2^k)^2}. \tag{9}$$

In particular, $e^{(k)}$ and $f^{(k)}$ are always *orthogonal* and clearly *do not* in general correspond to the stable and unstable eigenspaces of Df^k. We define the *hyperbolic coordinates of order* k at the point z as the coordinates given by the most expanded and most expanded directions. Now suppose that for the point z we have $H_k(z) < 1$ for some $k \geq 1$. If the map is C^2 there will exist a neighbourhood

$$\mathcal{N}^{(k)}(z) = \{\xi : H_k(\xi) < 1\}$$

In particular hyperbolic coordinates are defined in the tangent spaces of every point in $\mathcal{N}^{(k)}(z)$. Moreover, it is easy to see from (9) that the coordinates have a C^1 dependence on the point since they depend only on the partial derivative of f. Thus they define two orthogonal integrable unit vectors fields in $\mathcal{N}^{(k)}(z)$. We let $\mathcal{E}^{(k)}$ and denote the *stable foliation of order* k formed by the integral curves of the vector field $\{e^{(k)}\}$ and $\mathcal{F}^{(k)}$ denote the *unstable foliation of order* k formed by the integral curves of the vector field $\{f^{(k)}\}$. When there is no possibility of confusion we shall talk about the *hyperbolic coordinates* $\mathcal{H}_k$ in $\mathcal{N}^{(k)}$ as the coordinates defined by these two foliations.

In this section we apply the notion of hyperbolic coordinates to our specific situation.

Lemma 4. *Hyperbolic coordinates $\mathcal{H}^{(n)}$ are defined in each $\mathcal{V}_n$ and are C^2 close to the horizontal and vertical directions if $\mathcal{U}$ is sufficiently small. Moreover, for all $z_0 \in \mathcal{V}_n$ we have $\|Df^n_{z_0}(e^{(n)})\| \leq (b/3)^n$ and $\|Df^n_{z_0}(f^{(n)})\| \geq 3^n$.*

Proof. All the statements follow from the fact that the orbit of the point z_0 remain in the neighbourhood $\mathcal{Q}$ of q for at least n iterates. During this time equation 4 applies and thus we have the contraction and expanding estimates as in the statement. In particular H_n decays exponentially in n and this implies that the stable leaves are C^2 close to the piece of stable manifold of q in $\mathcal{V}$ and, by the C^2 continuous dependence of compact parts of stable manifolds on the parameter, this is C^2 close to the vertical. □

3.2. **Critical points.**

Corollary 1. *Let $\gamma \subset \Delta_\varepsilon$ be a long admissible curve. Then, for all $n \geq 1$, $f(\gamma)$ has a unique point of (quadratic) tangency with the stable foliation $\mathcal{E}^{(n)}$.*

Proof. Let $\gamma \subset \Delta_\varepsilon$ be admissible. We will show that then $f(\gamma)$ is quadratic with respect to the foliations $\mathcal{E}^{(n)}$ wherever these are defined. This implies the statement.

By an explicit calculation we can show that $f(\gamma)$ has strictlyt positive curvature with respect to the horizontal and vertical coordinates. Lemma 4 then implies that the same is true for $\mathcal{U}$ sufficiently small. More specifically, using the same notation as in the proof of Lemma 3 we notice that for points outside Δ_ε we have $(f \circ \gamma)_y(t) \neq 0$ and therefore we can write $f(\gamma)$ as a graph over the $y - axis$: $(g(y), y)$. In this case we have

$$g''(y) = \frac{(-2a + y''(t))b}{b^3} = \frac{-2a + y''(t)}{b^2}.$$

This quantity is uniformly bounded away from 0. □

3.3. **Recovering hyperbolicity near the folds.**

Proposition 2. *For all $\varepsilon > 0$ sufficiently small, $(a, b) \in \mathcal{U}^+ = \mathcal{U}^+(\varepsilon)$ as in Proposition 3, $f = f_{a,b}$, $\Delta = \Delta_{a,b}$, $x \in \Delta_\varepsilon \setminus \Delta$, $v \in \mathcal{C}(x)$ and $k > 1$ such that $f^j(x) \in \mathcal{Q}$ for all $1 \leq j \leq k-1$ and $f^k(x) \notin \mathcal{Q}$, we have*

$$\|Df^k_z(v)\| \geq e^{\lambda k}\|v\|.$$

We prove Proposition 2 in a sequence of lemmas. We suppose that $z \in \Delta_\varepsilon \setminus \Delta$ and $v \in \mathcal{C}(x)$. We let $z_0 = f(z)$, $v_0 = Df_z(v)$ and $v_n = Df^n_{z_0}(v_0)$. By construction, there exists some n_z such that $z \in \mathcal{V}_{n_z} \setminus \mathcal{V}_{n_z+1}$. For simplicity we just write $n = n_z$ and consider the position of z_0 and the vector v_0 in hyperbolic coordinates $\mathcal{H}^{(n)}$. Let $v_0 = (h_0^{(n)}, v_0^{(n)})$ denote the decomposition of v_0 in the coordinates $\mathcal{H}_0^{(n)}(z_0)$ and let $h_j^n = Df^j_{z_0}(h_0^{(n)})$ and $v_j^n = Df^j_{z_0}(v_0^{(n)})$.

Lemma 5. *Suppose that* $z_0 \in \mathcal{V}_n \setminus \mathcal{V}_{n+1}$. *Then*

$$d(z_0, W^s(q)) \geq \delta 5^{-n}/2.$$

Proof. By construction, the fact that $z_0 \notin \mathcal{V}_{n+1}$ means that $d(z_{n+1}, W^s(q)) \geq \delta/2$. Since the maximum norm of the derivative in $\mathcal{D}$ is bounded above by 5, this implies $d(z_0, W^s(q)) \geq \delta 5^{-n}/2$. □

Lemma 6. *Suppose that* $z_0 \in \mathcal{V}_n \setminus \mathcal{V}_{n+1}$. *Then*

$$d(z, c^{(n)}) \geq \frac{1}{\sqrt{3}} \sqrt{d(z_0, c_0^{(n)})} \geq \frac{\sqrt{\delta}}{3} \left(\frac{1}{\sqrt{5}} \right)^n .$$

Proof. The first inequality follows easily from direct computation. the second follows by the observation that z_0 and the "real" critical valuec_0 lie on opposite sides of the stable manifold $W^s(q)$. A priori we do not know which side $c^{(n)}$ lies on. However we can write

$$d(z_0, c_0^{(n)}) \geq d(z_0, W^s(q)) + d(W^s(q), c_0) - d(c_0^{(n)}, c_0) \geq \frac{\delta}{2} 5^{-n} + \delta_0 - b^n \geq \frac{\delta}{3} 5^{-n}.$$

Here the constant δ_0 can be arbitrarily small depending on the parameter a and b^n is the estimate given by the rate of convergence of hyperbolic coordinates. □

Lemma 7. *Suppose that* $z_0 \in \mathcal{V}_n \setminus \mathcal{V}_{n+1}$. *Then*

$$|v_n| \geq |h_n^{(n)}| \geq e^{\lambda n} |v|, \quad |v_n^n| \leq \left(\frac{b}{3} \right)^n .$$

In particular

$$\|w_n\| \geq e^{\lambda n} \|w\|$$

and w_n *is an admissible vector.*

3.4. **Recovering admissibility.** Also, we say that a C^2 curve γ is called **admissible** if it can be written as the graph $(x, y(x))$ of a function y satisfying

$$|y'(x)|, |y''(x)| \leq \varepsilon$$

We say that $\tilde{\gamma} \subset \Delta_\varepsilon$ is a **long admissible curve** if it is admissible and it crosses the entire width of Δ_ε.

Proposition 3. *Moreover if* $\gamma \subset \mathcal{D} \setminus \Delta_\varepsilon$ *is an admissible curve, then* $f(\gamma)$ *is also an admissible curve.*

Proof. The hyperbolicity statement is standard and follows from the C^2 persistence of uniform hyperbolicity and from the analogous statement for the one-dimensional limit family of maps. See [1, 11] for details. To prove the second statement, let us suppose without loss of generality that $\gamma(t) = (t, y(t))$ is parametrized by its horizontal coordinate. Then, letting $(f \circ \gamma)_x(t)$ and $(f \circ \gamma)_y(t)$ denote the horizontal and vertical coordinate functions of $f \circ \gamma$ we can write

$$f(\gamma(t)) = ((f \circ \gamma)_x(t), (f \circ \gamma)_y(t)) = (1 - at^2 + y(t), bt).$$

As long as $(f\circ\gamma)'_x(t) = -2at + y'(t) \neq 0$, which is always the case if $|t| = |x| \geq \varepsilon$ and $|y'(t)| \leq \kappa < \varepsilon$ since $a \approx 2$, we can write $f(\gamma(t))$ as the graph of a function

$$f(\gamma(t)) = (x, g(x)) = ((f\circ\gamma)_x(t), (f\circ\gamma)_y(t)) = (1 - at^2 + y(t), bt).$$

The statement that $f(\gamma)$ is admissible then amounts to showing that $|g'(x)| \leq \kappa$ and $|g''(x)| \leq \kappa$. To show this, we write $|g'(x)| = |(f\circ\gamma)'_x(t)/(f\circ\gamma)_y(t)| = |b/(-2at + y'(t))| \leq b/(2a\varepsilon - \kappa)$ and $|g''(x)| = |b(-2a + y''(x))/(-2ax + y'(x))^3| \leq b(2a+\varepsilon)/(2a\varepsilon - \kappa)$. It is therefore sufficient to choose ε, κ sufficiently small. □

Lemma 8. *Moreover, if $\gamma \subset \Delta_\varepsilon \setminus \Delta$ is a curve which belongs to a long admissible curve $\tilde{\gamma}$ and $k > 0$ is such that $f^j(\gamma) \subset \mathcal{Q}$ for all $1 \leq j \leq k-1$ and $f^k(\gamma) \cap \mathcal{Q} = \emptyset$, then $f^k(\gamma)$ is also admissible.*

Corollary 2. *All the connected components of $W^u(p) \cap \Delta_\varepsilon$ are long admissible curves.*

Lemma 9. *Let $\gamma_0 : [0,1] \to \mathbb{R}^2$ be C^2 curve and let $\gamma_i(s) = f^i(\gamma_0(s))$ for $i \geq 1$. Let $\kappa_i(s)$ denote the curvature of γ_i at the point $\gamma_i(s)$. Assume that for every s, $\kappa_0(s) \leq \varepsilon$ and that $\|Df^j\dot{\gamma}_{n-j}(s)\| \geq e^{\lambda j}\|\dot{\gamma}_{n-j}(s)\|$ for all $j = 0,..,n-1$. Then $\kappa_n(s) < \varepsilon$ for all $s \in [0,1]$.*

Proof. Let $\gamma_i(s) = (x_i(s), y_i(s))$. Then $x_i(s) = 1 - ax_{i-1}^2(s) + y_i(x)$ and $y_i(s) = bx_{i-1}(s)$. Therefore

$$\begin{aligned}|\kappa_i(s)| =& \frac{|\ddot{x}_i\dot{y}_i - \dot{x}_i\ddot{y}_i|}{\|(x_i(s), y_i(s))\|^2}\\ =& \frac{b(\ddot{x}_{i-1}\dot{y}_{i-1} - \dot{x}_{i-1}\ddot{y}_{i-1}) + 2ab\dot{x}_{i-1}^3}{\|(\dot{x}_i(s), \dot{y}'(s))\|^2}\\ \leq& b|\kappa_{i-1}|\left\|\frac{\dot{\gamma}_{i-1}(s)}{\dot{\gamma}_i(s)}\right\|^3 + 2ab\left\|\frac{\dot{\gamma}_{i-1}(s)}{\dot{\gamma}_i(s)}\right\|^3\end{aligned}$$

We apply this inequality recursively and choose b small and we have the statement. □

Corollary 3. *If $\gamma \subset W^u \cap \Delta_\varepsilon$ is an admissible curve, k is the smallest positive integer such that $f^k(\gamma) \subset \Delta_\varepsilon$, then $f^k(\gamma)$ is an admissible curve.*

Proof. ¿From the above Lemma and the Proposition we have uniform expansion. □

Corollary 4. *All components of $W^u(p) \cap \Delta_\varepsilon$ are long admissible curves.*

Proof. By choosing $\mathcal{U}$ small we can ensure that $W^u_{loc}(p) \cap \Delta_\varepsilon$ is a long admissible curve. Every piece of $W^u(p) \cap \Delta_\varepsilon$ is the image of some curve in $W^u_{loc}(p) \cap \Delta_\varepsilon$. □

4. Hyperbolicity estimates

4.1. **Uniform hyperbolicity on $W^u(p)$.**

Proposition 4. *For all $\varepsilon > 0$ sufficiently small, $(a,b) \in \mathcal{U}^+ = \mathcal{U}^+(\varepsilon)$ as in Proposition 3, , there exists a constant $N = N(a,b)$ such that for $f = f_{a,b}$, $\Delta = \Delta_{a,b}$, $x \in W^{(u)}(p) \cap \Delta_\varepsilon \cap \Omega(f)$, and v a tangent vector to $W^u(p)$ at x, we have*

$$\|Df_x^N(v)\| \geq e^{\lambda N}\|v\|.$$

We remark that the constant N is *not* uniformly bounded in $\mathcal{U}^+$. This implies that we have uniform expansion estimates for all vectors in $W^u(p) \cap \Omega(f)$ for all time. It does not yet however complete the proof of the uniform hyperbolicity of f the estimates do not automatically pass to the closure of $W^u(p)$.

4.2. **Uniform hyperbolicity on $\overline{W^u(p)}$.** We say that $\Gamma \subset \Delta_\varepsilon$ is a C^{1+1} admissible curve if it is a curve with small first derivative and small Lipschitz constant of the first derivative.

Lemma 10. *Any curve Γ which is the pointwise limit of long admissible curves in $W^u(p) \cap \Delta_\varepsilon$ is a C^{1+1} admissible curve.*

Lemma 11. *For all $(a,b) \in \mathcal{U}^+$, the image Γ_0 of a C^{1+1} admissible curve which is the limit of C^2 long admissible curves, has a unique point of tangency with the foliation $\mathcal{E}^\infty$. In particular the point of tangency lies strictly outside $\mathcal{D}$.*

Lemma 12. *Every $z \in \overline{W^u(p)} \cap \Delta_\varepsilon$ lies on a C^{1+1} admissible curve Γ which is the limit of C^2 admissible curves in $W^u(p)$ and Γ contains a unique critical point $c(\Gamma) \neq z$. In particular $d(z,c) > 0$.*

Definition 4. Let

$$\mathcal{C} = \mathcal{C}_{a,b} = \{c(\Gamma) : \Gamma \text{ is in the closure of long admissible curves}\}$$

denote the set of all (dynamically defined) "critical points" of $f_{a,b}$.

Lemma 13. *The set of critical points $\mathcal{C}$ is compact and, for $a > a^*$, disjoint from the non-wandering set : $d(\mathcal{C}, \Omega) > 0$.*

Lemma 14. *There exists a constant $N = N(a)$ such that expansion and admissibility are recovered after returns to $\Delta_\varepsilon \setminus \Delta$ with a maximum of N iterates, possibly long before the orbit of the point escapes from the neighbourhood $\mathcal{Q}$.*

We now define two constants

$$C_N^- = \min\{\|(Df_x^j)^{-1}\|^{-1} : x \in \mathcal{D}, 1 \leq j \leq N\}$$

and

$$C_N^+ = \max\{\|Df_x^j\| : x \in \mathcal{D}, 1 \leq j \leq N\}$$

which are respectively the maximum possible contraction and the maximum possible expansion exhibited by any vector $v \in T_x\mathbb{R}^2$ for any point $x \in \mathcal{D}$ in at most N iterations. Letting C_ε denote the constant of hyperbolicity as in Proposition 3, we then let

$$C_a = \min\left\{C_\varepsilon, \frac{C_N^- e^{-\lambda N}}{C_N^+}\right\}.$$

Lemma 15. *For all $a > a^*$ and all $z \in W^u(p) \cap \Omega(f)$ and vectors w tangent to $W^u(p)$ at z, we have*

$$\|Df_x^n(w) \geq \|C_a e^{\lambda n}\|w\|$$

for all $n \geq 1$.

Corollary 5. *For all $a > a^*$, and all $z \in \Omega(f)$ we have*

$$\|Df_x(w^u) \geq \|C_a e^{\lambda n}\|w^u\|$$

for all $n \geq 1$.

To complete the constuction of the uniformly hyperbolic structure we use the definitions and estimates of section 3.1 to define an invariant contracting subbundle at each point of $\Omega(f)$.

4.3. **Lyapunov exponents for maps in $\mathcal{U}^*$.** Finally it remains to consider parameters in $\mathcal{U}^*$. Here we need to show that the Lyapunov exponents of all ergodic invariant measures are uniformly bounded away from 0.

By definition we have a non-transversal intersection between $W^u(p)$ and $W^s(q)$. However all estimates made above remain valid with the exception of the fact that the critical point which is well defined for all long admissible curves is outside $\mathcal{D}$. In this case it may be on the boundary of $\mathcal{D}$ but in this case it belongs to $W^s(q)$ and thus asymptotically converges to the fixed point q and never escapes the neighrbourhood $\mathcal{Q}$.

This immediately implies that any orbit which intersects Δ_ε infinitely often in forward time has a subsequence of iterates for which vectors grow exponentially fast. All orbits which are asymptotically outside Δ_ε also grow exponentially except for those which are in the stable manifolf of q, but these are not visible by any measure. This implies that all Lyapunov exponents are uniformly bounded away from 0.

Appendix

Notice that the only assumptions we have used so far, to define these concepts, has been the condition that $H_k(z) < 1$. In practical situations, these ides have some use mainly if we are able to control the size of the neighbourhood $\mathcal{N}^{(k)}$, the geometry of the hyperbolic coordinates $\mathcal{H}^{(k)}$ in relation to the geometry of $\mathcal{N}^{(k)}$, and the relation between the neighbourhoods and the hyperbolic coordinates for different orders k. The questions about the geometry of the neighbourhoods and the foliations depend very much on the specific information concerning the system under study and we postpone these to the next section. On the other hand there are some quite general estimates which give some conditions for the convergence of the pointwise contracting directions $e^{(k)}(\xi)$ and integral curves $\mathcal{E}^{(k)}(\xi)$ as $k \to \infty$. The generality of these estimates is of independent interest and thus we give the full calculations here. In the next section we shall be able to assume that $H_k \to \infty$ exponentially fast in certain domains.

We shall adopt the notation $e_j^{(k)} = Df^j(e^{(k)})$ and $f_j^{(k)} = Df^j(e^{(k)})$. Notice that $E_k(z) = \|(Df_z^k)^{-1}\|^{-1} = \|e_k^{(k)}(z)\| = \|Df_z^k(e^{(k)}(z))\|$ and $F_k(z) = \|Df_z^k\| =$

$\|f_k^{(k)}(z)\| = \|Df_z^k(f^{(k)}(z))\|$. We also define angles $\phi^{(k)}, \theta^{(k)}$ by $\phi^{(k)} = \measuredangle(e^{(k)}, e^{(k+1)})$, $\phi_j^{(k)} = \measuredangle(e_j^{(k)}, e_j^{(k+1)})$ and $e^{(k)} = (\cos\theta^{(k)}, \sin\theta^{(k)})$, $f^{(k)} = (-\sin\theta^{(k)}, \cos\theta^{(k)})$, $e_k^{(k)} = E_k(\cos\theta_k^{(k)}, \sin\theta_k^{(k)})$, $f_k^{(k)} = F_k(-\sin\theta_k^{(k)}, \cos\theta_k^{(k)})$. Notice that $\theta^{(k)} = \theta^{(1)} + \sum_{j=1}^{k-1}\phi^{(j)}$ and $De^{(k)} = (-D\theta^{(k)}(\sin\theta^{(k)}), D\theta^{(k)}(\cos\theta^{(k)}))$ which implies in particular that

$$\|De^{(k)}\| \le K\|D\theta^{(k)}\| \le K\|D\theta^{(1)}\| + K\sum_{j=1}^{k-1}\|D\phi^{(j)}\|. \tag{10}$$

where the constant K depends only on the choice of the norms.

We start with a lemma concerning the convergence of the sequence of most contracted direction $e^{(k)}(x)$ as $k \to \infty$.

Lemma 16. *$\exists\ K > 0$ such that $\forall\ k \ge 1$ and $x \in \mathcal{N}^{(k+1)}$ and $\phi^{(k)} = \phi^{(k)}(x)$, we have*

$$|\phi^{(k)}| \le KH_k$$

The next result says gives some control over the dependence of the most contracting directions on the base point.

Lemma 17. *$\exists\ K > 0$ such that $\forall\ k \ge 1\ \forall\ x \in \mathcal{N}^{(k+1)}$ and $\phi^{(k)} = \phi^{(k)}(x)$, we have*

$$\|D\phi^{(k)}\| \le KE_k \quad \text{and} \quad \|De^{(k)}\| \le K$$

Sublemma 17.1. *$\|DE_k\|, \|DF_k\| \le KF_k^2$ and $\|DH_k\| \le KE_k$*

We now show that these estimates imply some estimate on the convergence of the sequence of curves $\{\mathcal{E}^{(k)}(\xi)\}$. Let $z_t^{(k)}$ and $z_t^{(k+1)}$, be parametrizations by arclength of the two curves $\mathcal{E}^{(k)}(p)$ and $\mathcal{E}^{(k+1)}(p)$ respectively, with $z_0^{(k)} = z_0^{(k+1)} = \xi$ and choose t_0 so that both $\{z_t^{(k+1)}\}_{t=-t_0}^{t_0}$ and $\{z_t^{(k)}\}_{t=-t_0}^{t_0}$ are both contained in $\mathcal{N}^{(k+1)}$.

Lemma 18. *$\exists\ K > 0$ such that $\forall\ k \ge 1$ and $-t_0 \le t \le t_0$ we have*

$$|z_t^{(k)} - z_t^{(k+1)}| \le Kt\bar{H}_k e^{tK} \le Kt_0\bar{H}_k e^{t_0K}$$

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Department of Mathematics, Suzhou University, Suzhou 215006, Jiangsu, P.R. China
E-mail address: ylcao@suda.edu.cn

Department of Mathematics, Imperial College, 180 Queen's Gate, London SW7 2AZ, UK
E-mail address: Stefano.Luzzatto@imperial.ac.uk
URL: http://www.ma.ic.ac.uk/ luzzatto

Universidade Federal Fluminense, Niteroi, RJ, Brazil.
E-mail address: rios@mat.uff.br