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DYNAMIC EQUATIONS ON TIME SCALES

SUNG KYU CHOI AND NAM JIP KOO

ABSTRACT. This is an introductory article about the dynamic equations on time scales. The foundation of dynamic equations on time scales is calculus on time scales by Hilger [4]. He generalized differential and integral calculus by replacing the range of definition of the functions under consideration $\mathbb{R}$ by an arbitrary time scale $\mathbb{T}$.

We explain two basic results : the Gronwall's inequality and stability theorem by comparison. Our basic references are [3] and [7].

1. PRELIMINARIES

Definition 1.1. A *time scale* $\mathbb{T}$ means any closed subset of $\mathbb{R}$ with order topology, that is, generated by open intervals

$$(r, s) = \{t \in \mathbb{T} \, : \, r < t < s, r, s \in \mathbb{T} \cup \{\pm\infty\}\}.$$

Examples of time scales are $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $h\mathbb{Z}$ for $h > 0$, and the Cantor set.

Definition 1.2. The mapping $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ defined as $\sigma(t) = \inf\{s \in \mathbb{T} \, : \, s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T} \, : \, s < t\}$ are called *jump operators*.

Definition 1.3. A non-maximal element $t \in \mathbb{T}$ is said to be *right dense* (rd) if $\sigma(t) = t$, *right scattered* if $\sigma(t) > t$, *left dense* if $\rho(t) = t$, and *left scattered* if $\rho(t) < t$.

Definition 1.4. The mapping $\mu : \mathbb{T} \to \mathbb{R}_+$ defined by $\mu(t) = \sigma(t) - t$ is called *graininess*.

Example 1.5. *When* $\mathbb{T} = \mathbb{R}$*,*

$$\sigma(t) = \inf(t, \infty) = t, \rho(t) = \sup(-\infty, t) = t.$$

Thus every $t \in \mathbb{R}$ *is dense and* $\mu(t) = \sigma(t) - t = t - t = 0$.
When $\mathbb{T} = \mathbb{Z}$*,*

$$\begin{aligned} \sigma(t) &= \inf\{t+1, t+2, \cdots\} = t+1, \\ \rho(t) &= \sup\{t-1, t-2, \cdots\} = t-1. \end{aligned}$$

Every $t \in \mathbb{Z}$ *is isolated, that is,* $\rho(t) = t - 1 < t < t + 1 = \sigma(t)$. *Moreover,* $\mu(t) = \sigma(t) - t = t + 1 - t = 1$.

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The graininess function plays a central role in analysis on time scales. For instance, we consider the scalar Riccati dynamic equation

$$z^{\Delta} + q(t) + \frac{z^2}{p(t) + \mu(t)z} = 0$$

becomes

$$z' + q(t) + \frac{z^2}{p(t)} = 0 \text{ when } \mathbb{T} = \mathbb{R}$$

and

$$\Delta z + q(t) + \frac{z^2}{p(t) + z} = 0 \text{ when } \mathbb{T} = \mathbb{Z}.$$

Definition 1.6. The mapping $\mu : \mathbb{T} \to X$, where X is a Banach space is called *rd-continuous* if

(i) it is continuous at each right dense $t \in \mathbb{T}$,
(ii) at each left dense point t, $\lim u(t^-) = \lim_{\substack{s \to t \\ s < t}} u(s)$ exists.

Remark 1.7. (i) Let $Crd(\mathbb{T}, X)$ denote the set of rd-continuous mapping from $\mathbb{T}$ to $\mathbb{X}$.

(ii) Clearly, a continuous mapping is rd-continuous.
However, if $\mathbb{T}$ contains left dense and right scattered points, then rd-continuity does not imply continuity. But on a discrete time scale the two notions coincide.

Definition 1.8. A mapping $u : \mathbb{T} \to X$ is said to be *(delta) differentiable at* $t \in \mathbb{T}$ *if there exists an* $\alpha \in X$ *such that for any* $\varepsilon > 0$ *there exists a neighborhood* N *of* t *satisfying*

$$|u(\sigma(t)) - u(s) - \alpha[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|, \ s \in N.$$

The derivative can also be defined in terms of a limit as follows :

$$\begin{aligned} u^{\Delta}(t) &= \lim_{\substack{s \to t \\ \sigma(s) \neq t}} \frac{u(\sigma(s)) - u(t)}{\mu(\sigma(s), t)} \\ &= \lim_{\substack{s \to t \\ s \neq \sigma(t)}} \frac{u(\sigma(t)) - u(s)}{\mu(\sigma(t), s)} \in X. \end{aligned}$$

Remark 1.9. [3] (i) Let $u^{\Delta}(t)$ denote the derivative of u. If $\mathbb{T} = \mathbb{R}$, then we have

$$\alpha = u^{\Delta} = \frac{du(t)}{dt}$$

and if $\mathbb{T} = \mathbb{Z}$, then

$$\alpha = u^{\Delta} = u(t+1) - u(t) = \Delta u(t).$$

(ii) If u is differentiable at t, then it is continuous at t.

(iii) If u is continuous at t and t is right scattered, then u is differentiable at t and

$$u^{\Delta}(t) = \frac{u(\sigma(t)) - u(t)}{\sigma(t) - t}.$$

(iv) If u is differentiable at t and t is right-dense, then

$$u^{\Delta}(t) = \lim_{s \to t} \frac{u(t) - u(s)}{t - s}.$$

Definition 1.10. The mapping $f : \mathbb{T} \times X \to X$ is *rd-continuous* if

(i) it is continuous at each (t, x) with right dense t,
(ii) $\lim_{(s,y) \to (t^-, x)} f(s, y) = f(t^-, x)$ exist,
(iii) at each (t, x) with left dense t, $\lim_{y \to x} f(t, y)$ exists.

Definition 1.11. Let h be a mapping from $\mathbb{T}$ to X. The mapping $g : \mathbb{T} \to X$ is called the *antiderivative* of h on $\mathbb{T}$ if

(i) it is differentiable on $\mathbb{T}$
(ii) $g^{\Delta}(t) = h(t)$ for $t \in \mathbb{T}$.

The *Cauchy integral* is defined as

$$\int_a^t h(s) \Delta s = g(t) - g(a).$$

Definition 1.12. The mapping $p : \mathbb{T} \to \mathbb{R}$ is said to be *regressive* if

$$1 + \mu(t) p(t) \neq 0, \ t \in \mathbb{T}^{\kappa},$$

where $\mathbb{T}^{\kappa} = \mathbb{T} - \{m\}$ when $\mathbb{T}$ has a left scattered maximum m, and $\mathbb{T}^{\kappa} = \mathbb{T}$

otherwise. The set of all regressive and rd-continuous functions is denoted by $\mathrm{R}(\mathbb{T}, \mathbb{R})$.

For $p \in \mathrm{R}(\mathbb{T}, \mathbb{R})$, we define the *exponential function* by

$$e_p(t, s) = \exp\left[\int_s^t \xi_{\mu(\tau)}(p(\tau)) \Delta\tau\right], \ s, t \in \mathbb{T},$$

where the cylinder transformation $\xi_h(z) = \frac{1}{h}\mathrm{Log}(1 + zh)$ is introduced in [3].

Example 1.13. [3] *The exponential function* $e_{\alpha}(t, t_0)$ *becomes* $e^{\alpha(t - t_0)}$ *when* $\mathbb{T} = \mathbb{R}$, *and* $(1 + \alpha)^{t - t_0}$ *when* $\mathbb{T} = \mathbb{Z}$.

In particular, for $p(t) = 1$ *and* $t_0 = 0, e_p(t, t_0)$ *becomes* e^t *if* $\mathbb{T} = \mathbb{R}$, *and* 2^t *if* $\mathbb{T} = \mathbb{Z}$.

Remark 1.14. $e_p(t, t_0)$ satisfies the dynamic equation

$$x^{\Delta} = p(t) x, \ x(t_0) = 1,$$

where $p \in \mathrm{R}(\mathbb{T}, \mathbb{R})$ and $t_0 \in \mathbb{T}$.

2. Gronwall's Inequality

The following is the well-known Gronwall's inequality for the differential version.

I. Let $m, v \in C(\mathbb{R}_+, \mathbb{R}_+)$. Suppose that, for some $c \geq 0$, we have

$$m(t) \leq c + \int_{t_0}^t v(s) m(s) ds, \ t \geq t_0 \geq 0. \tag{2.1}$$

Then

$$m(t) \leq c \exp\left[\int_{t_0}^{t} v(s)ds\right],\ t \geq t_0. \tag{2.2}$$

The most elementary type is the case "$v(s) = k,$ constant" and (2.2) becomes

$$m(t) \leq ce^{kt},\ t \geq t_0$$

[5].

The type I is generallized as the following :

II. Let $m, v, h \in C(\mathbb{R}_+, \mathbb{R}_+)$ and suppose that

$$m(t) \leq h(t) + \int_{t_0}^{t} v(s)m(s)ds,\ t \geq t_0 \geq 0. \tag{2.3}$$

Then

$$m(t) \leq h(t) + \int_{t_0}^{t} v(s)h(s) \exp\left[\int_{s}^{t} v(\tau)d\tau\right] ds,\ t \geq t_0. \tag{2.4}$$

The following is the discrete version of type II.

III. Let u, f, p, q be the scalar valued functions defined on $\mathbb{N}(a) = \{a, a+1, \cdots\},\ a \in \mathbb{N}$. Suppose that

$$u(k) \leq p(k) + q(k)\sum_{l=a}^{k-1} f(l)u(l).$$

Then

$$u(k) \leq p(k)) + q(k)\sum_{l=a}^{k-1} p(l)f(l)\prod_{\tau=l+1}^{k-1}\left[1 + q(\tau)f(\tau)\right], k \in \mathbb{N}(a). \tag{2.5}$$

Now, the basic inequality for the unified Gronwall's inequality is the following :

IV. Let $y, f \in Crd(\mathbb{T}, \mathbb{R})$ and $p \in \mathrm{R}_+(\mathbb{T}, \mathbb{R})$, i.e., p satisfies $1 + \mu(t)p(t) > 0$ for $t \in \mathbb{T}$. Suppose that

$$y^{\Delta}(t) \leq p(t)y(t) + f(t),\ t \in \mathbb{T}.$$

Then

$$y(t) \leq y(t_0)e_p(t, t_0) + \int_{t_0}^{t} e_p(t, \sigma(\tau))f(\tau)\Delta\tau,\ t \in \mathbb{T}.$$

Here

$$e_p(t, \sigma(\tau)) = \begin{cases} \exp\int_{\tau}^{t} p(s)ds & \text{if } \mathbb{T} = \mathbb{R}, \\ \prod_{s=\tau+1}^{t-1}\left[1 + p(s)\right] & \text{if } \mathbb{T} = \mathbb{Z}. \end{cases}$$

V. Let $y, f \in Crd(\mathbb{T}, \mathbb{R}),\ p \in \mathrm{R}_+(\mathbb{T}, \mathbb{R})$ and $p \geq 0$. Assume that

$$y(t) \leq f(t) + \int_{t_0}^{t} y(\tau)p(\tau)\Delta\tau,\ t \in \mathbb{T}.$$

Then

$$y(t) \leq f(t) + \int_{t_0}^{t} e_p(t, \sigma(\tau))f(\tau)\Delta\tau,\ t \in \mathbb{T}.$$

3. Stability by Comparison

We consider the nonlinear differential system

$$x' = f(t, x),\ x(t_0) = x_0, \tag{3.1}$$

where $f \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n),\ x_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}_+$. For the function $V \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$, we define

$$D^+V(t, x) = \lim_{h \to 0^+} \sup \frac{V(t + h, x + hf(t, x)) - V(t, x)}{h}. \tag{3.2}$$

The basic comparison theorem for (3.1) is the following.

Theorem 3.1. [8, Theorem 3.1.1] *Let $V(t, x)$ be locally Lipschitzian in $x \in \mathbb{R}^n$ for each $t \in \mathbb{R}_+$. Assume that*

$$D^+V(t, x) \leq g(t, V(t, x)),\ (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n,$$

where $g \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R})$. Let $r(t) = r(t, t_0, u_0)$ be the maximal solution of the scalar differential equation

$$u' = g(t, u),\ u(t_0) = u_0 \geq 0 \tag{3.3}$$

existing t_0 the right of t_0. If $x(t) = x(t, t_0, x_0)$ is any solution of (3.1) existing for $t \geq t_0$ such that

$$V(t_0, t_0) \leq u_0,$$

then

$$V(t, x(t)) \leq r(t),\ t \geq t_0.$$

The stability theorem using Theorem 3.1 can be found in [8, Theorem 3.5.2] :

Theorem 3.2. *Suppose that*

(i) *$V(t, x)$ is Lipschitzian in x for each $t \in \mathbb{R}_+$,*
(ii) *$b(|x|) \leq V(t, x) \leq a(|x|),\ (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$,*

where $a, b \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $a(0) = 0 = b(0)$ and a, b are nondecreasing in u,

(iii) *$D^+V(t, x) \leq g(t, V(t, x)),\ (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$,*

where $g \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R})$ with $g(t, 0) = 0$. Then the zero solution of (3.1) is stable provided the zero solution of (3.3) is stable.

Now, we consider the dynamic equation

$$x^{\Delta} = f(t,x),\ x(t_0) = x_0, \tag{3.4}$$

where $f \in \mathrm{Crd}(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n)$ with $f(t,0) = 0$ and x^{Δ} is the derivative of x with respect to $t \in \mathbb{T}$. We assume that the solutions $x(t) = x(t,t_0,x_0)$ of (3.4) exist and unique for $t \geq t_0$. For the Liapunov-like function $V \in \mathrm{Crd}(\mathbb{T}^{\kappa} \times \mathbb{R}^n, \mathbb{R}_+)$, we recall the following definition :

Definition 3.3. [9] We define the generalized derivative $D^+V^{\Delta}(x,x(t))$ of $V(t,x)$ relative to (3.4) as follows: given $\varepsilon > 0$, there exists a neighborhood $N(\varepsilon)$ of $t \in \mathbb{T}$ such that

$$\begin{aligned} \frac{1}{\mu(t,s)}[V(\sigma(t),x(\sigma(t)) \quad &- \quad V(s,x(s)) - \mu(t,s)f(t,x(t))] \\ &< \quad D^+V^{\Delta}(t,x(t)) + \varepsilon,\ s \in N(\varepsilon),\ s > t, \end{aligned} \tag{3.5}$$

where $\mu(t,s) = \sigma(t) - s$ and $x(t)$ is any solution of (3.4).

Then the basic comparison theorem for the differential system (3.1) becomes the following :

Theorem 3.4. [9, Theorem 3.1.1] *Let* $V \in \mathrm{Crd}(\mathbb{T}\times\mathrm{R}^{\mathrm{n}}, \mathbb{R}_+)$ *be locally Lipschitziaan in* x *for each* $t \in \mathbb{T}$. *Assume that*

$$D^+V^{\Delta}(t,x) \leq g(t,V(t,x)),\ (t,x) \in \mathbb{T} \times \mathbb{R}^n,$$

where $g \in \mathrm{Crd}(\mathbb{T} \times \mathbb{R}_+, \mathbb{R}_+)$ *and* $g(t,u)\mu(t)$ *is nondecreasing in* u *for each* $t \in \mathbb{T}$. *Let* $r(t) = r(t,t_0,u_0)$ *be the maximal solution of the scalar dynamic equation*

$$u^{\Delta} = g(t,u),\ u(t_0) = u_0 \geq 0 \tag{3.6}$$

existing on $\mathbb{T}$. *Then* $V(t_0,x_0) \leq u_0$ *implies that*

$$V(t,x(t)) \leq r(t,t_0,u_0), t \in \mathbb{T}, t \geq t_0.$$

Finally, the stability theorem for (3.4) can be obtained as follows.

Theorem 3.5. [9, Theorem 3.2.1] *Assume that*

(i) $V(t,x) \in \mathrm{Crd}(\mathbb{T} \times \mathrm{R}^{\mathrm{n}}, \mathbb{R}_+)$ is locally Lipschitzian in x for each right dense $t \in \mathbb{T}$,

(ii) $b(|x|) \leq V(t,x) \leq a(|x|),\ (t,x) \in \mathbb{T} \times \mathbb{R}^n$,
where $a, b \in C(\mathbb{R}_+, \mathbb{R}_+), a(0) = 0 = b(0)$ and a, b are nondecreasing in u,

(iii) $D^+V^{\Delta}(t,x) \leq g(t,V(t,x)), (t,x) \in \mathbb{T} \times \mathbb{R}^n$,
where $g \in \mathrm{Crd}(\mathbb{T} \times \mathbb{R}_+, \mathbb{R})$ with $g(t,u) = 0$ and $g(t,u)\mu(t)$ is nondecreasing in u for each $t \in \mathbb{T}$.

Then the zero solution of (3.4) is stable when the zero solution of (3.6) is stable.

Proof. Let $\varepsilon > 0$ and $t \in \mathbb{T}$ be given. Since $u(t) = 0$ is stable, we have, for any $b(\varepsilon) > 0$, there exists $\delta_1 = \delta_1(t_0,\varepsilon) > 0$ such that $u_0 < \delta_1$ implies $u(t) < b(\varepsilon)$ for all $t \in \mathbb{T}$. Choose $\delta = \delta(t_0,\varepsilon) > 0$ such that $a(\delta) < \delta_1$. We show that

$$|x_0| < \delta \text{ implies } |x(t)| < \varepsilon,\ t \in \mathbb{T}.$$

If this is not true, then there exist $t_1 > t_0$ and a solution $x(t)$ such that

$$|x(t)| < \varepsilon \text{ for } t_0 \leq t < t_1 \text{ and } |x(t)| \geq \varepsilon.$$

Putting $m(t) = V(t, x(t))$ for $t_0 \leq t \leq t_1$, we have

$$V(t, x(t)) \leq r(t, t_0, u_0),\ t_0 \leq t \leq t_1$$

in view of Theorem 3.2. Thus we obtain

$$b(\varepsilon) \leq b(|x(t_1)|) \leq V(t, x(t_1)) \leq r(t) < b(\varepsilon),$$

which is a contradiction. □

For the illustrative example to show the application of Theorem 3.5, see [7, p. 162].

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Department of Mathematics, Chungnam National University, Daejeon 305-764, Korea
E-mail address: skchoi{njkoo}@math.cnu.ac.kr