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SHIFT SPACES WITH COUNTABLE STATES AND THEIR C^*-ALGEBRAS

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Abstract. If E is a directed graph with the vertex set E^0 and the edge set E^1 it is known that there exists a universal C^*-algebra $C^*(E)$ generated by a family of partial isometries $\{s_e\}_{e\in E^1}$ and a family of mutually orthogonal projections $\{p_v\}_{v\in E^0}$ satisfying certain relations. This class of C^*-algebras includes the Cuntz-Krieger algebras. In this paper we first review some known results on the structure of graph C^*-algebras, and then for a locally finite infinite graph E we consider a dynamical system on $C^*(E)$ given by the canonical completely positive map ϕ_E whose restriction to the commutative C^*-subalgebra $\mathcal{D}_E$ corresponds to the one-sided shift map σ_E on the infinite path space X_E. We discuss the relation between the topological conjugacy classes of shift spaces $\{X_E\}_E$ and isomorphism classes of graph C^*-algebras $\{C^*(E)\}_E$.

1. Introduction

Since Cuntz and Krieger [7] introduced a C^*-algebra $\mathcal{O}_A$ associated with an $n \times n$ $\{0,1\}$ matrix A for the study of topological Markov chains, the class of Cuntz-Krieger algebras has been generalized in various ways (see [19] for a survey). Graph C^*-algebras can also be viewed as generalized Cuntz-Krieger algebras and in this paper we will focus on this class of C^*-algebras. Besides the relation with symbolic dynamics, there are many other interesting problems regarding the class of graph C^*-algebras and actually in recent years valuable results have been obtained, for example, see [2, 9, 11, 10, 13].

The graph C^*-algebra $C^*(E)$ of a directed graph E is generated by a universal CK (Cuntz-Krieger) E-family [20, 21, 2, 19]. Given an $n \times n$ $\{0,1\}$ matrix A, one can draw the finite graph E with the vertex matrix A and it can be shown that the graph C^*-algebra $C^*(E)$ is isomorphic to the Cuntz-Krieger algebra $\mathcal{O}_A$. Moreover if B is the edge matrix of E, $C^*(E)$ is also isomorphic to $\mathcal{O}_B$ by definition.

Let A be an $n \times n$ $\{0,1\}$ matrix and X_A denote its one-sided shift space with the shift map σ_A. Then X_A is compact in product topology of $\Pi_{i=1}^{\infty} E^1$, the infinite product space of the finite edge set E^1 of E, and the C^*-algebra $\mathcal{O}_A$ contains a commutative subalgebra $\mathcal{D}_A$ which is isomorphic to the commutative C^*-algebra $C(X_A)$ of all continuous functions on X_A in such a way that the restriction of the canonical completely positive map ϕ_A to $\mathcal{D}_A$ is transformed to the $*$-homomorphism σ_A^* induced by the shift $\sigma_A : X_A \to X_A$. Therefore the study of shift spaces (X_A, σ_A) of finite type is closely related with that of Cuntz-Krieger algebras $(\mathcal{O}_A, \phi_A)$ with the cp map ϕ_A, and more generally one might expect that the graph C^*-algebras associated with infinite directed graphs will paly an important role in our understanding the shift spaces of countable states.

Cuntz and Krieger [7] prove that if (X_A, σ_A) and (X_B, σ_B) are topologically conjugate then there is a $*$-isomorphism $\rho : \mathcal{O}_A \to \mathcal{O}_B$ transforming $\phi_A|_{\mathcal{D}_A}$ into $\phi_B|_{\mathcal{D}_B}$, which can be extended to graph C^*-algebras of row finite (possibly infinite) graphs with no sinks, that is, if (X_E, σ_E) and (X_F, σ_F) are topologically conjugate shift spaces of row finite graphs E and F with no sinks, there exists an isomorphism $\theta : C^*(E) \to C^*(F)$ such that $\theta \circ \phi_E = \phi_F \circ \theta$ on the commutative subalgebra $\mathcal{D}_E$ of $C^*(E)$ (see section 4). Note here that if E is infinite then X_E is a locally compact space and $\mathcal{D}_E$ is isomorphic to $C_0(X_E)$. Also in [7, 6] it is known that $(\mathcal{O}_A \otimes K(H), \mathcal{D}_A \otimes K(H))$ is an invariant under flow equivalence if A is an irreducible matrix or a reducible matrix satisfying certain condition, which is extended later in fully generalized situation by Huang [12].

Voiculescu [27] defined the topological entropy for an automorphism ϕ on a nuclear C^*-algebra, and Brown [4] extended the notion to automorphisms on exact C^*-algebras. But it also applies easily to cp maps on exact C^*-algebras as described in [1]. [29] is a good reference for nuclear or exact C^*-algebras. Since a graph C^*-algebra is always nuclear and for a locally finite graph E there exists a canonical cp map ϕ_E on the C^*-algebra $C^*(E)$, the topological entropy $ht(\phi_E)$ can be considered. In case where E is a finite graph possibly with sinks or sources, it is known [5, 1, 14, 22] that $ht(\phi_E) = \log r(A_E)$ provided E contains an infinite path. Here $r(A_E)$ is the spectral radius of the edge matrix A_E of E. Furthermore it turns out that $ht(\phi_E) = ht(\phi_E|_{\mathcal{D}_E})$, and from $\mathcal{D}_E \cong C(X_E)$ we see that $ht(\phi_E|_{\mathcal{D}_E})$ is again equal to $h_{top}(X_A, \sigma_A)$, the topological entropy of the continuous map σ_A on the compact space X_A. In this paper we review some definitions and basic properties of graph C^*-algebras, and then see some results known for the topological entropy of the map ϕ_E when E is a locally finite irreducible infinite graph.

2. Preliminaries

Positive linear maps on C^*-algebras. A C^*-algebra is a Banach $*$-algebra which is isometrically isomorphic to a (operator) norm closed $*$-subalgebra of $B(H)$, the algebra of all bounded linear operators acting on a Hilbert space H. A Banach $*$-algebra A with norm $\|\cdot\|$ is a C^*-algebra if and only if $\|a^*a\| = \|a\|^2$ for $a \in A$. An element $a \in A$ is said to be *self-adjoint* if $a = a^*$. We call $a \in A$ *positive* $(a \geq 0)$ if $a = b^*b$ for some $b \in A$, or equivalently if $a = c^2$ for a self-adjoint element $c \in A$. Recall that a *projection* p is a self-adjoint idempotent$(p = p^2)$, hence projections are positive operators. A *partial isometry* s is an operator such that s^*s (or ss^*) is a projection. It is useful to note that $s = (ss^*)s = s(s^*s)$ holds. The projections ss^* and s^*s are called the range and initial projections of s, respectively. For two self-adjoint elements $a, b \in A$, we write $a \geq b$ if $a - b$ is positive. If $p, q \in A$ are projections with $p \geq q$ then $q = pq = qp$ and $p = q + (p - q)$ is the sum of two mutually orthogonal sub-projections q and $p - q$ $(q(p - q) = (p - q)q = 0)$.

A linear map $\phi : A \to A$ is *positive* if $\phi(a) \geq 0$ whenever $a \geq 0$. If A is a C^*-subalgebra of $B(H)$, the matrix algebra $M_n(A)$ over A can be realized as a C^*-algebra again since it is a subalgebra of $B(H^n)$. Then ϕ extends to a linear map $\phi_n : M_n(A) \to M_n(A)$, $\phi_n([a_{ij}]) = [\phi(a_{ij})]$. If ϕ_n is positive for all $n = 1, 2, \dots,$ we call ϕ *completely positive* (cp) map. Every $*$-homomorphism (a multiplicative $*$-preserving linear map) is a cp map. One typical example of a cp map which is not necessarily a $*$-homomorphism is the map given by, with a fixed $x \in A$, $a \mapsto xax^* : A \to A$.

Graphs and graph C^*-algebras. A directed graph (or simply a graph) E is a quadruple $E = (E^0, E^1, r, s)$ consisting of the vertex set E^0, the edge set E^1 and the range, source maps $r, s : E^1 \to E^0$. If each vertex emits only finitely many edges, E is *row finite*, and a row finite graph E is *locally finite* if each vertex receives only finitely many edges. A *finite* graph is a locally finite graph with a finite vertex set. By E^n we denote the set of all finite paths $\alpha = e_1 \cdots e_n$ $(r(e_i) = s(e_{i+1}), 1 \leq i \leq n-1)$ of *length* n $(|\alpha| = n)$. Vertices are regarded as finite paths of length 0. The set $\cup_{n \geq 0} E^n$ of all finite paths will be denoted by E^*. Similarly, one can define the (one-sided) infinite paths E^∞.

Given a graph E, we call a family $\{s_e, p_v \mid e \in E^1, v \in E^0\}$ of operators a *CK E-family* if $\{s_e\}_{e \in E^1}$ are partial isometries and $\{p_v\}_{v \in E^0}$ are mutually orthogonal projections such that

(i) $s_e^* s_e = p_{r(e)}$,

(ii) $s_e s_e^* \leq p_{s(e)}$,

(iii) $p_v = \displaystyle\sum_{s(e)=v} s_e s_e^*$, if v emits finitely many edges.

It is now well known that for an arbitrary graph E there exists a C^*-algebra $C^*(E)$ generated by a CK E-family $\{s_e, p_v \mid e \in E^1, v \in E^0\}$ which is universal in the following sense; if B is a C^*-algebra generated by a CK E-family $\{t_e, q_v\}$, there exists a $*$-homomorphism $\pi_{t,q} : C^*(E) \to B$ such that $\pi_{t,q}(s_e) = t_e$ and $\pi_{t,q}(p_v) = q_v$ for each $e \in E^1$ and $v \in E^0$. We simply write $C^*(E) = C^*(s_e, p_v)$ if $\{s_e, p_v \mid e \in E^1, v \in E^0\}$ is a CK E-family which generates $C^*(E)$. If $C^*(E) = C^*(s_e, p_v)$, then $\{zs_e, p_v\}$ is a CK E-family for each $z \in \mathbb{T}$, and so by the universal property of $C^*(E)$ there exists a $*$-homomorphism $\gamma_z : C^*(E) \to C^*\{zs_e, p_v\}$ such that $\gamma_z(s_e) = zs_e$ and $\gamma_z(p_v) = p_v$. But clearly γ_z is an automorphism of $C^*(E)$ and moreover $\gamma : \mathbb{T} \to Aut(C^*(E))$ is a strongly continuous action of the unit torus $\mathbb{T}$ on the graph C^*-algebra $C^*(E)$ which we call the *gauge action*.

1. *Uniqueness Theorem.* Let $C^*(E) = C^*(s_e, p_v)$ and B be a C^*-algebra generated by another CK E-family $\{t_e, q_v\}$ such that $q_v \neq 0$. For the canonical $*$-homomorphism $\pi_{t,q} : C^*(E) \to B$ it is known [2] that if there is a strongly continuous action α of $\mathbb{T}$ on B with $\alpha_z \circ \pi_{t,q} = \pi_{t,q} \circ \gamma_z$ for $z \in \mathbb{T}$, then $\pi_{t,q}$ is an isomorphism. This fact is referred to as the gauge-invariant uniqueness theorem.

2. *Ideal structure of* $C^*(E)$. Let E be a locally finite graph. A finite path $\alpha \in E^*$ with $s(\alpha) = r(\alpha)$ is called a *loop* if $|\alpha| > 0$. We say that E satisfies condition (K) if every vertex v which is a source of a loop is a source of at least two distinct loops. A subset $H \subset E^0$ is *hereditary* if $v \in H$, $v = s(e)$ for some edge $e \in E^1$ implies $r(e) \in H$. A hereditary subset H is *saturated* if H contains every vertex $v \in E^0$ satisfying $r(s^{-1}(v)) \subset H$. Then the set $\mathcal{H}(E^0)$ of saturated vertex subsets has a lattice structure, and if E satisfies condition (K), there is a lattice isomorphism between the closed two-sided ideals of $C^*(E)$ and $\mathcal{H}(E^0)$ [21]. In fact, more general results on the ideal structure of graph algebras have been known, but here we just stated a special case of locally finite graphs which is enough to get the picture.

In this paper we are mainly interested in row finite (or locally finite) graphs because we deal with several entropies of certain maps including the cp map ϕ_E which is well defined on a graph C^*-algebra $C^*(E) = C^*(s_e, p_v)$ of a locally finite graph E: $\phi_E(x) = \sum_{e\in E^1} s_e x s_e^*$.

Let $C^*(E) = C^*(s_e, p_v)$ be a graph C^*-algebra, and $\alpha = e_1 \cdots e_n$ be a finite path of length $n = |\alpha|$. Then $s_\alpha := s_{e_1} \cdots s_{e_n}$ becomes a partial isometry. It is useful to note that the linear span of the partial isometries of the form $s_\alpha s_\beta^*$ ($\alpha, \beta \in E^*$) is dense in $C^*(E)$, that is,

$$C^*(E) = \overline{span}\{s_\alpha s_\beta^* \mid \alpha, \beta \in E^*\}.$$

The graph algebra $C^*(E)$ contains two important C^*-subalgebras

$$\begin{aligned}\mathcal{A}_E : &= \overline{span}\{s_\alpha s_\beta^* \mid \alpha, \beta \in E^*, |\alpha| = |\beta|\},\\ \mathcal{D}_E : &= \overline{span}\{s_\alpha s_\alpha^* \mid \alpha \in E^*\}.\end{aligned}$$

It is easy to see that $\mathcal{A}_E$ is an AF(approximately finite dimensional) algebra, and $\mathcal{D}_E$ is a commutative subalgebra of $\mathcal{A}_E$. These are ϕ_E-invariant ($\phi_E(\mathcal{A}_E) \subset \mathcal{A}_E$, $\phi_E(\mathcal{D}_E) \subset \mathcal{D}_E$) C^*-subalgebras.

The graph C^*-algebras are known to be nuclear so that the following definition of topological entropy for cp maps on graph algebras makes sense.

Topological entropy of a cp map. Let A be a C^*-subalgebra of $B(H)$ and $\mathcal{F}(A)$ denote the set of all finite subsets of A. For $\omega \in \mathcal{F}(A)$ and $\delta > 0$, put

$$\begin{aligned}CPA(A) = \{(\phi, \psi, B) \mid\ &\phi : A \to B, \psi : B \to B(H) \text{ are contractive cp maps}\\ &\text{and } B \text{ is a } C^*\text{-algebra with} \dim B < \infty\},\\ rcp(\omega, \delta) = \inf\{&rank(B) \mid (\phi, \psi, B) \in CPA(A),\ \|\psi \circ \phi(x) - x\| < \delta\ \forall x \in \omega\},\end{aligned}$$

where $rank(B)$ denotes the dimension of a maximal abelian subalgebra of B. If B is isomorphic to a direct sum $M_{n_1} \oplus \cdots \oplus M_{n_k}$ of matrix algebras M_{n_j}, the rank of B is $rank(B) = n_1 + \cdots + n_k$ (every finite dimensional C^*-algebra is of this form up to isomorphism). For a cp map $\Phi : A \to A$ we put

$$\begin{aligned}ht(\Phi, \omega, \delta) &= \limsup_{n\to\infty} \frac{1}{n} \log\left(rcp(\cup_{i=0}^{n-1} \Phi^i(\omega), \delta)\right)\\ ht(\Phi, \omega) &= \sup_{\delta>0} ht(\Phi, \omega, \delta).\end{aligned}$$

Then the *topological entropy* of Φ is defined to be $ht(\Phi) := \sup_{\omega\in\mathcal{F}(A)} ht(\Phi,\omega)$ (see [4] and [1]).

Remark 2.1. For a cp map $\Phi : A \to A$ on a nuclear (or more generally exact) C^*-algebra the following are known to hold:

(a) If $\theta : A \to B$ is a C^*-isomorphism then $ht(\Phi) = ht(\theta\Phi\theta^{-1})$.
(b) If $A_0 \subset A$ is a Φ-invariant C^*-subalgebra of A, $ht(\Phi|_{A_0}) \le ht(\Phi)$.
(c) If $\{\omega_\lambda\}_{\lambda\in\Lambda}$ is a net (partially ordered by inclusion) of finite subsets in A such that the linear span of $\cup_{\lambda,l\in\mathbb{Z}^+}\Phi^l(\omega_\lambda)$ is dense in A then

$$ht(\Phi) = \sup_\lambda ht(\Phi,\omega_\lambda).$$

Topological entropy of a continuous map. Let $T : X \to X$ be a continuous map on a compact space X. If $\mathcal{U}$ is an open cover of X then so is $T^{-1}\mathcal{U}$. By $N(\mathcal{U})$ we denote the number of sets in a finite subcover of $\mathcal{U}$ with smallest cardinality. Then the *entropy of T relative* to $\mathcal{U}$ is given by

$$h_{top}(T,\mathcal{U}) := \lim_{n\to\infty}\frac{1}{n}\log(N(\vee_{i=0}^{n-1}T^{-i}\mathcal{U})),$$

where $\mathcal{U}\vee\mathcal{V}$ denotes the join of $\mathcal{U}$ and $\mathcal{V}$, and the *topological entropy* of (X,T) is defined to be

$$h_{top}(X,T) = \sup_{\mathcal{U}} h_{top}(T,\mathcal{U}),$$

where the supremum is taken over the set of all open covers (see [28]).

Shift spaces and graph groupoids. Let E be a row-finite graph. By (X_E,σ_E) we denote the one-sided shift space of infinite paths E^∞ with the shift map σ_E, $\sigma_E(x)_i = x_{i+1}$. Then X_E is a locally compact Hausdorff space in the product topology (see [14] or [21] for example) and the shift map σ_E is continuous. Moreover the compact open cylinder sets $\{Z(\alpha)\}_{\alpha\in E^*}$, $Z(\alpha) = \{\alpha x \mid x\in E^\infty,\ s(x) = r(\alpha)\}$, form a neighborhood base for the topology. Recall that if there is a homeomorphism $\rho : X_E \to X_F$ such that $\sigma_F\circ\rho = \rho\circ\sigma_E$, two shift spaces (X_E,σ_E) and (X_F,σ_F) are said to be *topologically conjugate*.

Let E be a row finite graph. It is known in [21] that the graph algebra $C^*(E)$ is isomorphic to the groupoid C^*-algebra $C^*(\mathcal{G}_E)$ [23], where

$$\mathcal{G}_E = \{(x,k,y) \mid x\sim_k y,\ x,y\in E^\infty,\ k\in\mathbb{Z}\}$$

is the groupoid with operations

$$(x,k,y)(y,l,z) = (x,k+l,z),\quad (x,k,y)^{-1} = (y,-k,x).$$

$x\sim_k y$ means that there is an $N\in\mathbb{N}$ such that $x_i = y_{i+k}$, for all $i\ge N$. $\mathcal{G}_E$ has the locally compact Hausdorff topology with the neighborhood base consisting of compact open sets $\{Z(\alpha,\beta) \mid \alpha,\beta\in E^*,\ r(\alpha) = r(\beta)\}$, where

$$Z(\alpha,\beta) = \big\{(x,k,y)\,\big|\, x\in Z(\alpha), y\in Z(\beta), k = |\beta|-|\alpha|,\ x_i = y_{i+k},\ \text{for}\, i>|\alpha|\big\}.$$

Furthermore $\mathcal{G}_E$ is r-discrete in this topology and its unit space $\mathcal{G}_E^0 = \{(x,0,x) \mid x\in E^\infty\}$ can be identified with the infinite path space X_E [21].

3. Topological conjugacy and flow equivalence

The following theorem extends a result [7] where E and F are finite graphs with no sinks. It can be proved by using the fact that $C^*(E)$ is isomorphic to $C^*(\mathcal{G}_E)$.

Theorem 3.1. ([16]) *Let E and F be two row finite infinite graphs with no sinks. If the shift spaces (X_E, σ_E) and (X_F, σ_F) are topologically conjugate then there exists a $*$-isomorphism $\theta : C^*(E) \to C^*(F)$. If in addition E and F are locally finite then $\theta \circ \phi_E|_{\mathcal{D}_E} = \phi_F|_{\mathcal{D}_F} \circ \theta$ holds.*

Let (Σ_A, σ_A) denote the two-sided shift space associated with a matrix A (hence σ_A is a homeomorphism). Recall that two shifts of finite type (Σ_A, σ_A) and (Σ_B, σ_B) are *flow equivalent* if there is a homeomorphism between their suspension spaces carrying flow lines onto flow lines and preserving orientation. Under a mild condition, the following is known (A and B are finite $\{0, 1\}$ matrices).

Theorem 3.2. ([7, 6]) *If (Σ_A, σ_A) and (Σ_B, σ_B) are flow equivalent, the Cuntz-Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_B$ are stably isomorphic.*

Let $K(H)$ be the C^*-algebra of compact operators acting on the separable infinite dimensional Hilbert space H. Two C^*-algebras A and B are said to be *stably isomorphic* if $A \otimes K(H) \cong B \otimes K(H)$. It is known that there is a unique norm for which the completion of the algebraic tensor product $A \bigodot K(H)$ becomes a C^*-algebra. If two C^*-algebras A and B are stably isomorphic, the lattices of closed two-sided ideals are isomorphic ([24]). In particular, if A is simple (A has no non-trivial closed two-sided ideals), so is B.

If a Cuntz-Krieger algebra $\mathcal{O}_A$ is simple, it is purely infinite or AF. In the 1970's Elliott proved a complete classification theorem of AF algebras by their ordered K_0-groups and the class of Kirchberg algebras (purely infinite, simple, separable and nuclear C^*-algebras) are classified by Kirchberg and Phillips in the early 1990's. For these theorems and other classifications for C^*-algebras, we refer the reader to [26].

In [3] Bates and Pask considered four graphical constructions which naturally arise in the study of flow equivalence for topological Markov chains. From an arbitrary graph E, one can construct a (proper) out-split graph E_{out-sp}, an out-delayed graph, an in-delayed graph and a (proper) in-split graph. It turns out that $C^*(E) \cong C^*(E_{out-sp})$ and the graph algebras of the rest three graphs are stably isomorphic to $C^*(E)$.

4. Topological entropy of the canonical CP map ϕ_E

E will be a locally finite irreducible infinite graph throughout this section. For ease of notation we will use the following notation with a fixed vertex $v \in E^0$.

$$E^n_r(v) = \{\alpha \in E^* \mid r(\alpha) = v,\ |\alpha| = n\},$$
$$E^n_s(v) = \{\alpha \in E^* \mid s(\alpha) = v,\ |\alpha| = n\},$$
$$E^n_{lp}(v) = \{\alpha \in E^* \mid s(\alpha) = r(\alpha) = v,\ |\alpha| = n\}.$$

Then *loop entropy* $h_l(E)$ and *block entropy* $h_b(E)$ of E are defined as follows;

$$h_l(E) = \limsup_{n\to\infty} \frac{1}{n} \log |E^n_{lp}(v)|,$$
$$h_b(E) = \limsup_{n\to\infty} \frac{1}{n} \log |E^n_s(v)|.$$

Since E is irreducible, these values are independent of the choice of the vertex v, and moreover $h_l(E) \le h_b(E)$ is immediate from $|E^n_{lp}(v)| \le |E^n_s(E)|$. If E^t is the transposed graph of E, clearly $h_l(E) = h_l(E^t)$, but $h_b(E) \ne h_b(E^t)$ in general.

The following has been known recently [14, 15, 16]: If E is a finite graph, any two of the entropies $ht(\phi_E)$, $ht(\phi_E|_{\mathcal{A}_E})$, $h_l(E)$, $h_b(E)$ and $h_b(E^t)$ coincide with each other.

Theorem 4.1. *Let E be a locally finite irreducible infinite graph. Then*

$$h_l(E) \le\ ht(\phi_E|_{\mathcal{A}_E})\ \le\ h_b(E^t).$$

The lower bound $h_l(E)$ is obtained from the fact that $ht(\phi_E|_{\mathcal{D}_E}) = h_l(E)$ and Remark 2.1(b). Applying the (non-trivial) fact that $ht(\phi_E|_{\mathcal{A}_E(v)}) \le h_b(E^t)$ for each vertex $v \in E^0$, one can show that $h_b(E^t)$ is an upper bound for the entropy $ht(\phi_E|_{\mathcal{A}_E})$, where $\mathcal{A}_E(v) := \overline{span}\{s_\alpha s_\beta^* \in \mathcal{A}_E \mid r(\alpha) = r(\beta) = v\}$ is a ϕ_E-invariant AF subalgebra of $\mathcal{A}_E$. On the other hand the commutative subalgebra $\mathcal{D}_E(v) := \overline{span}\{s_\alpha s_\alpha^* \mid \alpha \in E^*, r(\alpha) = v\}$ of $\mathcal{A}_E(v)$ is ϕ_E-invariant and $\mathcal{D}_E(v) \subset \mathcal{D}_E$. Thus from the monotonicity of entropy it follows that $ht(\phi|_{\mathcal{D}_E(v)}) \le h_l(E)$. Hence it is natural to ask if we really have the equality $ht(\phi|_{\mathcal{D}_E(v)}) = h_l(E)$, that is, $h_l(E)$ is also a lower bound for $ht(\phi_E|_{\mathcal{A}_E(v)})$. The rest of this section is devoted to this problem. For detailed proofs of the results stated in what follows refer the paper [16]. For this purpose we will seek a space Y such that $\mathcal{D}_E(v) \cong C_0(Y)$ (Y is the maximal ideal space of $\mathcal{D}_E(v)$) and the restriction map $\phi_E|_{\mathcal{D}_E(v)}$ can be viewed as the induced $*$-homomorphism of certain continuous map on the space Y (actually, a continuous map on the compactification of Y).

Note 4.2. Put

$$\begin{aligned}
E^*_r(v) &= \{\alpha \in E^* \mid r(\alpha) = v\},\\
E^*_{lp}(v) &= \{\alpha \in E^* \mid s(\alpha) = r(\alpha) = v,\ |\alpha| > 0\},\\
E^\infty_{lp}(v) &= \{\beta = \beta^1\beta^2\cdots \in E^\infty \mid \beta^j \in E^*_{lp}(v) \text{ for all } j\},\\
E^\infty_r(v) &= \{\alpha\beta \mid \alpha \in E^*_r(v),\ \alpha \ne v,\ \beta \in E^\infty_{lp}(v)\},\\
[\alpha] &= \{\alpha\} \cup \{\alpha\gamma \mid \gamma \in E^*_{lp}(v) \cup E^\infty_{lp}(v)\},\ \alpha \in E^*_r(v).
\end{aligned}$$

Note that if $\alpha \notin [\gamma]$ and $\gamma \notin [\alpha]$, the projections $s_\alpha s_\alpha^*$ and $s_\gamma s_\gamma^*$ are orthogonal to each other.

Maximal ideal space of $\mathcal{D}_E(v)$. The commutative C^*- algebra $\mathcal{D}_E(v) = \overline{\cup_{n\ge 0} D^n_E}$ is the inductive limit of the increasing sequence of finite dimensional C^*-algebras $D^n_E = \text{span}\{p_\alpha \mid \alpha \in E^*_r(v),\ |\alpha| \le n\}$. Since E is irreducible and infinite the projections $\{p_\alpha \mid r(\alpha) = v,\ |\alpha| \le n\}$ are linearly independent for each n. Hence a linear map $f : D_E^n \to \mathbb{C}$ is determined by $f(p_\alpha)$ at each projection p_α. Fix a path $\beta \in E_r^*(v) \cup E_r^\infty(v)$ and define a linear map $f_\beta : D_E^n(v) \to \mathbb{C}$ by

$$f_\beta(p_\alpha) = \begin{cases} 1, & [\alpha] \text{ contains } \beta, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

It can be shown that $f_\beta : D_E^n \to \mathbb{C}$ is a $*$-homomorphism for each n, and so it extends to the inductive limit $D_E(v)$. Thus each $\beta \in X_v := E_r^*(v) \cup E_r^\infty(v)$ give rise to a $*$-homomorphism $f_\beta : \mathcal{D}_E(v) \to \mathbb{C}$. Furthermore the maximal ideal space $\mathcal{M}(\mathcal{D}_E(v))$ of $\mathcal{D}_E(v)$ is identified with the set X_v via the map $\beta \mapsto f_\beta : X_v \to \mathcal{M}(\mathcal{D}_E(v))$, and the Gelfand transform $\rho : \mathcal{D}_E(v) \to C_0(X_v)$ satisfies

$$\rho(p_\alpha)(\beta) = f_\beta(p_\alpha),\ \alpha \in E_r^*(v),\ \beta \in X_v.$$

Since the linear span of the projections $\{p_\alpha \mid \alpha \in E_r^*(v)\}$ is dense in $\mathcal{D}_E(v)$, it follows that $f_{\beta_i} \to f_\beta \in \mathcal{D}_E(v)^*$ if and only if $f_{\beta_i}(p_\alpha) \to f_\beta(p_\alpha),\ \alpha \in E_r^*(v)$.

Now let $\overline{X}_v$ be the one point compactification of X_v. Then from the isomorphism $C(\overline{X}_v) \cong \widetilde{C_0(X_v)}$ we see that the multiplicative homomorphism of $C(\overline{X}_v)$ corresponding to the point in $\overline{X}_v \setminus X_v$ is the zero map when restricted to $C_0(X_v)$. Hence we may write $\overline{X}_v = X_v \cup \{0\}$. Also each function $g \in C_0(X_v)$ will be regarded as a function in $C(\overline{X}_v)$ with $g(0) = 0$.

Definition 4.3. Define a map $T : \overline{X}_v \to \overline{X}_v$ as follows:

$$\begin{aligned} &T(v) = 0,\ T(0) = 0, \\ &T(e) = v,\ e \in E_r^1(v), \\ &T(\beta) = \beta' \ \text{ if } \beta = e\beta',\ e \in E^1,\ \beta' \in X_v,\ |\beta| > 1. \end{aligned}$$

Proposition 4.4. *$T : \overline{X}_v \to \overline{X}_v$ is continuous. Let $T^* : C(\overline{X}_v) \to C(\overline{X}_v)$ be the $*$-homomorphism induced by T, that is, $T^*(f)(\beta) = f(T\beta)$, $f \in C(\overline{X}_v)$, $\beta \in \overline{X}_v$. Then for each $a \in \mathcal{D}_E(v)$,*

$$\rho(\phi_E(a)) = T^*(\rho(a)), \tag{2}$$

where $\rho : \mathcal{D}_E(v) \to C_0(X_v)$ is the Gelfand transform. In particular, $T^*(C_0(X_v)) \subset C_0(X_v)$ since ρ is a $*$-isomorphism.

Proposition 4.5. $ht(\phi_E|_{\mathcal{D}_E(v)}) = ht(T^*) = h_{top}(\overline{X}_v, T)$.

Proof. Since the Gelfand transform $\rho : \mathcal{D}_E(v) \to C_0(X_v)$ is a $*$-isomorphism satisfying (2), we know that $ht(\phi_E) = ht(T^*|_{C_0(X_v)})$. On the other hand, for a non-unital exact C^*-algebra A and a cp map $\psi : A \to A$ it is known [4] that $ht(\psi) = ht(\widetilde{\psi})$, where $\widetilde{\psi} : \widetilde{A} \to \widetilde{A}$ is the extension of ψ to the smallest unitization $\widetilde{A}$ of A. Therefore $ht(\phi_E|_{\mathcal{D}_E(v)}) = ht(T^*)$ follows since T^* on $C(\overline{X}_v)$ corresponds to $\widetilde{(T^*|_{C_0(X_v)})}$ on $\widetilde{C_0(X_v)}$ under the isomorphism $C(\overline{X}_v) \cong \widetilde{C_0(X_v)}$. See [8] for the fact that $ht(T^*) = h_{top}(\overline{X}_v, T)$. □

For each $n \in \mathbb{N}$, put $A_n = \cup_{\alpha \in \cup_{i=0}^n E_r^i(v)}[\alpha]$. Then A_n is a compact open subset of $\overline{X}_v$ and so $A_n^c := \overline{X}_v \setminus A_n$ is open in $\overline{X}_v$. Consider an open cover $\mathcal{U}_n$ of $\overline{X}_v$, where

$$\mathcal{U}_n := \{\, [\alpha] \mid \alpha \in \cup_{i=0}^n E_r^i(v)\} \cup \{A_n^c\}.$$

Then $\mathcal{U}_n \prec \mathcal{U}_{n+1}$. Each path $\alpha \in X_v$ can be decomposed uniquely as $\alpha = \alpha^{(0)}\alpha^{(1)}\cdots\alpha^{(k)}$, where $\alpha^{(0)}$ meets v only at its range (if $s(\alpha) = v$, then $\alpha^{(0)} = v$ and so we write $\alpha = \alpha^{(1)}\cdots\alpha^{(k)}$) and each $\alpha^{(j)}$ is a simple loop at v for $\forall j \geq 1$. If $[\alpha] \in \mathcal{U}_n$ then clearly $[\alpha^{(0)}] \in \mathcal{U}_n$ and $[\alpha] \subset [\alpha^{(0)}]$. Hence for the computation of $h_{top}(T, \mathcal{U}_n)$ we only need to consider open sets of the form $[\alpha]$ and A_n^c where α is either a simple loop at v or a path with $\alpha = \alpha^{(0)}$.

The following proposition is useful to calculate the entropy $h_{top}(\overline{X}_v, T)$. We call a loop $\beta \in E_{lp}^*(v)$ *simple* if β never meets v except at its source and range.

Proposition 4.6. *Let U_j, $V_j \in \mathcal{U}_n$ be open sets of the form A_n^c or $[\alpha]$ for a path α with $|\alpha| \leq n$ such that α is a simple loop at v or $\alpha = \alpha^{(0)}$. Put*

$$U := \cap_{j=0}^{m-1} T^{-j}(U_j), \quad V := \cap_{j=0}^{m-1} T^{-j}(V_j).$$

Then

$$U \cap V \neq \emptyset \iff U_j = V_j, \quad \forall j = 0, \ldots, m-1.$$

Theorem 4.7. *Let E be a locally finite irreducible infinite graph. Then*

$$ht(\phi_E|_{\mathcal{D}_E(v)}) = h_l(E).$$

Sketch of proof. For an $\varepsilon > 0$, find an open cover $\mathcal{U}_{N_\varepsilon}$ satisfying

$$h_{top}(T, \mathcal{U}_{N_\varepsilon}) \geq h_l(E) - \varepsilon,$$

which then implies that $h_{top}(\overline{X}_v, T) \geq h_l(E)$. The assertion follows from Proposition 4.5. □

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