RELATIVE ENTROPY FUNCTIONS FOR FACTOR MAPS BETWEEN SUBSHIFTS

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If (X,S) is a topological dynamical system, i.e., X is a compact metric space with a homeomorphism $S:X\to X$, then $M(X)$ is the set of all S-invariant Borel probability measures on X. For $\mu\in M(X)$, let $h(\mu)$ be the measure-theoretic entropy of S relative to μ. Denote by $C(X)$ the set of all real-valued continuous functions on X.

Let $\pi:(X,S)\to(Y,T)$ be a *factor map* between topological dynamical systems, i.e., a continuous surjection with $\pi\circ S=T\circ\pi$. For a given compact subset K of X, for $n\geq 1$ and $\delta>0$, denote by $\Delta_{n,\delta}(K)$ the set of (n,δ)-separated sets of X contained in K. Let $f\in C(X)$. Fix $\delta>0$ and $n\geq 1$. For each $y\in Y$, let

$$P_n(\pi,f,\delta)(y)=\sup\left\{\sum_{x\in E}\exp\sum_{i=0}^{n-1}f(S^ix)\,\middle|\,E\in\Delta_{n,\delta}\big(\pi^{-1}\{y\}\big)\right\}.$$

Define $P(\pi,f):Y\to\mathbb{R}$ by

$$P(\pi,f)(y)=\lim_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{n}\ln P_n(\pi,f,\delta)(y)$$

which we call the *relative pressure function* corresponding to f. In case $f\equiv 0$, it is called the *relative entropy function* for π.

We denote also by π the naturally induced (onto) map from $M(X)$ to $M(Y)$. For $\nu\in M(Y)$, let $M(\nu)$ denote the set of measures in $M(X)$ that project to ν. For $f\in C(X)$, the associated relative pressure function $P(\pi,f):Y\to\mathbb{R}$ satisfies the *relative variational principle* [2], that is, for each $\nu\in M(Y)$,

$$\int P(\pi,f)d\nu=\sup\left\{h(\mu)+\int fd\mu\,\middle|\,\mu\in M(\nu)\right\}-h(\nu).$$

Given $g\in C(Y)$, we have for $\nu\in M(Y)$,

$$\int P(\pi,g\circ\pi)d\nu=\sup_{\mu\in M(\nu)}h(\mu)-h(\nu)+\int gd\nu$$

and particularly, for $g \equiv 0$,

$$\int P(\pi,0)d\nu = \sup_{\mu\in M(\nu)} h(\mu) - h(\nu).$$

Relative pressure functions are connected with the notion of compensation functions. A continuous function $F \in C(X)$ is called a *compensation function* for π if the pressure functions satisfy

$$P_S(F + \phi\circ\pi) = P_T(\phi) \qquad \text{for all } \phi \in C(Y).$$

A compensation function of type $G\circ\pi \in C(X)$ with $G \in C(Y)$ is said to be *saturated*. There always exists a compensation function for factor maps between shifts of finite type [7]. There is, however, an example of a factor map between shifts of finite type for which no saturated compensation function exists [4]. In this work we study relative entropy functions for factor maps between subshifts, relating to saturated compensation functions. A subshift is accompanied by the shift map, denoted σ, to represent a topological dynamical system.

For $y \in Y$, let

$$\mathcal{T}(y) = -P(\pi,0)(y).$$

Theorem 1. [7] *Let X and Y be subshifts and let $\pi : X \to Y$ be a factor map. Let $g \in C(Y)$. Then $g\circ\pi \in C(X)$ is a compensation function for π if and only if for all $\nu \in M(Y)$,*

$$\int g d\nu = \int \mathcal{T} d\nu.$$

Using this and ergodic decomposition, one can prove the following.

Proposition 2. [5] *Let X and Y be subshifts and let $\pi : X \to Y$ be a factor map. Let $g \in C(Y)$. Then $\int g d\nu = \int \mathcal{T} d\nu$ for all $\nu \in M(Y)$ if and only if $\int g d\nu = \int \mathcal{T} d\nu$ for all ergodic $\nu \in M(Y)$.*

Hereinafter, let X and Y denote one-step shifts of finite type and let $\pi : X \to Y$ be a factor code that is represented by a one-block map. For a block $b_1\cdots b_n$ of X, $n \geq 1$, let $[b_1\cdots b_n]$ be a cylinder set in X with b_1 on the 0-th coordinate. Fix $y \in Y$. For each $n \geq 1$, let $D_n(y)$ consist of one point from each nonempty set $\pi^{-1}(y)\cap[x_0x_1\cdots x_{n-1}]$. It is known [7] that

$$\mathcal{T}(y) = -\limsup_{n\to\infty} \frac{1}{n} \ln |D_n(y)|.$$

A point $y \in Y$ is called *generic* if the limit

$$\mu_y = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{\sigma^i y}$$

exists, in which case $\mu_y \in M(Y)$. Denote by Δ_Y the set of all generic points in Y. For each $k \geq 1$, let $P_k(Y) = \{y \in Y | \sigma^k y = y\}$. Then $\mathcal{P}(Y) = \bigcup_{k\geq 1} P_k(Y) \subset \Delta_Y$. For the proof of the following result, we refer to [5].

Proposition 3. *Let $y \in \Delta_Y$ and $y^{(s)} \in P_{l_s}(Y)$ with $l_s \geq s$ for $s \geq 1$. Suppose there is $N \geq 1$ such that $y^{(s)}_{[0,l_s-N]} = y_{[0,l_s-N]}$ for all $s \geq 1$. Then $\mathcal{T}(y) \leq \liminf_{s\to\infty} \mathcal{T}(y^{(s)})$ and $\mu_{y^{(s)}} \to \mu_y$ as $s \to \infty$. If there is a saturated compensation function, then $\mathcal{T}(y^{(s)}) \to \mathcal{T}(y)$ as $s \to \infty$.*

Example. Let X and Y be the shifts of finite type determined by allowing the transitions marked on Figure 1 and the one-block code $\pi : X \to Y$ map 1 to 1, and $2, 3, 4, 5$ to 2.

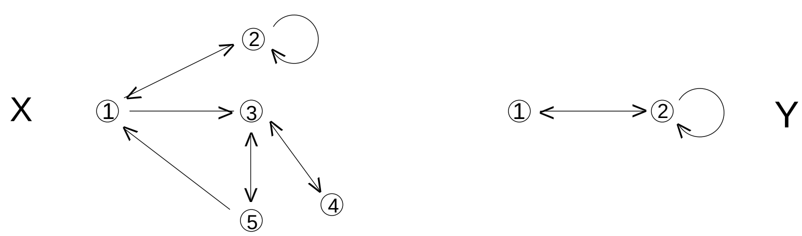

FIGURE 1

Let $y = \cdots 22.122\cdots \in Y$, i.e., $y_i = 1$ if $i = 0$ and $y_i = 2$ if $i \neq 0$. For each $s \geq 1$, let

$$y^{(s)} = \cdots 12^s.12^s12^s1\cdots \in P_{s+1}(Y).$$

Then $y \in \Delta_Y$ and $y^{(s)}_{[0,s]} = y_{[0,s]}$ for each $s \geq 1$. If s is odd, then $|D_n(y^{(s)})| = 1$ for all $n \geq 1$, so $\mathcal{T}(y^{(s)}) = 0$. Fix $s = 2m+2$, $m \geq 0$. Then $|D_s(y^{(s)})| = 2^m + 1$ and hence

$$\begin{aligned}\mathcal{T}(y^{(s)}) &= -\lim_{p\to\infty} \frac{1}{p(s+1)} \ln |D_{p(s+1)}(y^{(s)})| \\ &= \lim_{p\to\infty} \frac{-1}{p(s+1)} \ln(2^m+1)^p = \frac{-1}{s+1} \ln(2^{s/2-1}+1).\end{aligned}$$

Thus $\mathcal{T}(y^{(2m)}) \to -\ln\sqrt{2}$ as $m \to \infty$, so that $\mathcal{T}(y^{(s)})$ does not converge as $s \to \infty$. By Proposition 3 no compensation function is saturated.

The set

$$Y_0 = \Big\{ y \in Y \Big| \mathcal{T}(y) = - \lim_{n\to\infty} \frac{1}{n} \ln \big| D_n(y) \big| \Big\}.$$

is a total probability set, i.e., $\nu(Y_0) = 1$ for all $\nu \in M(Y)$ [7].

Proposition 4. *For $y \in \Delta_Y \cap Y_0$, there exist $y^{(s)} \in \mathcal{P}(Y)$, $s \geq 1$, such that $\mu_{y^{(s)}} \to \mu_y$ as $s \to \infty$ and $\limsup_{s\to\infty} \mathcal{T}(y^{(s)}) \leq \mathcal{T}(y)$.*

See [5] for the proof.

Theorem 5. *Let X and Y be irreducible shifts of finite type and let $\pi : X \to Y$ be a factor map. Let $g \in C(Y)$. Then $g \circ \pi \in C(X)$ is a compensation function if and only if $\int g d\mu_y = \mathcal{T}(y)$ for all $y \in \mathcal{P}(Y)$.*

Proof. Suppose $\int g d\mu_y = \mathcal{T}(y)$ for all $y \in \mathcal{P}(Y)$. If $\nu \in M(Y)$ is ergodic, then the set of all points $y \in \Delta_Y$ with $\mu_y = \nu$ is of full measure with respect to ν (see [1] [6]). So there is $y \in \Delta_Y \cap Y_0$ such that $\nu = \mu_y$ and $\int \mathcal{T} d\nu = \mathcal{T}(y)$. By Proposition 4 there exist $y^{(s)} \in \mathcal{P}(Y)$, $s \geq 1$, such that $\mu_{y^{(s)}} \to \mu_y$ as $s \to \infty$ and $\limsup_{s\to\infty} \mathcal{T}(y^{(s)}) \leq \mathcal{T}(y)$. Meanwhile, it follows from Proposition 3 that $\mathcal{T}(y) \leq \liminf_{s\to\infty} \mathcal{T}(y^{(s)})$. Thus $\lim_{s\to\infty} \mathcal{T}(y^{(s)}) = \mathcal{T}(y)$. Since $\int g d\mu_{y^{(s)}} = \mathcal{T}(y^{(s)})$ for all $s \geq 1$ and $\int g d\mu_{y^{(s)}} \to \int g d\nu$ as $s \to \infty$, it follows that $\int g d\nu = \int \mathcal{T} d\nu$. From Theorem 1 and Proposition 2 we conclude that $g \circ \pi \in C(X)$ is a compensation function. $\square$

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