RELATIVE ENTROPY FUNCTIONS
FOR FACTOR MAPS BETWEEN SUBSHIFTS

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If \((X, S)\) is a topological dynamical system, i.e., \(X\) is a compact metric space with a homeomorphism \(S : X \to X\), then \(M(X)\) is the set of all \(S\)-invariant Borel probability measures on \(X\). For \(\mu \in M(X)\), let \(h(\mu)\) be the measure-theoretic entropy of \(S\) relative to \(\mu\). Denote by \(C(X)\) the set of all real-valued continuous functions on \(X\).

Let \(\pi : (X, S) \to (Y, T)\) be a factor map between topological dynamical systems, i.e., a continuous surjection with \(\pi \circ S = T \circ \pi\). For a given compact subset \(K\) of \(X\), for \(n \geq 1\) and \(\delta > 0\), denote by \(\Delta_{n,\delta}(K)\) the set of \((n, \delta)\)-separated sets of \(X\) contained in \(K\). Let \(f \in C(X)\). Fix \(\delta > 0\) and \(n \geq 1\). For each \(y \in Y\), let

\[
P_n(\pi, f, \delta)(y) = \sup \left\{ \sum_{x \in E} \exp \left(\sum_{i=0}^{n-1} f(S^i x) \right) \middle| E \in \Delta_{n,\delta}(\pi^{-1}\{y\}) \right\}.
\]

Define \(P(\pi, f) : Y \to \mathbb{R}\) by

\[
P(\pi, f)(y) = \lim_{\delta \to 0} \lim_{n \to \infty} \frac{1}{n} \ln P_n(\pi, f, \delta)(y)
\]

which we call the relative pressure function corresponding to \(f\). In case \(f \equiv 0\), it is called the relative entropy function for \(\pi\).

We denote also by \(\pi\) the naturally induced (onto) map from \(M(X)\) to \(M(Y)\). For \(\nu \in M(Y)\), let \(M(\nu)\) denote the set of measures in \(M(X)\) that project to \(\nu\). For \(f \in C(X)\), the associated relative pressure function \(P(\pi, f) : Y \to \mathbb{R}\) satisfies the relative variational principle [2], that is, for each \(\nu \in M(Y)\),

\[
\int P(\pi, f)d\nu = \sup \left\{ h(\mu) + \int hf \middle| \mu \in M(\nu) \right\} - h(\nu).
\]

Given \(g \in C(Y)\), we have for \(\nu \in M(Y)\),

\[
\int P(\pi, g \circ \pi)d\nu = \sup_{\mu \in M(\nu)} h(\mu) - h(\nu) + \int gd\nu
\]
and particularly, for $g \equiv 0$,

$$
\int P(\pi, 0) d\nu = \sup_{\mu \in M(\nu)} h(\mu) - h(\nu).
$$

Relative pressure functions are connected with the notion of compensation functions. A continuous function $F \in C(X)$ is called a compensation function for $\pi$ if the pressure functions satisfy

$$
P_S(F + \phi \circ \pi) = P_T(\phi) \quad \text{for all } \phi \in C(Y).
$$

A compensation function of type $G \circ \pi \in C(X)$ with $G \in C(Y)$ is said to be saturated. There always exists a compensation function for factor maps between shifts of finite type [7]. There is, however, an example of a factor map between shifts of finite type for which no saturated compensation function exists [4]. In this work we study relative entropy functions for factor maps between subshifts, relating to saturated compensation functions. A subshift is accompanied by the shift map, denoted $\sigma$, to represent a topological dynamical system.

For $y \in Y$, let

$$
T(y) = -P(\pi, 0)(y).
$$

**Theorem 1.** [7] Let $X$ and $Y$ be subshifts and let $\pi : X \to Y$ be a factor map. Let $g \in C(Y)$. Then $g \circ \pi \in C(X)$ is a compensation function for $\pi$ if and only if for all $\nu \in M(Y)$,

$$
\int g d\nu = \int T d\nu.
$$

Using this and ergodic decomposition, one can prove the following.

**Proposition 2.** [5] Let $X$ and $Y$ be subshifts and let $\pi : X \to Y$ be a factor map. Let $g \in C(Y)$. Then $\int g d\nu = \int T d\nu$ for all $\nu \in M(Y)$ if and only if $\int g d\nu = \int T d\nu$ for all ergodic $\nu \in M(Y)$.

Hereinafter, let $X$ and $Y$ denote one-step shifts of finite type and let $\pi : X \to Y$ be a factor code that is represented by a one-block map. For a block $b_1 \cdots b_n$ of $X$, $n \geq 1$, let $[b_1 \cdots b_n]$ be a cylinder set in $X$ with $b_1$ on the 0-th coordinate. Fix $y \in Y$. For each $n \geq 1$, let $D_n(y)$ consist of one point from each nonempty set $\pi^{-1}(y) \cap [x_0x_1 \cdots x_{n-1}]$. It is known [7] that

$$
T(y) = -\lim_{n \to \infty} \frac{1}{n} \ln |D_n(y)|.
$$
A point \( y \in Y \) is called \textit{generic} if the limit

\[
\mu_y = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{\sigma^i y}
\]

exists, in which case \( \mu_y \in M(Y) \). Denote by \( \Delta_Y \) the set of all generic points in \( Y \). For each \( k \geq 1 \), let \( P_k(Y) = \{ y \in Y | \sigma^k y = y \} \). Then \( P(Y) = \bigcup_{k \geq 1} P_k(Y) \subset \Delta_Y \). For the proof of the following result, we refer to [5].

**Proposition 3.** Let \( y \in \Delta_Y \) and \( y^{(s)} \in P_{l_s}(Y) \) with \( l_s \geq s \) for \( s \geq 1 \). Suppose there is \( N \geq 1 \) such that \( y^{(s)}_{[0,l_s-N]} = y_{[0,l_s-N]} \) for all \( s \geq 1 \). Then \( T(y) \leq \liminf_{s \to \infty} T(y^{(s)}) \) and \( \mu_{y^{(s)}} \to \mu_y \) as \( s \to \infty \). If there is a saturated compensation function, then \( T(y^{(s)}) \to T(y) \) as \( s \to \infty \).

**Example.** Let \( X \) and \( Y \) be the shifts of finite type determined by allowing the transitions marked on Figure 1 and the one-block code \( \pi : X \to Y \) map 1 to 1, and 2, 3, 4, 5 to 2.

Let \( y = \cdots 22.122 \cdots \in Y \), i.e., \( y_i = 1 \) if \( i = 0 \) and \( y_i = 2 \) if \( i \neq 0 \). For each \( s \geq 1 \), let

\[
y^{(s)} = \cdots 12^s.12^s 12^s 1 \cdots \in P_{s+1}(Y).
\]

Then \( y \in \Delta_Y \) and \( y^{(s)}_{[0,s]} = y_{[0,s]} \) for each \( s \geq 1 \). If \( s \) is odd, then \( |D_n(y^{(s)})| = 1 \) for all \( n \geq 1 \), so \( T(y^{(s)}) = 0 \). Fix \( s = 2m + 2 \), \( m \geq 0 \). Then \( |D_s(y^{(s)})| = 2^m + 1 \) and hence

\[
T(y^{(s)}) = -\lim_{p \to \infty} \frac{1}{p(s+1)} \ln |D_{p(s+1)}(y^{(s)})|
\]

\[
= \lim_{p \to \infty} \frac{-1}{p(s+1)} \ln(2^m + 1)^p = \frac{-1}{s+1} \ln(2^{s/2-1} + 1).
\]

Thus \( T(y^{(2m)}) \to -\ln \sqrt{2} \) as \( m \to \infty \), so that \( T(y^{(s)}) \) does not converge as \( s \to \infty \). By Proposition 3 no compensation function is saturated.
The set
\[ Y_0 = \left\{ y \in Y \mid T(y) = -\lim_{n \to \infty} \frac{1}{n} \ln |D_n(y)| \right\}. \]
is a total probability set, i.e., \( \nu(Y_0) = 1 \) for all \( \nu \in M(Y) \) [7].

**Proposition 4.** For \( y \in \Delta_Y \cap Y_0 \), there exist \( y^{(s)} \in \mathcal{P}(Y) \), \( s \geq 1 \), such that \( \mu_{y^{(s)}} \to \mu_y \) as \( s \to \infty \) and \( \limsup_{s \to \infty} T(y^{(s)}) \leq T(y) \).


**Theorem 5.** Let \( X \) and \( Y \) be irreducible shifts of finite type and let \( \pi : X \to Y \) be a factor map. Let \( g \in C(Y) \). Then \( g \circ \pi \in C(X) \) is a compensation function if and only if \( \int gd\mu_y = T(y) \) for all \( y \in \mathcal{P}(Y) \).

**Proof.** Suppose \( \int gd\mu_y = T(y) \) for all \( y \in \mathcal{P}(Y) \). If \( \nu \in M(Y) \) is ergodic, then the set of all points \( y \in \Delta_Y \) with \( \mu_y = \nu \) is of full measure with respect to \( \nu \) (see [1] [6]). So there is \( y \in \Delta_Y \cap Y_0 \) such that \( \nu = \mu_y \) and \( \int T d\nu = T(y) \). By Proposition 4 there exist \( y^{(s)} \in \mathcal{P}(Y) \), \( s \geq 1 \), such that \( \mu_{y^{(s)}} \to \mu_y \) as \( s \to \infty \) and \( \limsup_{s \to \infty} T(y^{(s)}) \leq T(y) \). Meanwhile, it follows from Proposition 3 that \( T(y) \leq \liminf_{s \to \infty} T(y^{(s)}) \). Thus \( \lim_{s \to \infty} T(y^{(s)}) = T(y) \). Since \( \int gd\mu_{y^{(s)}} = T(y^{(s)}) \) for all \( s \geq 1 \) and \( \int gd\mu_y \to \int gd\nu \) as \( s \to \infty \), it follows that \( \int gd\nu = \int T d\nu \). From Theorem 1 and Proposition 2 we conclude that \( g \circ \pi \in C(X) \) is a compensation function. \( \square \)

**References**

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