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A SURVEY ON ENTROPY AND SEQUENCE ENTROPY PAIRS

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1. Introduction

Properties of measurable and topological dynamics often have been studied together[11, 12, 18]. It is known that there are strong similarities and also sharp differences between them. Ergodicity and strongly mixing property in ergodic theory correspond respectively to transitivity and strongly mixing property in topological dynamical systems(TDS). It is well known that a K-mixing system in measurable dynamics is strongly mixing of all orders. And many of the K-properties are well understood for $\mathbb{Z}$-actions and $\mathbb{Z}^2$-actions[7, 8]. It has been an open question if in topological setting, there exist some topological property of entropy which implies topological mixing, moreover topological mixing of all orders.

The property K-mixing is equivalent to completely positive entropy(CPE): every non-trivial factor of a system has positive entropy. There are examples of topological CPE, but without transitivity. This is unlike measurable dynamical systems(MDS) where CPE implies ergodicity.

The notion of entropy pairs was introduced by F. Blanchard in a motivation to study topological analogue of K-mixing property[2]. A pair $(x, y) \in X \times X, x \neq y$ is called a topological entropy pair if any open cover $\{U, V\}$ with $x \in U^c$ and $y \in V^c$ has positive topological entropy. He introduced the notion of uniformly positive entropy(UPE) of $\mathbb{Z}$-actions : every pair $(x, y) \in X \times X,\ x \neq y$ is an entropy pair. By the definition of entropy pairs it is clear that UPE implies CPE. Blanchard showed that UPE implies weakly mixing[1]. However he has constructed an interesting example which has UPE but not strongly mixing. There are also weakly mixing flows with CPE, but without UPE[1].

We investigate the parallel properties for $\mathbb{Z}^2$-actions. We show that UPE also implies weakly mixing, but UPE does not necessarily imply strongly mixing. That is, there is not much relation between UPE and mixing properties for $\mathbb{Z}^2$-actions.

Metric entropy pairs for a $\mathbb{Z}$-action (X, T, μ) was defined in [4]: $(x, y') \in X \times X, x \neq y$ is a μ-entropy pair if any measurable partition $\{Q, Q^c\}$ such that $x \in \mathrm{int}(Q)$ and $y \in \mathrm{int}(Q^c)$ has positive entropy. It is shown in [4] that every μ-entropy pair is always topological entropy pair and the converse is true if (X, T) is uniquely ergodic. Moreover it is shown that a topological entropy pair is a μ-entropy pair for some invariant measure μ.

The proof demands the study of the relation between entropy of covers and entropy of partitions. Also entropy pairs for a measure allowed to construct the maximal topological factor of zero measure theoretic entropy. Characterizing the set of entropy pairs for an invariant measure as the support of some measure, Glasner has shown that the product of two UPE systems is UPE[9, 10].

Recently concepts of topological and measure theoretic entropy pairs have been generalized in several directions. In order to study systems of zero entropy, sequence entropy pairs defined over a subsequence of $\mathbb{Z}$ has been introduced[14]. Also entropy n-tuples on $X \times \cdots \times X (n$ times) have been investigated in relation with multiple mixing property[13]. We find relative entropy pairs are useful to investigate the dynamics of X relative to a given factor[24] and study the relative topological Pinsker factor of X.

Weakly mixing pair was defined and it turns out that a system is weakly mixing if and only if any pair not in the diagonal is a sequence entropy pair if and only if any pair not in the diagonal is a weakly mixing pair. And a weakly mixing system has a maximal null factor. It is known fact that a weakly mixing system is disjoint from any null minimal system[14]. The topological concepts of maximal null factor and maximal equicontinuous factor are related with the Kronecker factor in ergodic theory. In [13] they explore topological factors in between the Kronecker and the maximal equicontinuous factor and the maximal null factor. Sequence entropy n-tuples was defined by Huang et al and they proved sequence entropy n-tuples for a measure is contained in the set of topological sequence entropy n-tuples but the reciprocal is not true[13].

2. Basic notions

We consider a dynamical system (X, T) where X is a compact metric space and T is a homeomorphism of X. We denote a dynamical system with measurable structure by $(X, \mathcal{B}, \mu, T)$ where μ is preserved, that is,

$$\mu(T^{-n}A) = \mu(A) \text{ for all } n \in \mathbb{Z}.$$

2.1 *Ergodicity, weakly mixing, strongly mixing.* A MDS $(X, T, \mathcal{B}, \mu)$ is ergodic if for any measurable sets A and B with $\mu(A) > 0$ and $\mu(B) > 0$ there exists $n \in \mathbb{N}$ such that $\mu(T^{-n}A \cap B) > 0$. It is weakly mixing if for any measurable sets A and B

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |\mu(\Phi^{-k}A \cap B) - \mu(A)\mu(B)| = 0.$$

It is strongly mixing if for any measurable sets A and B

$$\lim_{n\to\infty} \mu(T^{-n}A \cap B) = \mu(A)\mu(B).$$

Note that if a MDS $(X, T, \mathcal{B}, \mu)$ is strongly mixing(weakly mixing) then it is weakly mixing(ergodic).

The next notions are defined in a TDS and correspond to the above definitions in a MDS.

2.2 *Transitivity, weakly mixing, strongly mixing.* A TDS (X, T) is transitive (ergodicity in a MDS) if for any two nonempty open sets $U, V \subset X$, there is $n \in \mathbb{N}$ such that $T^{-n}U \cap V \neq \emptyset$. It is weakly mixing if the cartesian product $(X \times X, T \times T)$ is transitive. It is strongly mixing if for any two nonempty open sets $U, V \subset X$, there is $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$, $T^{-n}U \cap V \neq \emptyset$. Similarly, strongly mixing(weakly mixing) implies weakly mixing(transitivity) in a TDS.

2.3 *Kolmogorov-mixing.* An invertible measure preserving transformation T of a probability space $(X, T, \mathcal{B}, \mu)$ is Komogorov mixing(K-automorphism or K-mixing) if there exists a sub σ-algebra $\mathcal{K}$ of $\mathcal{B}$ such that:

(1) $\mathcal{K} \subset T\mathcal{K}$
(2) $\bigvee_{n \in \mathbb{N}^+} T^n \mathcal{K} \doteq \mathcal{B}$, where $\mathbb{N}^+ = \mathbb{N} \cup 0$.
(3) $\bigcap_{n \in \mathbb{Z} \setminus \mathbb{N}} T^{-n} \mathcal{K} \doteq \{X, \emptyset\}$.

Theorem 2.1. *The following conditions are equivalent.*

(1) T is a Komogorov automorphism.
(2) It has the property of completely positive entropy(CPE): $h_\mu(T, \mathcal{P}) > 0$ for all finite partition $\mathcal{P} \neq \{X, \emptyset\}$, that is the Pinsker factor is trivial, $\{\emptyset, X\}$.
(3) T is disjoint from any zero entropy system (Y, S), that is, any ergodic invariant measure under $T \times S$ is the product measure on $X \times Y$.
(4) T is uniform stronly mixing.

These results are extended to amenable group actions by D. Roudolph and B. Weiss[25]. Their method is surprising in the sense that the properties are related with entropy or proven via orbit equivalence. It has been an open question if in a TDS, there exist some topological property of entropy which implies topological mixing, moreover topological mixing of all orders. The next definitions are defined to investigate a topological property simliar to a K-automorphism in a TDS. The following definitions and notations were introduced by F. Blanchard[1].

2.4 *Uniformly positive entropy, completely positive entropy.* A TDS (X, T) has uniformly positive entropy(UPE) if for every two nondense open cover $\{U, V\}$ of X, $h_{top}(\{U, V\})$ is positive. Let (X, T) and (Y, S) be two topological dynamical systems. A factor map ϕ is continuous, onto map and commutative, that is, $\phi \circ T = S \circ \phi$. The system (Y, S) is called a factor of (X, T). It has completely positive entropy(CPE) if every non trivial factor of (X, T) has positive entropy.

Note that UPE implies that CPE by definitions but the converse is not true. Blanchard has shown that UPE implies weakly mixing[1] and is disjoint from all minimal zero entropy systems[2]. He constructed the example that is UPE without strongly mixing. And CPE does not implies transitivity.

3. Entropy pairs

3.1 *Topological entropy pairs and Metric entropy pairs.* Let (X, T) be a topological dynamical system(TDS) and $M(X, T)$ be the set of T invariant measures. Denote by $E(X, T)$ the set of topological entropy pairs of (X, T). A TDS (X, T) has UPE if and only if

$$E(X, T) = X \times X \setminus \triangle_X.$$

Let $\mu \in M(X, T)$. The set of μ-entropy pairs of (X, T) is denoted by $E_\mu(X, T)$.

A dynamical system with positive entropy has an entropy pair in a TDS and $E(X, T) = \emptyset$ if and only if $h_{top}(T) = 0$. The set of entropy pairs is a nonempty finite set. These are also true in a MDS. Let (X, T) and (Y, S) be $\mathbb{Z}$-actions. Let $\phi : (X, T) \to (Y, S)$ be a factor map. If $(x, x') \in E(X, T)$ and $y = T(x) \neq T(x') = y'$, then (y, y') is an entropy pair of (Y, S). Conversely if (y, y') belongs to $E(Y, S)$, there exists a pair (x, x') in $X \times X$ such that

$$T(x) = y, T(x') = y' \text{ and } (x, x') \in E(X, T).$$

Like a topological case, the above also holds for a MDS. And topological entropy pairs are used to construct the maximal zero topological entropy factor, which corresponds to the notion of the Pinsker factor. The maximal topological zero entropy factor is defined using entropy pairs, that is the topological Pinsker factor of X is the quotient flow $X/\langle E(X, T)\rangle$, where $\langle E(X, T)\rangle$ denotes the smallest T-invariant closed equivalence relation containing $E(X, T)$. If (X, T) has UPE and $h(Y, S) = 0$ and the union of the minimal subsets of Y is dense in Y. Then (Y, S) is a maximal zero entropy factor of $(X \times Y, T \times S)$.

Theorem 3.1 (The variational principle for entropy pairs[3])**.** *Let (X, T) be a topological dynamical system. There exists a measure $\mu \in M(X, T)$ such that*

$$E_\mu(X, T) = E(X, T)$$

Topological and metric entropy pairs of $\mathbb{Z}^2$-actions have been defined and their properties have been investigated analogous to $\mathbb{Z}$-actions[17]. We extend the results based on the works [3] and [4]. We prove the variational principle for the entropy pairs for $\mathbb{Z}^2$-actions. And we show that there exists a measure $\mu \in M(X, \Phi)$ such that $E_\mu(X, \Phi) = E(X, \Phi)$. We mention that most of our arguments work for $\mathbb{Z}^n$ for any $n \geq 2$.

3.2 *Directional entropy pairs.* The notion of the directional entropy was introduced by Milnor in 1988[19] to study the cellular automaton map together with the Bernoulli shift. The directional entropy(topological and metric) was used by Boyle and Lind[6] to study expansive $\mathbb{Z}^2$-actions. This concept was extended to the class of arbitrary $\mathbb{Z}^2$-actions on a Lebesque space[21]. Many of its properties are further studied in [15, 21, 22]. Directional systems can be regarded as non cocompact subgroup actions and hence the directional entropy as an isomorphism invariant is a useful tool to investigate 0 entropy $\mathbb{Z}^2$-actions.

We define entropy pairs for directional systems for a TDS[17]. For a given direction $\vec{v}$ a pair $(x, x') \in X \times X$ is called a $\vec{v}$-entropy pair if every nondense cover $\mathcal{U} = (U, V)$

with $x \in int(U^c)$ and $x' \in int(V^c)$ has positive(including ∞) entropy for this direction $\vec{v}$. We show that the directional entropy pairs relate weakly mixing property of a $\mathbb{Z}^2$-action. We consider several examples to study the behaviors of directional entropy pairs. For instance Ledrappier's example has zero entropy of a $\mathbb{Z}^2$-action, hence $E(X, \Phi) = \emptyset$ but it has UPE for all directions $(X, \Phi^{(i,j)})$ as a $\mathbb{Z}$-action. The second example is as follows[5]. Let (X, T) have UPE and Y be the compact space $\mathbb{Z} \cup \{\infty\}$ with S the translation by 1 on Y. Note that (Y, S) has zero entropy. Let $\Phi : X \times Y \to X \times Y$ be defined by

$$\Phi^{(i,j)}(x, n) = (T^i x, n + j).$$

Then the set of directional entropy pairs is,

$$E(X \times Y, \Phi^{\vec{v}}) \cup \triangle_{X \times Y} = \begin{cases} \emptyset & \text{if } \vec{v} = (0,1), \\ \{((x,y),(x',y)) : x, x' \in X,\ y \in Y\} & \text{if } \vec{v} = (1,0), \\ (X \times \{\infty\}) \times (X \times \{\infty\}) & \text{otherwise} \end{cases}$$

The third example shows the different behavior[6]. Let X have UPE and (Y, S) have entropy zero. But the behaviors of directional entropy pair are quite different from previous one. Let $X = \{0,1\}^{\mathbb{Z}}$ and $Y = [0, 1)$. And T is the shift map in X and $S(x) = \alpha x$ is an irrational rotation by α in Y. We define $\Phi : X \times Y \to X \times Y$ by

$$\Phi^{(i,j)}(x, y) = T^i S^j(x, y) = (T^i x, \alpha^j y).$$

For every direction (i, j) with $i \neq 0$ the the directional entropy pairs are

$$E(X \times Y, \Phi^{(i,j)}) \cup \triangle_{X \times Y} = \{((x, y), (x', y)) : x, x' \in X, y \in Y\}.$$

In fact, the directional entropy can be regarded as a special type of sequence entropy.

4. Sequence entropy pairs

Sequence entropy pairs were introduced by Huang, Shao and Ye[14] to localize the notion sequence entropy. For a given sequence $\mathcal{A} = \{t_0, t_1, \ldots,\}$ of $\mathbb{Z}_+$ the sequence entropy of T with respect to $\mathcal{A}$ and a finite open cover $\mathcal{U}$ is

$$h_{\mathcal{A}}(T, \mathcal{U}) = \limsup_{n \to \infty} \frac{1}{n} \log \mathcal{N}(\bigvee_{i=0}^{n-1} T^{-t_i} \mathcal{U}),$$

where $\mathcal{N}(\bigvee_{i=0}^{n-1} T^{-t_i} \mathcal{U})$ is the minimal cardinality among all cardinalities of sub-covers of $\bigvee_{i=0}^{n-1} T^{-t_i} \mathcal{U}$. The topological sequence entropy of (X, T) with respect to $\mathcal{A}$ is

$$h_{\mathcal{A}}(T) = \sup_{\mathcal{U}} h_{\mathcal{A}}(T, \mathcal{U}).$$

Analogously, given $\mu \in M(X, T)$ and $\mathcal{P}$ a finite measurable partition of X we define the sequence entropy of $\mathcal{P}$ with respect to (X, T, μ) along $\mathcal{A}$ by

$$h^{\mathcal{A}}_{\mu}(T, \mathcal{P}) = \limsup_{n \to \infty} -\frac{1}{n} \sum_{A \in \bigvee_{i=0}^{n-1} T^{-t_i} \mathcal{P}} \mu(A) \log \mu(A).$$

The sequence entropy of (X, T, μ) along $\mathcal{A}$ is

$$h_\mu^{\mathcal{A}}(T) = \sup_{\mathcal{P}} h_\mu^{\mathcal{A}}(T, \mathcal{P}),$$

where supremum is taken over all finite measurable partitions.

We note that the definition of the sequence entropy pairs can be directly extended to a $\mathbb{Z}^2$-action by taking $\mathcal{A} = \{t_0, t_1, ...\} \subset \mathbb{Z}^2$ where $|t_i| \to \infty$.

4.1 *Sequence entropy pairs* [14], *sequence entropy n-tuples* [13]. Let (X, T) be a dynamical system, $X^{(2)} = X \times X$ and $\triangle_X = \{(x, x) : x \in X\}$. We say that $(x_1, x_2) \in X^{(2)} \setminus \triangle_X$ is a sequence entropy pair if whenever U_i are closed mutually disjoint neighborhoods of points $x_i, i = 1, 2$, there exists a sequence $\mathcal{A} \subset \mathbb{Z}_+$ such that $h_{\mathcal{A}}(T, (U_1^c, U_2^c)) > 0$. We say that (X, T) has sequence uniform positive entropy(for short SUPE), if every point $(x_1, x_2) \in X^{(2)} \setminus \triangle_X$ is a sequence entropy pair. Note that (X, T) has SUPE if and only if for any cover $\mathcal{R} = \{U, V\}$ of X by two non-dense open sets, one has $h_{\mathcal{A}}(T, \mathcal{R}) > 0$ for some sequence $A \subset \mathbb{Z}_+$.

Let $X^{(n)} = X \times \cdots \times X$(n times). We say an n-tuple $(x_1, \cdots, x_n) \in X^{(n)}$ is a sequence entropy n-tuple if at least two of the points $\{x_i\}_1^n$ are different and if whenever U_j are closed mutually disjoint neighborhoods of distinct points x_j, the open cover $\mathcal{U} = \{U_j^c : 1 \leq j \leq n\}$ satisfies the condition that $h_{\mathcal{A}}(T, \mathcal{U}) > 0$ for some sequence $\mathcal{A} \subset \mathbb{Z}_+$.

4.2 *Weakly mixing pairs* [14]. For a TDS (X, T), $(x_1, x_2) \in X \times X \setminus \triangle$ is a weakly mixing pair if for every open neighborhood U_i of $x_i, i = 1, 2$ there is $(m, n) \in \mathbb{N} \times \mathbb{N}$ such that $U_1 \cap T^{-(m,n)}U_1 \neq \emptyset$ and $U_1 \cap T^{-(m,n)}U_2 \neq \emptyset$. We denote by $WM(X, T)$ the set of weakly mixing pairs. It is clear that a TDS is weakly mixing iff $WM(X, T) = X^{(2)} \setminus \triangle_X$ iff (X, T) is SUPE. And weakly ming system is disjoint from any null minimal system. For a dynamical system the topological Pinsker factor, i.e. the maximal factor with zero topological entropy is defined using entropy pairs. Similarly, a dynamical system (X, T) the smallest closed invariant equivalence relation containing $SE(X, T)$ induces the maximal null factor. $WM(X, T) = \emptyset$ if and only if (X, T) is equicontinuous. The factor (Y, S) of (X, T) induced by the smallest invariant equivalence relation containing $WM(X, T)$ is equicontinuous.

5. Sequence UPE extension

5.1 *Relative sequence entropy pairs.* The factor map ϕ determines a closed $T \times T$-invariant equivalence relation $R_\phi \subset X^{(2)}$ on X,

$$R_\phi = \{(x_1, x_2) : \phi(x_1) = \phi(x_2)\}$$

We denote by $SE_\phi(X, T) = SE(X, T) \cap R_\phi$ the set of relative sequence entropy pair with respect to (Y, S). We say that (X, T) is a SUPE extension with respect to (Y, S) if $SE_\phi(X, T) = R_\phi \setminus \Delta_X$.

It is easy to see that (X, T) is a SUPE extension with respect to (Y, S) if and only if for $y \in Y$ and for any cover and $\mathcal{R} = \{U, V\}$ with $U^c \cap \phi^{-1}(y) \neq \emptyset$ and $V^c \cap \phi^{-1}(y) \neq \emptyset$, one has $h_{\mathcal{A}}(T, \mathcal{R}) > 0$ for some sequence $\mathcal{A} \subset \mathbb{Z}_+$.

5.2 *Transitive extension, weakly mixing extension, relative minimal.* We say that (X, T) is a transitive extension with respect to (Y, S), if for any $y \in Y$ and for each pair of open subsets U and V of X with $U \cap \phi^{-1}(y) \neq \emptyset$, $V \cap \phi^{-1}(y) \neq \emptyset$, there is $n \in \mathbb{N}$ such that $U \cap T^{-n}V \neq \emptyset$.

We say (X, T) is a weakly mixing extension with respect to (Y, S) if $(X \times_Y X, T^{(2)})$ is a transitive extension with respect to (Y, S), that is, for any $y \in Y$ and for any open subsets U_i of X with $U_i \cap \phi^{-1}(y) \neq \emptyset$, $i = 1, 2, 3, 4$, there is $n \in \mathbb{N}$ such that $U_1 \cap T^{-n}U_3 \neq \emptyset$, $U_2 \cap T^{-n}U_4 \neq \emptyset$.

5.3 *Weakly mixing extension pair.* For a dynamical system (X, T), $(x_1, x_2) \in R_\phi \setminus \triangle_X$ is a weakly mixing extension pair with respect to (Y, S) if for any open neighborhood U_i of x_i, $i = 1, 2$, there is $n \in \mathbb{N}$ such that $U_1 \cap T^{-n}U_1 \neq \emptyset$ and $U_1 \cap T^{-n}U_2 \neq \emptyset$. We denote by $WM_\phi(X, T)$ the set of all weakly mixing extension pair with respect to (Y, S).

Theorem 5.1. *Let (X, T) be a dynamical system. Then the following statements are equivalent*

(1) (X, T) is a weakly mixing extension with respect to (Y, S).
(2) $WM_\phi(X, T) = R_\phi \setminus \triangle_X$.
(3) (X, T) is a SUPE extension.

Remark. If (Y, S) is a trivial one point factor, then the above theorem is for an absolute case.

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