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GENERIC DIFFEOMORPHISMS AWAY FROM HOMOCLINIC BIFURCATIONS

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Abstract. We introduce the C^1 density conjecture of Palis, and illustrate some partial results we obtained towards the conjecture.

1. The conjecture and the results

Let M be a compact manifold without boundary, and $\mathrm{Diff}(M)$ be the set of diffeomorphisms of M, endowed with the C^1 topology. A diffeomorphism $f : M \to M$ is called *hyperbolic* if the limit set $L(f)$ of f is a hyperbolic set, where $L(f)$ is by definition the closure of the union of the ω-limit set $\omega(x)$ and α-limit set $\alpha(x)$, for all $x \in M$. Hyperbolic systems include many nice systems such as structurally stable systems, Axiom A systems, etc. However, contrary to a common expectation, hyperbolic systems are found not dense in $\mathrm{Diff}(M)$. That is, there are diffeomorphisms that can not be C^1 approximated by hyperbolic systems (see [AS] [N1] [Si] for early examples of this nature). Which bifurcation phenomena could be typical in this robustly non-hyperbolic world? Palis [P, PT] has the following famous conjecture:

The C^1 Density Conjecture. Diffeomorphisms of M exhibiting either a homoclinic tangency or a heterodimensional cycle are C^1 dense in the complement of the C^1 closure of hyperbolic systems.

Here a point $x \in M$ is called a *homoclinic tangency* of f, if there is a hyperbolic periodic orbit P of f such that $x \in W^u(P) \cap W^s(P) - P$, and such that the intersection of $W^u(P)$ and $W^s(P)$ at x is not transverse. A *heterodimensional cycle* of f consists of two hyperbolic periodic orbits P and Q of f of different indices such that $W^u(P) \cap W^s(Q) \neq \emptyset$, and $W^u(Q) \cap W^s(P) \neq \emptyset$. Here by the *index* of a hyperbolic periodic orbit P we mean the integer $\dim(W^s(P))$. Note that at least one of the two intersections, $W^u(P) \cap W^s(Q)$ or $W^u(Q) \cap W^s(P)$, is not transverse, due to inadequate dimensions. Homoclinic tangency and heterodimensional cycle are two bifurcation phenomena that go beyond the Kupka-Smale systems. This makes the conjecture of Palis even more striking.

In dimension 2 the conjecture is proved recently by Pujals and Sambarino in a remarkable paper [PS]. Note that in dimension 2, a priori, there can be no heterodimensional cycles (a heterodimensional cycle involves periodic saddles of different indices, while in dimension 2 all periodic saddles have the same index 1). Hence in dimension 2, what was conjectured by Palis, and now proved by Pujals and Sambarino, is that any diffeomorphism can be C^1 approximated either by a hyperbolic diffeomorphism, or by one with a homoclinic tangency.

In dimension higher than 2, there are systems that can be C^1 approximated neither by hyperbolic systems, nor by systems with a homoclinic tangency [Sh1] [M1] [BD] [BV], and the conjecture says such a system must be C^1 approximated by one with a heterodimensional cycle. In this note we illustrate some partial results we obtained towards the conjecture for higher dimensions. Note that the conjecture can be alternately stated as that *Hyperbolic diffeomorphisms are C^1 dense in the complement of the C^1 closure of diffeomorphisms exhibiting either a homoclinic tangency or a heterodimensional cycle.* We can prove that there is a C^1 residual subset $\mathcal{R}$ in the complement of the C^1 closure of diffeomorphisms exhibiting either a homoclinic tangency or a heterodimensional cycle such that every $f \in \mathcal{R}$ is "nearly" hyperbolic, in the sense as described in Theorem A and B below. The precise statements use the notion of the so called minimally non-hyperbolic set we now introduce.

A compact invariant set Λ of f is called *minimally non-hyperbolic* of f if Λ is not a hyperbolic set of f but any nonempty compact invariant proper subset of Λ is a hyperbolic set of f. This notion plays an important role in Pujals and Sambarino [PS]. It resembles the notion of minimally rambling sets, studied intensively by Liao [L2].

Proposition 1.1. ([L2, P.4], [PS, P.983]) *Any non-empty non-hyperbolic compact invariant set of f contains at least one minimally non-hyperbolic set of f.*

Proof. Let A be a non-empty non-hyperbolic compact invariant set of f. Let $\mathcal{C}$ be the set of non-empty non-hyperbolic compact invariant subset of A, respecting f. $\mathcal{C}$ is partially ordered by the inclusion. It is easy to see any linearly ordered subset of $\mathcal{C}$ has a lower bound in $\mathcal{C}$. By Zorn's lemma, $\mathcal{C}$ has a minimal element Λ. It is easy to see Λ is a minimally non-hyperbolic set of f, contained in A. This proves Proposition 1.1.

We follow Liao [L2] to divide minimally non-hyperbolic sets into two types, simple and non-simple, in a slightly different way by using the following characterization of hyperbolic sets that can be found in Selgrade [S], Sacker and Sell [SS], Mañé [M2] and Liao [L1]. Denote

$$D^s(x) = \{v \in T_x(M) \mid \ \|Df^n(v)\| \to 0,\ n \to +\infty\},$$

$$D^u(x) = \{v \in T_x(M) \mid \ \|Df^{-n}(v)\| \to 0,\ n \to +\infty\}.$$

These are Df-invariant (as family) linear subspaces of T_xM. By definition, vectors of D^s and D^u are asymptotic to zero under forward or backward iterates, respectively, but not necessarily exponentially fast. However, if the two subspaces form a direct sum at every point of a compact invariant set, exponential rates will follow:

Proposition 1.2. ([S, SS, M2, L1]) *A compact invariant set A of f is hyperbolic if and only if $D^s(x) \oplus D^u(x) = T_xM$, for all $x \in A$.*

Let us call a point $x \in M$ *resisting* of f if the equality $D^s(x) \oplus D^u(x) = T_xM$ does not hold. This means either $D^s(x) + D^u(x) \neq T_xM$, or $D^s(x) \cap D^u(x) \neq \{0\}$. The set of resisting points of f is f-invariant, but generally not closed. A minimally non-hyperbolic set Λ will be called of *simple type* if there is a resisting point $a \in \Lambda$ such that both $\omega(a)$ and $\alpha(a)$ are proper subsets of Λ. Otherwise the minimally non-hyperbolic set will be called of *non-simple type*. The following proposition describes the structure of a simple type minimally non-hyperbolic set.

Proposition 1.3. *A simple type minimally non-hyperbolic set Λ of f can be written as $\Lambda = \omega(a) \cup \mathrm{Orb}(a) \cup \alpha(a)$, where $a \in \Lambda$ is a resisting point of f, such that $\omega(a)$ and $\alpha(a)$ are both hyperbolic, and $a \notin \omega(a) \cup \alpha(a)$.*

Proof. Since $\omega(a)$ and $\alpha(a)$ are both proper subsets of Λ, they are hyperbolic. Hence $a \notin \omega(a) \cup \alpha(a)$. Also, being a non-hyperbolic (with a resisting) compact invariant subset, $\omega(a) \cup \mathrm{Orb}(a) \cup \alpha(a)$ must be the whole Λ. This proves Proposition 1.3.

Thus a simple type minimally non-hyperbolic set Λ has a clear structure. It is a non-transverse (more correctly, non-direct-sum) heteroclinic (if $\omega(a) \cap \alpha(a) = \emptyset$) or homoclinic (if $\omega(a) \cap \alpha(a) \neq \emptyset$) connection of two hyperbolic sets. In particular, in case $\omega(a) = \alpha(a) = \{p\}$ is a hyperbolic fixed point, this will be the familiar picture of homoclinic tangency.

On the other hand, there has been no general structure theorem available for a non-simple type minimally non-hyperbolic set. By definition a non-simple type minimally non-hyperbolic set Λ must be topologically transitive. Indeed, by definition, every resisting point $a \in \Lambda$ satisfies either $\omega(a) = \Lambda$, or $\alpha(a) = \Lambda$. A trivial example for a non-simple type minimally non-hyperbolic set would be a non-hyperbolic fixed point, or a non-hyperbolic periodic orbit, or any non-hyperbolic minimal set. Liao proves under the conditions of [L2] that a non-simple type minimally non-hyperbolic set must not be a minimal set. Pujals-Sambarino prove under the conditions of [PS] that the only non-simple type minimally non-hyperbolic set is an invariant circle with an irrational rotation. These highly non-trivial results form a key step in their work, and also justify the use of a general principle of Liao:

Principle. *To prove that a compact invariant set A is hyperbolic, it suffices to rule out the possibility of the existence of simple type and non-simple type minimally non-hyperbolic sets contained in A.*

This principle has an advantage that, to prove that a compact invariant set A (for instance the nonwandering set) is hyperbolic, we do not have to handle the whole set A globally, but only have to rule out the possibility of the existence of the two types of minimally non-hyperbolic sets in A, which may be of relatively less global nature.

Now we state our partial results toward the conjecture of Palis. Recall the conjecture can be alternately stated as that *Hyperbolic diffeomorphisms are C^1 dense in the complement of the C^1 closure of diffeomorphisms exhibiting either a homoclinic tangency or a heterodimensional cycle.*

Theorem A. *There is a C^1 residual subset $\mathcal{R}$ in the complement of the C^1 closure of diffeomorphisms exhibiting either a homoclinic tangency or a heterodimensional* *cycle, such that every* $f \in \mathcal{R}$ *has no simple type minimally non-hyperbolic sets contained in* $L(f)$.

To prove Palis conjecture it remains to rule out the possibility of the existence of non-simple type minimally non-hyperbolic sets Λ (contained automatically in $L(f)$ since Λ is topologically transitive), for a dense (or residual, if possible) subset of diffeomorphisms in the complement of the C^1 closure of diffeomorphisms exhibiting either a homoclinic tangency or a heterodimensional cycle. The following Theorem B has not achieved this, but asserts that such a non-simple type minimally non-hyperbolic set Λ, if it exists, must look somewhat special. We hope this would help to rule out its existence eventually. Since the intersection of (countably many) residual subsets is residual, we will use the same notation for several different residual subsets of the same space ($\mathcal{R}$ for a local one, and $\mathcal{A}$ for a global one).

Theorem B. *There is a* C^1 *residual subset* $\mathcal{R}$ *in the complement of the* C^1 *closure of diffeomorphisms exhibiting either a homoclinic tangency or a heterodimensional cycle, such that any non-simple type minimally non-hyperbolic set* Λ *of any* $f \in \mathcal{R}$ *has the following feature:*

(1) Λ *is the common Hausdorff limit of two sequences of hyperbolic periodic orbits of different indices. More precisely, there are two sequences of hyperbolic periodic orbits* $\{P_k\}$ *and* $\{Q_k\}$ *of* f *that both converge to* Λ *in the Hausdorff metric, such that* $ind(Q_k) = ind(P_k)+1$ *and* $W^u(P)$ *intersects* $W^s(Q)$ *transversely. In particular,* Λ *can not be contained in any normally hyperbolic arc or circle of* f.

(2) Λ *is partially hyperbolic with central bundle at most 2-dimensional. More precisely, either there is a three-ways* Df*-invariant splitting* $T_\Lambda M = E^s \oplus E^c \oplus E^u$, *where* E^s *is dominated by* E^c, *and* E^c *is dominated by* E^u, *such that* E^s *is contracting,* E^u *is expanding, and* E^c *is 1-dimensional and is neither contracting nor expanding, or, there is a four-ways* Df*-invariant splitting* $T_\Lambda M = E^s \oplus E^{cs} \oplus E^{cu} \oplus E^u$, *where* E^s *is dominated by* E^{cs}, E^{cs} *is dominated by* E^{cu}, *and* E^{cu} *is dominated by* E^u, *such that* E^s *is contracting,* E^u *is expanding, and* E^{cs} *and* E^{cu} *are each* 1*-dimensional and neither contracting nor expanding.*

Theorem A can be obtained quickly from the results of [W2] and [GW] (which are actually preparations for the present paper). We give the proof in the next section. The proof of Theorem B will depend in addition the elegant selecting lemma of Liao [L2]. The details are involved, and will be omitted.

§2. The proof for Theorem A

We start with recalling a characterization for diffeomorphisms that are C^1 away from homoclinic tangencies, by dominated splittings on the so called preperiodic sets. A point $x \in M$ is C^1 *preperiodic*, if for any C^1 neighborhood $\mathcal{U}$ of f in Diff(M) and any neighborhood U of x in M, there is $g \in \mathcal{U}$ and $y \in U$ such that y is periodic of g (see [W1]). We denote the set of C^1 preperiodic points by $P_*(f)$. It is easy to see that $P_*(f)$ is closed and f-invariant. Also,

$$\Omega(f) \subset P_*(f) \subset R(f),$$

where $\Omega(f)$ and $R(f)$ denote the nonwandering and chain recurrent sets of f, respectively. The first inclusion is by the C^1 closing lemma of Pugh [Pu, PR], and the second inclusion is just by definitions.

Note that in the definition of preperiodic points it is equivalent to replace the term "periodic " by "hyperbolic periodic", because any periodic point can be made hyperbolic by an arbitrarily small C^r perturbation. We call a point $x \in M$ C^1 *i-preperiodic* of f, $0 \le i \le d$, if for any C^1 neighborhood $\mathcal{U}$ of f in Diff(M) and any neighborhood U of x in M, there is $g \in \mathcal{U}$ and $y \in U$ such that y is a hyperbolic periodic point of g of index i. Denote by $P_*^i(f)$ the set of C^1 i-preperiodic points of f. Then

$$P_*(f) = \bigcup_{i=0}^{d} P_*^i(f).$$

$P_*^i(f)$ is closed and f-invariant, for each $0 \le i \le d$. Generally $P_*^i(f)$ and $P_*^j(f)$ are not disjoint for $i \ne j$. In fact, according to Liao [L2] and Mañé [M3], $P_*^i(f)$ are mutually disjoint for $0 \le i \le d$ if and only if f is Axiom A and no-cycle.

Let Λ be a compact invariant set of f. A continuous invariant splitting $T_\Lambda M = \Delta^s \oplus \Delta^u$ on Λ is called *dominated of index* i, $1 \le i \le d-1$, if dim$\Delta^s(x) = i$ for all $x \in \Lambda$, and if there are two constants $0 < \lambda < 1$ and $C > 0$ such that

$$\|Df^n|_{\Delta^s(x)}\| \cdot \|Df^{-n}|_{\Delta^u(f^n(x))}\| \le C\lambda^n$$

for all $x \in \Lambda$ and $n \ge 0$. We may call $\Delta^s \oplus \Delta^u$ specifically a (C, λ)-dominated splitting. This is equivalent to that for some constants $\iota \in \mathbb{N}$ and $0 < \mu < 1$,

$$\|Df^\iota|_{\Delta^s(x)}\| \cdot \|Df^{-\iota}|_{\Delta^u(f^\iota(x))}\| \le \mu$$

for all $x \in \Lambda$. We may also call $\Delta^s \oplus \Delta^u$ specifically an (ι, μ)-dominated splitting. A compact invariant set may have more than one dominated splitings. Nevertheless, for fixed i, dominated splitting of index i is unique.

Proposition 2.1 ([W2]) *Let $f : M \to M$ be a diffeomorphism. The following three conditions are equivalent:*

(1) *f has dominated splitting of index i on its C^1 i-preperiodic set $P_*^i(f)$, for all $1 \le i \le d-1$.*

(2) *f can not be C^1 approximated by a system that exhibits a homoclinic tangency associated with some hyperbolic periodic point of some index $1 \le i \le d-1$.*

(3) *There is a C^1 neighborhood $\mathcal{U}$ of f and a number $\gamma > 0$ such that for any hyperbolic periodic point p of any $g \in \mathcal{U}$ of any index $1 \le i \le d-1$, $\angle(E^s(p,g), E^u(p,g)) \ge \gamma$.*

Another major issue to us will be C^1 generic properties about orbits-connecting. There are a number of recent work along this direction, see [Ab1], [Ab2], [Ar], [CMP] and [GW]. We state three propositions of this type. The first one concerns a weak form of transitivity. It is to generalize the notion of topological transitive sets as much as possible, but still to keep certain recurrence so that the C^1 connecting lemma applies on such a set to yield various connections. We say $y \in M$ is *attainable from $x \in M$ respecting f*, if for any neighborhood U of x in M and any neighborhood V of y in M, there is $z \in U$ such that $f^n(z) \in V$ for some integer $n \ge 1$. Thus a nonwandering point of f is attainable from itself. We say x and y are *bi-attainable* to each other if y is attainable from x and x is attainable from y. A compact invariant set Λ is called *weakly transitive* if every pair of points of Λ are bi-attainable to each other. The main examples in our mind for weakly transitive sets are a single ω-limit set $\Lambda = \omega(x)$ or α-limit set $\Lambda = \alpha(x)$ (this is more general than a transitive set because x may not be in Λ), or the Hausdorff limit Λ of a sequence of periodic orbits P_k of f.

Proposition 2.2. ([Ar], [GW]) *There is a C^1 residual subset $\mathcal{A} \subset$ Diff(M) such that for every $f \in \mathcal{A}$, bi-attainability is a closed equivalence relation on the nonwandering set $\Omega(f)$.*

Thus for every $f \in \mathcal{A}$, $\Omega(f)$ decomposes into closed, f-invariant equivalence classes (generally infinite in number). We call each equivalence class a *weakly transitive component* of f. Note that bi-attainability is not an equivalence relation in general.

The second generic property concerns the possibility of creation of a heterodimensional cycle by C^1 perturbations. Recall for a hyperbolic set H with constant dimension i of stable subspaces, we define its *index* to be i. Any hyperbolic set decomposes into at most $\dim(M) + 1$ pieces of hyperbiolic subsets, of which the index is well defined.

Proposition 2.3. ([GW]) *There is a C^1 residual subset $\mathcal{A} \subset$ Diff(M) such that for every $f \in \mathcal{A}$, if a weakly transitive set Λ of f contains two hyperbolic sets H_1 and H_2 of different indices, then f can be C^1 approximated by g that has a heterodimensional cycle.*

Note that H_1 and H_2 in the assumption must be both of saddle type. That is, $1 \le \text{ind}(H_1) \le d-1$, $1 \le \text{ind}(H_2) \le d-1$. This is because a hyperbolic set of index 0 (or d) consists of finitely many periodic sources (or sinks), and because if a weakly transitive set Λ contains a periodic source (or sink) P, Λ must reduce to P.

The problem stated in Proposition 2.3 is very natural. Being contained in the same weakly transitive set Λ, the two hyperbolic sets of different indices are loosely connected each other. The problem is then to create a true connection between two hyperbolic periodic orbits of different indices. Interference of orbits appears in the perturbations however, which makes it unclear if the problem could be solved by the C^1 connecting lemma alone, even if the set Λ in question is not only weakly transitive, but transitive. See [GW] for a detailed illustration about this subtle point. Some C^1 generic assumptions then are added to avoid the interference of orbits.

The third generic property concerns how much a homoclinic class spreads. It involves a similar subtle point and needs some generic assumptions, as stated in Proposition 2.4 next. Recall that the *homoclinic class* $H(P)$ of a hyperbolic periodic orbit P of f is defined to be the closure of the union of hyperbolic periodic orbits of f that are H-related to P. Here two hyperbolic periodic orbits P and Q of f are called *H-related* if $W^u(P) \cap W^s(Q) \neq \emptyset$ with a transverse intersection, and $W^u(Q) \cap W^s(P) \neq \emptyset$ with a transverse intersection. A homoclinic class $H(P)$ is called *trivial* if it consists of the orbit P only. If $H(P)$ is non-trivial, it coincides with the closure of transverse homoclinic points of P. If $\text{ind}(P) = i$, we will simply call $H(P)$ an *i-homoclinic class.* Note that i may not be uniquely assigned to $H(P)$, because generally it is possible that $H(P) = H(Q)$ for a hyperbolic periodic orbit Q with $\text{ind}(P) \neq \text{ind}(Q)$.

Proposition 2.4. ([Ar], [GW]) *There is a C^1 residual subset $\mathcal{A} \subset \mathit{Diff}(M)$ such that for every $f \in \mathcal{A}$, if a weakly transitive set Λ of f contains a hyperbolic set K of index i, then $\Lambda \subset H(P)$ for a hyperbolic periodic orbit P of index i. In particular, if this Λ is a weakly transitive component, then $\Lambda = H(P)$.*

Note that K must be of saddle type. That is, $1 \leq i \leq d-1$. This is because otherwise Λ would reduces to a periodic source or sink, contradicting that Λ is non-trivial.

In particular, Proposition 2.4 says that, for generic f, any homoclinic class $H(P)$ is itself a weakly transitive component, — the component that contains P. Of course, being transitive, a homoclinic class can not go beyond the weakly transitive component it lies in. Thus, generically, every homoclinic class spreads as much as it can.

We also need the following two simple facts. Recall $D^s(x)$ and $D^u(x)$ are the two linear subspaces of vectors in T_xM that tend to zero asymptotically under positive and negative Df-iterates, respectively, defined in §1. Also, note that if $\omega(x)$ happens to be hyperbolic for some $x \in M$, then $\text{ind}(\omega(x))$ is well defined.

Lemma 2.5. *Let a be any point of M. If $\omega(a)$ is hyperbolic, then* $\text{dim}D^s(a) = \text{ind}(\omega(a))$. *Likewise, if $\alpha(a)$ is hyperbolic, then* $\text{dim}D^u(a) = d - \text{ind}(\alpha(a))$.

Proof. The proof is easy by the shadowing lemma. We take the case of $\omega(a)$. Write $\text{ind}\omega(a) = i$. Take a large integer m such that the positive f-orbit of $b = f^m(a)$ remains close to $\omega(a)$. Together with the negative f-orbit of a point $y \in \omega(a)$ that is close to b, it gives a pseudo orbit, which is hence shadowed by the f-orbit of some point z, which remains entirely in a small neighborhood of $\omega(a)$ hence is hyperbolic of index i. Moreover, by shadowing, $b \in W^s(z)$. Hence $a \in W^s(z)$. Then it is easy to see $\text{dim}D^s(a) = i$. This proves Lemma 2.5.

The following elementary lemma is due to Liao. We omit the proof, which is cited in [W2, Lemma 2.2].

Lemma 2.6. ([L2]) *Assume Λ is a compact invariant set of f with a dominated splitting $\Delta^s \oplus \Delta^u$ on Λ, and $x \in \Lambda$. Then either $\Delta^s(x) \subset D^s(x)$, or $D^s(x) \subset \Delta^s(x)$. Likewise, either $\Delta^u(x) \subset D^u(x)$, or $D^u(x) \subset \Delta^u(x)$.*

Now we prove Theorem A.

Theorem A. *There is a C^1 residual subset $\mathcal{R}$ in the complement of the C^1 closure of diffeomorphisms exhibiting either a homoclinic tangency or a heterodimensional cycle, such that every $f \in \mathcal{R}$ has no simple type minimally non-hyperbolic sets contained in $L(f)$.*

Proof. Let $\mathcal{R}$ be the set of diffeomorphisms in the complement of the C^1 closure of diffeomorphisms exhibiting either a homoclinic tangency or a heterodimensional cycle that satisfy the C^1 generic conditions stated in Propositions 2.2 through 2.4, as well as the well known C^1 generic condition $L(f) = \overline{P(f)}$ (a consequence of the C^1 closing lemma of Pugh). Let $f \in \mathcal{R}$. We prove f has no simple type minimally non-hyperbolic set contained in $L(f) = \overline{P(f)}$. Suppose f has a simple type minimally non-hyperbolic set Λ in $\overline{P(f)}$. By Proposition 1.3,

$$\Lambda = \omega(a) \cup \mathrm{Orb}(a) \cup \alpha(a),$$

where $a \in \Lambda$ is a resisting point of f such that $\omega(a)$ and $\alpha(a)$ are both hyperbolic, and $a \notin \omega(a) \cup \alpha(a)$. Since $a \in \overline{P(f)}$, there is a sequence of periodic orbits of f that converge in the Hausdorff metric to a compact f-invariant set Γ with $a \in \Gamma$. Thus $\Gamma \supset \Lambda$. Being the Hausdorff limit of a sequence of periodic orbits, Γ is weakly transitive. If $\omega(a)$ and $\alpha(a)$ have different indices, by Proposition 2.3, a heterodimensional cycle can be created by C^1 perturbation, giving a contradiction. Thus $\omega(a)$ and $\alpha(a)$ have the same index, say i. Note that Γ is non-trivial, because even the subset Λ of it does not reduce to a periodic orbit. In particular, $1 \leq i \leq d-1$. By Proposition 2.4, Γ is contained in an i-homoclinic class of f, hence contained in $\overline{P^i(f)} \subset P^i_*(f)$ and hence, by Proposition 2.1, has a dominated splitting

$$T_\Gamma(M) = \Delta^s \oplus \Delta^u$$

of index i. By Lemma 2.5,

$$\dim D^s(a) = i, \quad \text{and} \quad \dim D^u(a) = d - i.$$

Hence by Lemma 2.6, $D^s(a) = \Delta^s(a)$, $D^u(a) = \Delta^u(a)$. Thus

$$T_a(M) = D^s(a) \oplus D^u(a),$$

contradicting that a is a resisting point. This proves Theorem A.

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