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OPERATOR INEQUALITIES DERIVED FROM NEW FORMULA THAT SPECHT RATIO $S(1)$ CAN BE EXPRESSED BY GENERALIZED KANTOROVICH CONSTANT $K(p)$: $S(1) = e^{K'(1)}$

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Abstract. We obtained a basic new formula between Specht ratio $S(1)$ and generalized Kantorovich constant $K(p)$, that is, Specht ratio $S(1)$ can be expressed by generalized Kantorovich constant $K(p)$: $S(1) = e^{K'(1)}$. We shall announce operator inequalities derived from new formula without proofs.

1. Introduction

An operator A is said to be positive operator (denoted by $T \geq 0$) if $(Ax, x) \geq 0$ for all x in a Hilbert space H and also A is said to be strictly positive operator (denoted by $A > 0$) if A is invertible positive operator.

Definition 1. Let $h > 1$. $S(h,p)$ is defined by

$$S(h,p) = \frac{h^{\frac{p}{h^p-1}}}{e \log h^{\frac{p}{h^p-1}}} \quad \text{for any real number } p \tag{1.1}$$

and $S(h,p)$ is denoted by $S(p)$ briefly. Especially $S(1) = S(h,1) = \dfrac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}}$ is said to be *Specht ratio. The generalized Kantorovich constant* $K(h,p)$ is defined by

$$K(h,p) = \frac{(h^p - h)}{(p-1)(h-1)}\left(\frac{(p-1)}{p}\frac{(h^p-1)}{(h^p-h)}\right)^p \quad \text{for any real number } p \tag{1.2}$$

and $K(h,p)$ is denoted by $K(p)$ briefly.

Basic Property [13]. *The following basic property among $S(1)$, $K'(1)$ and $K'(0)$ holds*:

$$S(1) = e^{K'(1)} = e^{-K'(0)}. \tag{1.3}$$

Another nice relation between $K(p)$ and $S(1)$ is in [26]. We cite Figure 1 relation between $K(p)$ and $S(p)$ before the References. The relation (1.3) is quite important, so we state its proof for the sake of convenience.

$$(*) \quad K'(p) = \frac{\left(\frac{(p-1)}{p}\frac{(h^p-1)}{(h^p-h)}\right)^p}{(h-1)(h^p-1)}\left\{\frac{h^p(h^p-1+p-hp)\log h + (h^p-1)(h^p-h)\log\frac{(p-1)(h^p-1)}{p(h^p-h)}}{p-1}\right\}.$$

By using L'Hospital theorem to (*), we have

$$\lim_{p\to 1} K'(p) = \frac{h-1}{h\log h}\frac{1}{(h-1)^2}\Big\{h\log h(h\log h+1-h)+(h-1)h\log h\log\Big[\frac{h-1}{h\log h}\Big]\Big\}$$
$$= \frac{h}{h-1}\log h - 1 + \log\Big[\frac{h-1}{h\log h}\Big]$$
$$= \log\Big[\frac{h^{\frac{1}{h-1}}}{e\log h^{\frac{1}{h-1}}}\Big] = \log S(1)$$

so that we have $S(1) = e^{K'(1)}$ and also $S(1) = e^{-K'(0)}$ by the same way.

Let A be strictly positive operator satisfying $MI \geq A \geq mI > 0$, where $M > m > 0$. Put $h = \frac{M}{m} > 1$. The celebrated Kantorovich inequality asserts that

$$\frac{(1+h)^2}{4h}(Ax,x)^{-1} \geq (A^{-1}x,x) \geq (Ax,x)^{-1} \tag{1.4}$$

holds for every unit vector x and this inequality is just equivalent to the following one

$$\frac{(1+h)^2}{4h}(Ax,x)^{2} \geq (A^{2}x,x) \geq (Ax,x)^{2} \tag{1.5}$$

holds for every unit vector x. We remark that $K(h,p)$ in (1.2) is an extension of $\dfrac{(1+h)^2}{4h}$ in (1.4) and (1.5) , in fact, $K(h,-1) = K(h,2) = \dfrac{(1+h)^2}{4h}$ holds.

Many papers on Kantorovich inequality have been published. Among others, there is a long research series by Mond-Pečarić, we cite [21][22] and [23] for examples.

We state the Jensen inequality as follows. (c.f. [Theorem 4, 1],[3, 4],[Theorem 2.1, 17])

Jensen inequality. Let f be an operator concave function on an interval I. If Φ is normalized positive linear map,then

$$f(\Phi(A)) \geq \Phi(f(A))$$

for every self adjoint operator A on a Hilbert space H whose spectrum is contained in I.

On the other hand, the relative operator entropy $S(X|Y)$ for $X > 0$ and $Y > 0$ is defined in [7] as an extension of the operator entropy $S(X|I) = -X\log X$

$$S(X|Y) = X^{\frac{1}{2}}[\log(X^{\frac{-1}{2}}YX^{\frac{-1}{2}})]X^{\frac{1}{2}}. \tag{1.6}$$

By using this $S(X|Y)$, we define $T(X|Y)$ for $X > 0$ and $Y > 0$;

$$T(X|Y) = (X\sharp Y)X^{-1}S(X|Y)X^{-1}(X\sharp Y) \tag{1.7}$$

where $X\sharp Y = X^{\frac{1}{2}}(X^{\frac{-1}{2}}YX^{\frac{-1}{2}})^{\frac{1}{2}}X^{\frac{1}{2}}$. The power mean $X\sharp_p Y = X^{\frac{1}{2}}(X^{\frac{-1}{2}}YX^{\frac{-1}{2}})^{p}X^{\frac{1}{2}}$ for $p \in [0,1]$ is in [16] as an extension of $X\sharp Y$. We shall verify that $T(X|Y) = \lim_{p\to 1}(X\sharp_p Y)'$ and we remark that $S(X|Y) = \lim_{p\to 0}(X\sharp_p Y)'$ shown in [7].

2. Inequalities derived from new formula and related Kantorovich type inequalities

We shall state several inequalities derived from new formula without proofs and related results together with proofs will appear elsewhere.

Theorem 2.1. *Let A be strictly positive operator satisfying $MI \geq A \geq mI > 0$, where $M > m > 0$. Put $h = \frac{M}{m} > 1$. Then the following inequalities* (i), (ii) *and* (iii) *hold for every unit vector x and follow from each other:*

(i) $$K(h,p)(Ax,x)^p \geq (A^p x,x) \geq (Ax,x)^p \quad \text{for any } p > 1.$$

(ii) $$(Ax,x)^p \geq (A^p x,x) \geq K(h,p)(Ax,x)^p \quad \text{for any } 1 > p > 0.$$

(iii) $$K(h,p)(Ax,x)^p \geq (A^p x,x) \geq (Ax,x)^p \quad \text{for any } p < 0.$$

We remark that the latter half inequality in (i) or (iii) of Theorem 2.1 and the former half one of (ii) are called *Hölder-McCarthy inequality* and the former one of (i) or (iii) and the latter half one of (ii) can be considered as *generalized Kantorovich inequality* and the reverse inequalities to *Hölder-McCarthy inequality.* (i) and (iii) are in [11] and the equivalence relation among (i),(ii) and (iii) is shown in [Theorem 3, 14] and several extensions of Theorem 2.1 are shown, for example,[Theorem 3.2, 17].

Related results to Theorem 2.1 and operator inequalities associated with Kantorovich type inequaloities are in Chapter III of [12].

Theorem 2.2 [13]. *Let A be strictly positive operator satisfying $MI \geq A \geq mI > 0$, where $M > m > 0$. Put $h = \frac{M}{m} > 1$. Then the following inequalities hold for every unit vector x*:

(i) $$\begin{aligned} &[\log S(1)](Ax,x) + (Ax,x)\log(Ax,x) \\ &\geq ((A\log A)x,x) \\ &\geq (Ax,x)\log(Ax,x). \end{aligned}$$

(ii) $$\begin{aligned} &\frac{mh\log h}{h-1}(S(1)-1) + (Ax,x)\log(Ax,x) \\ &\geq ((A\log A)x,x) \\ &\geq (Ax,x)\log(Ax,x). \end{aligned}$$

(iii) $$[\log S(1)] + ((\log A)x,x) \geq \log(Ax,x) \geq ((\log A)x,x).$$

Theorem 2.3 [15]. *Let A_j be strictly positive operator satisfying $MI \geq A_j \geq mI > 0$ for $j = 1,2,...,n$, where $M > m > 0$ and $h = \frac{M}{m} > 1$. Also λ_1, λ_2,...,λ_n be any positive numbers such that $\sum_{j=1}^n \lambda_j = 1$. Then the following inequalities hold*:

(i) $$\begin{aligned} [\log S(1)]\sum_{j=1}^n \lambda_j A_j + \Big(\sum_{j=1}^n \lambda_j A_j\Big)\log\Big(\sum_{j=1}^n \lambda_j A_j\Big) &\\ \geq \sum_{j=1}^n \lambda_j A_j \log A_j &\\ \geq \Big(\sum_{j=1}^n \lambda_j A_j\Big)\log\Big(\sum_{j=1}^n \lambda_j A_j\Big) & \end{aligned}$$

(ii) $$\begin{aligned} \frac{mh\log h}{h-1}(S(1)-1) + \Big(\sum_{j=1}^n \lambda_j A_j\Big)\log\Big(\sum_{j=1}^n \lambda_j A_j\Big) &\\ \geq \sum_{j=1}^n \lambda_j A_j \log A_j & \end{aligned}$$

$$\geq \Big(\sum_{j=1}^{n}\lambda_j A_j\Big)\log\Big(\sum_{j=1}^{n}\lambda_j A_j\Big).$$

(iii) $$[\log S(1)] + \sum_{j=1}^{k}\lambda_j \log A_j \geq \log(\sum_{j=1}^{k}\lambda_j A_j) \geq \sum_{j=1}^{k}\lambda_j \log A_j.$$

We remark (iii) for $n = 2$ of Theorem 2.3 is shown in [9]. Interesting closely related results to Theorem 2.2 and Theorem 2.3 are in [24].

Theorem 2.4. *Let A be strictly positive operator on a Hilbert space H satisfying*

$MI \geq A \geq mI > 0$, where $M > m > 0$ and $h = \frac{M}{m} > 1$ and Φ be a normalized positive linear map on $B(H)$. Then the following inequalities hold:

(i)
$$\begin{aligned}&[\log S(1)]\Phi(A) + \Phi(A)\log\Phi(A)\\ &\geq \Phi(A\log A)\\ &\geq \Phi(A)\log\Phi(A)\end{aligned}$$

(ii)
$$\begin{aligned}&\frac{mh\log h}{h-1}(\log S(1)-1) + \Phi(A)\log\Phi(A)\\ &\geq \Phi(A\log A)\\ &\geq \Phi(A)\log\Phi(A).\end{aligned}$$

(iii)
$$\log S(1) + \Phi(\log A) \geq \log\Phi(A) \geq \Phi(\log A),$$

where $S(1)$ is defined in (1.1).

We remark that the first inequality of (i) in Theorem 2.4 is the reverse inequality of the second one which is known by [Theorem 4, 1] and also the first inequality of (ii) is the reverse inequality of the second one , and the first inequality of (iii) in Theorem 2.4 is the reverse inequality of the second one which is known by Jensen inequality. Closely related results are in [17][18][19][20].

Theorem 2.5. *Let A and B be strictly positive operator on a Hilbert space H such that*
$M_1I \geq A \geq m_1I > 0$ and $M_2I \geq B \geq m_2I > 0$. Put $m = m_1m_2$, $M = M_1M_2$ and $h = \frac{M}{m} = \frac{M_1M_2}{m_1m_2} > 1$. Then the following inequalities hold:

(i)
$$\begin{aligned}&[\log S(1)](A*B) + (A*B)\log(A*B)\\ &\geq A*(B\log B) + (A\log A)*B\\ &\geq (A*B)\log(A*B)\end{aligned}$$

(ii)
$$\begin{aligned}&\frac{mh\log h}{h-1}(\log S(1)-1) + (A*B)\log(A*B)\\ &\geq A*(B\log B) + (A\log A)*B\\ &\geq (A*B)\log(A*B)\end{aligned}$$

(iii)
$$\begin{aligned}&\log S(1) + (\log A)*I + I*(\log B)\\ &\geq \log(A*B)\\ &\geq (\log A)*I + I*(\log B)\end{aligned}$$

where $S(1)$ is defined in (1.1).

We remark that the first inequality of (i) in Theorem 2.5 is the reverse inequality of the second one and also the first inequality of (ii) is the reverse inequality of the second one, and the first inequality of (iii) in Theorem 2.5 is the reverse inequality of the second one.

Theorem 2.6. *Let A,B, C and D be strictly positive operator on a Hilbert space H such that* $M_1 I \geq A \bigotimes B \geq m_1 I > 0$ *and* $M_2 I \geq C \bigotimes D \geq m_2 I > 0$. *Put* $m = \frac{m_2}{M_1}$, $M = \frac{M_2}{m_1}$ *and* $h = \frac{M}{m} = \frac{M_1 M_2}{m_1 m_2} > 1$. *Then the following inequalities hold*:

(i)
$$\begin{aligned} &[\log S(1)](C * D) + T(A * B|C * D) \\ &\geq T(A|C) * D + C * T(B|D) \\ &\geq T(A * B|C * D) \end{aligned}$$

(ii)
$$\begin{aligned} &\frac{mh \log h}{h-1}(\log S(1) - 1)(A * B) + T(A * B|C * D) \\ &\geq T(A|C) * D + C * T(B|D) \\ &\geq T(A * B|C * D) \end{aligned}$$

(iii)
$$\begin{aligned} &[\log S(1)](A * B) + S(A|C) * B + A * S(B|D) \\ &\geq S(A * B|C * D) \\ &\geq S(A|C) * B + A * S(B|D) \end{aligned}$$

where $S(X|Y)$ *and* $T(X|Y)$ *are defined in* (1.6) *and* (1.7) *and* $S(1)$ *is defined in* (1.1).

We remark that the first inequality of (i) in Theorem 2.6 is the reverse inequality of the second one and also the first inequality of (ii) is the reverse inequality of the second one, and the first inequality of (iii) in Theorem 2.6 is the reverse inequality of the second one.

Theorem 2.7. *Let A,B, C and D be strictly positive operator on a Hilbert space H such that* $M_1 I \geq A \geq m_1 I > 0$ *and* $M_2 I \geq B \geq m_2 I > 0$. *Put* $m = \frac{m_2}{M_1}$, $M = \frac{M_2}{m_1}$ *and* $h = \frac{M}{m} = \frac{M_1 M_2}{m_1 m_2} > 1$. *Let* Φ *be a normalized positive linear map on* $B(H)$. *Then the following inequalities hold*:

(i)
$$\begin{aligned} &[\log S(1)]\Phi(B) + T(\Phi(A)|\Phi(B)) \\ &\geq \Phi(T(A|B)) \\ &\geq T(\Phi(A)|\Phi(B)) \end{aligned}$$

(ii)
$$\begin{aligned} &\frac{mh \log h}{h-1}(\log S(1) - 1)\Phi(A) + T(\Phi(A)|\Phi(B)) \\ &\geq \Phi(T(A|B)) \\ &\geq T(\Phi(A)|\Phi(B)) \end{aligned}$$

(iii)
$$\begin{aligned} &\log S(1)\Phi(A) + \Phi(S(A|B)) \\ &\geq S(\Phi(A)|\Phi(B)) \\ &\geq \Phi(S(A|B)) \end{aligned}$$

where $S(X|Y)$ *and* $T(X|Y)$ *are defined in* (1.6) *and* (1.7) *and* $S(1)$ *is defined in* (1.1).

We remark that the first inequality of (i) in Theorem 2.7 is the reverse inequality of the second one and also the first inequality of (ii) is the reverse inequality of the second one, and the first inequality of (iii) in Theorem 2.7 is the reverse inequality of the second one in [Theorem 7, 7].

Theorem 2.8. *Let A and B be strictly positive operator on a Hilbert space H such that $M_1 I \geq A \geq m_1 I > 0$ and $M_2 I \geq B \geq m_2 I > 0$. Put $m = \frac{m_1}{M_2}$, $M = \frac{M_1}{m_2}$ and $h = \frac{M}{m} = \frac{M_1 M_2}{m_1 m_2} > 1$. Then the following inequalities hold*:

(i)
$$[\log S(1)](A * I) + A * \log B + (A * I)\log(A * I)$$
$$\geq (A \log A) * I + (A * I)\log(B * I)$$
$$\geq A * \log B + (A * I)\log(A * I)$$

(ii)
$$\frac{mh \log h}{h-1}(S(1) - 1) + A * \log B + (A * I)\log(A * I)$$
$$\geq (A \log A) * I + (A * I)\log(B * I)$$
$$\geq A * \log B + (A * I)\log(A * I)$$

(iii)
$$[\log S(1)](B * I) + (B * I)\log(B * I) + (\log A) * B$$
$$\geq I * (B \log B) + (\log(A * I))(B * I)$$
$$\geq (\log A) * B + (B * I)\log(B * I)$$

where $S(1)$ is defined in (1.1).

We remark that the first inequality of (i) in Theorem 2.8 is the reverse inequality of the second one and also the first inequality of (ii) is the reverse inequality of the second one, and the first inequality of (iii) is the reverse inequality of the second one.

Although there are some papers in References which are not referred in this survey, we need them to give proofs of our results in this paper.

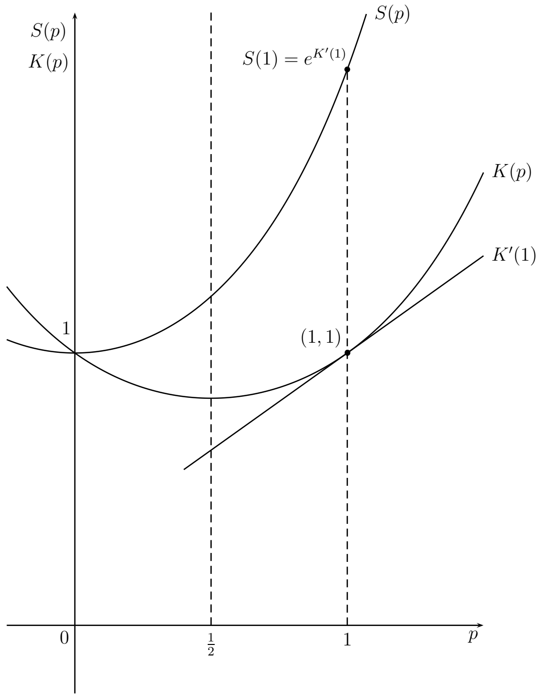

Figure 1. Relation between $K(p)$ and $S(p)$

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