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CANONICAL DECOMPOSITION OF OPERATORS CAUSED BY GIVEN OPERATOR INEQUALITIES

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ABSTRACT. It will be turned out that, given an operator inequality $\phi(\boldsymbol{T}) \geq \psi(\boldsymbol{T})$ for a tuple $\boldsymbol{T}$ of operators, it gives rise to a direct sum decomposition $\boldsymbol{T} = \boldsymbol{T}_0 \oplus \boldsymbol{T}'$ of $\boldsymbol{T}$, where $\boldsymbol{T}_0$ is the maximal part of $\boldsymbol{T}$ for which the given inequality holds.

Let $\mathcal{H}$ be a separable Hilbert space, $\mathcal{B}(H)$ the algebra of all bounded linear operators on $\mathcal{H}$. For any pair of self-adjoint operators $A, B \in \mathcal{B}(H)$, it is easily seen that there exists the maximal A- and B-reducing subspace $\mathcal{M}$ on which $A \leq B$ holds (Consider the spectral projection E for $B - A$ and put $\mathcal{M} = \mathrm{ran}E([0, +\infty))$). It also is seen that any $T \in \mathcal{B}(H)$ has the maximal subspace which reduces T to a positive operator. On the other hand, for a given family $\mathcal{G}$ of polynomials in two noncommuting variables, it is shown in [3] that any $T \in \mathcal{B}(H)$ has the maximal T-reducing subspace $\mathcal{M}$ on which T is $\mathcal{G}$-definite (resp., $\mathcal{G}$-semidefinite), i.e.,

$$p(T|_{\mathcal{M}}, (T|_{\mathcal{M}})^*) = O \quad (\text{resp., } p(T|_{\mathcal{M}}, (T|_{\mathcal{M}})^*) \geq O)$$

holds. We concerned in [4] with a larger family $\mathcal{G}$ than that of polynomials and made some considerations on the maximal subspaces on which previously given essentially $\mathcal{G}$-definite (resp., $\mathcal{G}$-semidefinite) tuples of operators are $\mathcal{G}$-definite (resp., $\mathcal{G}$-semidefinite). In this paper we will present, among others, a theorem which has led us to the following (For notations, see below):

Corollary 1 (Comparison). *If ϕ, ψ are in $\mathcal{E}_{\mathcal{S}}$, $\mathcal{S}$ a subset of $\mathcal{B}(H)$, then any $\boldsymbol{T} \in \mathcal{B}(H)^n$ has the maximal $\boldsymbol{T}$- and $\mathcal{S}$-reducing subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{H}$ such that*

(*a*) $\tau_j(\phi(\boldsymbol{T})), \tau_j(\psi(\boldsymbol{T}))(1 \leq j \leq n)$ *are self-adjoint and*

$$\tau_j(\phi(\boldsymbol{T})) \geq \tau_j(\psi(\boldsymbol{T})) \quad (1 \leq j \leq n)$$

hold on $\mathcal{M}$.

(*b*) $\tau_j(\phi(\boldsymbol{T})), \tau_j(\psi(\boldsymbol{T}))(1 \leq j \leq n)$ *are essentially self-adjoint and*

$$\tau_j(\phi(\boldsymbol{T})) \geq \tau_j(\psi(\boldsymbol{T})) \quad (1 \leq j \leq n)$$

hold essentially on any $\boldsymbol{T}$- and $\mathcal{S}$-reducing subspace of $\mathcal{N}$ on which the C^*-algebra generated by the terms of $\boldsymbol{T}$ and members of $\mathcal{S}$ is irreducible.

In the case, one has $\mathcal{M} \subseteq \mathcal{N}$.

Corollary 1 is well illustrateed by the following:

Example 1. It follows that, any pair of positive self-adjoint operators $A, B \in \mathcal{B}(H)$ has the maximal A- and B-reducing subspace on which an indicated operator inequality, e.g.,

$$e^A \le e^B, \quad \log A \le \log B, \quad \text{or} \quad A^p \le B^p \ (p > 0),$$

holds, and the maximal A- and B-reducing subspace on which an indicated operator inequality essentially holds on any A- and B-reducing subspace on which the C^*-algebra generated by $A,\ B$ is irreducible. To see this, consider the 2-tuple $(T_1,\ T_2)$ and apply Corollary 1 to the sets $\mathcal{G}_j = \{\phi_j\} (j = 1, 2, 3)$, where

$$\phi_1(T_1,\ T_2) := e^{T_1} - e^{T_2},\ \phi_2(T_1,\ T_2) := \log T_1 - \log T_2 \text{ and } \phi_3(T_1,\ T_2) := {T_1}^p - {T_2}^p.$$

Let $\mathcal{B}(H)^n$ be the algebra of all n-tuples of operators in $\mathcal{B}(H)$, and for a subset $\mathcal{S}$ in $\mathcal{B}(H)$, $\mathcal{S}^n$ the set of all n-tuples whose terms are in $\mathcal{S}$. For tuples $\boldsymbol{A} = (A_1,\ A_2,\ \ldots,\ A_n)$, $\boldsymbol{B} =(B_1,\ B_2,\ \ldots,\ B_n)\in \mathcal{B}(H)^n$, the map $\lambda_{\boldsymbol{A},\ \boldsymbol{B}}$ is defined by

$$\tau_j(\lambda_{\boldsymbol{A},\ \boldsymbol{B}}(\boldsymbol{T})) = A_j T_j B_j \quad (1 \le j \le n),$$

where

$$\tau_j(T_1,\ T_2,\ \ldots,\ T_n) := T_j \quad (1 \le j \le n),$$

and for a tuple $\boldsymbol{p} \in \mathcal{P}_n$, the set of all n-tuples of polynomials in $2n$ noncommuting variables, the map $\psi_{\boldsymbol{p}}$ is defined by

$$\tau_j(\psi_{\boldsymbol{p}}(\boldsymbol{T})) := p_j(T_1,\ T_2,\ \ldots,\ T_n,\ T_1^*,\ T_2^*,\ \ldots,\ T_n^*).$$

For a subset $\mathcal{S}$ in $\mathcal{B}(H)$, let $\mathcal{E}_{\mathcal{S}}$ be the pointwise norm closed subalgebra generated by the maps $\lambda_{\boldsymbol{A},\ \boldsymbol{B}}$, $\boldsymbol{A},\ \boldsymbol{B} \in \mathcal{S}^n$ and $\psi_{\boldsymbol{p}}$, $\boldsymbol{p} \in \mathcal{P}_n$.

For a subset $\mathcal{G}$ of $\mathcal{E}_{\mathcal{S}}$, a tuple $\boldsymbol{T}$ is said to be $\mathcal{G}$-*definite* (resp., $\mathcal{G}$-*semidefinite*) if

$$\phi(\boldsymbol{T}) = O \quad (\text{resp.,} \quad \tau_j(\phi(\boldsymbol{T})) \ge O \quad (1 \le j \le n))$$

for any $\phi \in \mathcal{G}$.

If a subspace $\mathcal{M}$ reduces the terms of $\boldsymbol{T}$, put $\boldsymbol{T}|_{\mathcal{M}} := (T_1|_{\mathcal{M}},\ T_2|_{\mathcal{M}},\ \ldots,\ T_n|_{\mathcal{M}})$, and if a subspace $\mathcal{M}$ reduces any operator in $\mathcal{S}$, then from any $\phi \in \mathcal{G}$ the map $\phi_{\mathcal{M}}$ from $\mathcal{B}(M)^n$ into itself is induced by a familiar procedure. Moreover, if $\mathcal{M}$ also reduces $\boldsymbol{T}$ (i.e., $\mathcal{M}$ reduces each term of $\boldsymbol{T}$), then the concepts of $\mathcal{G}_{\mathcal{M}}$-definiteness and $\mathcal{G}_{\mathcal{M}}$-semidefiniteness for $\boldsymbol{T}|_{\mathcal{M}} := (T_1|_{\mathcal{M}},\ T_2|_{\mathcal{M}},\ \ldots,\ T_n|_{\mathcal{M}})$ make sense.

Corollary 1 immediately follows from:

Theorem 1. *Let $\mathcal{G}$ be a subset of $\mathcal{E}_{\mathcal{S}}$, $\mathcal{S}$ a subset of $\mathcal{B}(H)$. Then any tuple $\boldsymbol{T} \in \mathcal{B}(H)^n$ has the maximal $\boldsymbol{T}$- and $\mathcal{S}$-reducing subspace $\mathcal{M}$ of $\mathcal{H}$ on which $\boldsymbol{T}$ is $\mathcal{G}$-definite (resp., $\mathcal{G}$-semidefinite), and the maximal $\boldsymbol{T}$- and $\mathcal{S}$-reducing subspace $\mathcal{N}$ of $\mathcal{H}$ such that $\boldsymbol{T}$ is essentially $\mathcal{G}$-definite (resp., $\mathcal{G}$-semidefinite) on any $\boldsymbol{T}$- and $\mathcal{S}$-reducing subspace of $\mathcal{N}$ on which the C^*-algebra generated by the terms of $\boldsymbol{T}$ and members of $\mathcal{S}$ is irreducible.*

In the case, $\mathcal{M} \subseteq \mathcal{N}$ holds and the projection onto $\mathcal{M}$ is contained in the center of the von Neumann algebra generated by the terms of $\boldsymbol{T}$ and members of $\mathcal{S}$.

Proof. We first show that the subspace

$$\mathcal{M} = \bigcap \Big\{ \mathrm{Ker} \Big(\big(\tau_j(\phi(\boldsymbol{T})) - |\tau_j(\phi(\boldsymbol{T}))| \big) A \Big) : \phi \in \mathcal{G}, A \in \mathcal{A}, 1 \le j \le n \Big\},$$

where $\mathcal{A}$ is the C^*-algebra generated with the identity I by the terms of $\boldsymbol{T}$ and members of $\mathcal{S}$, is the maximal $\boldsymbol{T}$- and $\mathcal{S}$-reducing subspace $\mathcal{M}$ of $\mathcal{H}$ on which $\boldsymbol{T}$ is $\mathcal{G}$-semidefinite. It is obvious that $\boldsymbol{T}|_{\mathcal{M}}$ is $\mathcal{G}_{\mathcal{M}}$-semidefinite. Let $\mathcal{M}^{'}$ be arbitrary $\boldsymbol{T}$- and $\mathcal{S}$-reducing subspace of $\mathcal{H}$ on which $\boldsymbol{T}$ is $\mathcal{G}$-semidefinite. Hence we have

$$\big(\tau_j(\phi(\boldsymbol{T})) - |\tau_j(\phi(\boldsymbol{T}))| \big) A\xi = o \quad \text{for} \quad \xi \in \mathcal{M}^{'}.$$

Therefore we have $\mathcal{M}^{'} \subseteq \mathcal{M}$ and hence $\mathcal{M}$ is the maximal T- and $\mathcal{S}$-reducing subspace. For $B \in \mathcal{A}^{'}$ and $\xi \in \mathcal{M}$, we see that

$$\big(\tau_j(\phi(\boldsymbol{T})) - |\tau_j(\phi(\boldsymbol{T}))| \big) AB\xi = B\big(\tau_j(\phi(\boldsymbol{T})) - |\tau_j(\phi(\boldsymbol{T}))| \big) A\xi = o.$$

Then $B\xi \in \mathcal{M}$ and hence $\mathcal{M}$ reduces B. Consequently, the projection onto $\mathcal{M}$ is contained in the center of the von Neumann algebra generated by the terms of $\boldsymbol{T}$ and members of $\mathcal{S}$. Similary, for the $\mathcal{G}$-definite case, we can show that the subspace

$$\mathcal{M} = \bigcap \Big\{ \mathrm{Ker}\, (\tau_j(\phi(\boldsymbol{T}))A) : \phi \in \mathcal{G}, A \in \mathcal{A}, 1 \le j \le n \Big\}$$

is the one stated in the theorem. To prove the essentially $\mathcal{G}$-definite (resp., $\mathcal{G}$-semidefinite) case, decompose $\mathcal{A}$ to the direct sum $\oplus \mathcal{A}_k$ (on $\mathcal{H} = \oplus \mathcal{H}_k$) of irreducible algebra and let $\mathcal{N}$ be the direct sum of $\mathcal{H}_k$ on which $\boldsymbol{T}$ is essentially $\mathcal{G}$-definite (resp., $\mathcal{G}$-semidefinite). Then $\mathcal{N}$ is nothing but the subspace stated in the theorem. □

Example 2. It is known that an operator S is subnormal if and only if

$$\phi_{A_1,\ A_2,\ \ldots,\ A_n}(S) := \sum_{0 \le j,k \le n} A_j^* S^{*k} S^j A_k \ge O$$

for any $A_1,\ A_2,\ \ldots,\ A_n (n \ge 1)$ in the C^*-algebra generated by S and I (see [2]). Then, applying Theorem 1 to the functions $\{\phi_{A_1,\ A_2,\ \ldots,\ A_n}\}$, we see that any operator T has the maximal subnormal part T_s and completely nonsubnormal part $T^{'}$ such that $T = T_s \oplus T^{'}$. Moreover, T has the maximal T-reducing subspace $\mathcal{N}_s$ such that T is "essentially subnormal" on any T-reducing subspace of $\mathcal{N}_s$ on which T is irreducible.

Corollary 2. *Let $\mathcal{S}$ be a subset of $\mathcal{B}(H)$, $\{\phi_{i,j} : 1 \le i,j \le n\}$ a subset of $\mathcal{E}_{\mathcal{S}}$ and $T \in \mathcal{B}(H)$. Then, there exists the maximal T-reducing subspace $\mathcal{M}$ such that $(\phi_{i,j}(T))|_{\mathcal{M}^n} \ge O$ on the direct sum $\mathcal{M}^n$ of n copies of $\mathcal{M}$.*

Proof. Put $M(T) = (\phi_{ij}(T))$ and

$$q_{ij}(T) = \sum_k \phi_{ki}(T)\phi_{jk}(T)^*.$$

Then $q_{ij} \in \mathcal{E}_{\mathcal{S}}$ and $M(T)^*M(T) = (q_{ij}(T))$. Choose a sequence $\{p_n\}$ of polynomials in single variable such that $||p_n(A) - A^{1/2}|| \to 0$ for any positive operator A. Let

$$p_n(M(T)^*M(T)) = (p_{ij,n}(T)) \quad \text{and} \quad |M(T)| = (\psi_{ij}(T)).$$

Then $p_{ij,n} \in \mathcal{E}_{\mathcal{S}}$ and

$$||p_{ij,n}(T) - \psi_{ij}(T)|| \to 0$$

for any $1 \leq i, j \leq n$ because

$$||p_n(M(T)^*M(T)) - |M(T)||| \to 0.$$

Therefore ψ_{ij} is the pointwise norm limit of $\{p_{ij,n}\}$ and hence $\psi_{ij} \in \mathcal{E}_{\mathcal{S}}$. Now we apply Theorem 1 to

$$\mathcal{G} = \{\phi_{ij} - \psi_{ij} \ : \ 1 \leq i, j \leq n\}$$

to have the maximal T- and $\mathcal{S}$-reducing subspace $\mathcal{M}$ on which T is $\mathcal{G}$-definite, or equivalently, $M(T) = |M(T)|$ holds. Therefore $\mathcal{M}$ is the maximal subspace such that $(\phi_{ij}(T)) \geq O$ holds on $\mathcal{M}^n$. □

We will proceed with the following:

Theorem 2. *Let $\mathcal{S}$ be a subset of $\mathcal{B}(H)$, $\{\phi_{i,j} : 1 \leq i, j \leq n\}$ be a subset of $\mathcal{E}_{\mathcal{S}}$. If $T \in \mathcal{B}(H)$ satisfies that $\pi((\phi_{ij}(T))) \geq O$, π the Calkin map for $\mathcal{B}(\mathcal{H}^n)$, then there exists an orthogonal family $\{\mathcal{H}_m : m \geq 0\}$ of T- and $\mathcal{S}$-reducing subspaces of $\mathcal{H}$ which satisfies the following statements*:

(a) $\mathcal{H} = \bigoplus_{m=0}^{\infty} \mathcal{H}_m$*, and there is no nontrivial T- and $\mathcal{S}$-reducing subspace of $\mathcal{H}_m$ if $m \geq 1$.*

(b) *$\mathcal{H}_0$ is the maximal T- and $\mathcal{S}$-reducing subspace such that*

$$(\phi_{ij}(T))|_{\mathcal{H}_0^n} \geq O$$

holds.

Therefore, if $m \geq 1$,

$$(\phi_{ij}(T))|_{\mathcal{H}_m^n} \geq O$$

holds essentially, but there is no T- and $\mathcal{S}$-reducing subspace of $\mathcal{H}_m$ on which $(\phi_{ij}(T)) \geq O$ holds.

Proof. Put $M(T) = (\phi_{ij}(T))$, and

$$|M(T)| = (\psi_{ij}(T)), \quad \{\psi_{ij}\} \subset \mathcal{E}_{\mathcal{S}}.$$

and put $\mathcal{G} = \{\phi_{ij} - \psi_{ij}\}$. Since $\pi((\phi_{ij}(T))) = (\pi(\phi_{ij}(T)))$ and $\pi(|T|) = |\pi(T)|$, it follows by the similar argument used in the proof of Corollary 2 that $\pi((\phi_{ij}(T))) \geq O$ if and only if T is essentially $\mathcal{G}$-definite. So, apply Theorem 1 in [4], we obtain the faimily $\{\mathcal{H}_m\}$. □

Therefore, for any given $T \in \mathcal{B}(H)$ and $\{\phi_{ij}\}, \{\psi_{ij}\} \subset \mathcal{E}_{\mathcal{S}}$, the claim on $(\phi_{ij}(T))$ and $(\psi_{ij}(T))$ corresponding to that of Corollary 1 should be true.

Example 3. An operator $T \in \mathcal{B}(H)$ called k-hyponormal $(1 \leq k \leq \infty)$ if the operator matrix

$$(T^{*(j-1)}T^{i-1})_{1 \leq i,j \leq (k+1)}$$

is positive (cf. [1]). Then apply the preceding corollaries to this matrix, we conclude that any T has the maximal k-hyponormal part, and any essentially k-hyponormal operator T can be decomposed into the direct sum

$$T = T_0 \oplus T_1 \oplus T_2 \oplus \cdots$$

of the maxmal k-hyponormal part T_0 and irreducible essentially k-hyponormal, but non k-hyponormal operators $T_1,\ T_2,\ \ldots$.

Let $\boldsymbol{H}_k (1 \le k \le \infty)$ be the set of all k-hyponormal operators, then it is clear thet $\boldsymbol{H}_{k+1} \subseteq \boldsymbol{H}_k (1 \le k \le \infty)$ and $\bigcap \boldsymbol{H}_k$ coincides with the set of all subnormal operators. It follows that any $T \in \mathcal{B}(H)$ can be decomposed into

$$T = T_0 \oplus T_1 \oplus T_2 \oplus \ldots \oplus T_s \quad \text{on} \quad \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \ldots \oplus \mathcal{H}_s,$$

where T_0 is completely nonhyponormal, T_k $(1 \le k < \infty)$ is k-hyponormal but completely non $(k+1)$-hyponormal and T_s is subnormal. To show this, first we decompose $T = T_s \oplus T^{'}$ where T_s is the maximal subnormal part (acting on the subspace $\mathcal{H}_s$) and $T^{'}$ is the completely non subnormal part (acting on $\mathcal{H}_0$). Next, decompose $T^{'} = T_0 \oplus T_1^{'}$ where T_0 is the completely non hyponormal part and $T_1^{'}$ is the maximal hyponormal part (acting on $\mathcal{H}_1^{'}$) of $T^{'}$. Then, we also decompose $T_1^{'} = T_1 \oplus T_2^{'}$ where T_1 is completely non 2-hyponormal but hyponormal and $T_2^{'}$ is the maximal 2-hyponormal part (acting on $\mathcal{H}_2^{'}$) of $T_1^{'}$. Recursively, if $T_k^{'}$ is maximal k-hyponormal part (acting on $\mathcal{H}_k^{'}$) of $T_{k-1}^{'}$, then $T_k^{'}$ is decomposed into the direct sum $T_k^{'} = T_k \oplus T_{k+1}^{'}$ of k-hyponormal but non $(k+1)$-hyponormal operator T_k and maximal $(k+1)$-hyponormal part $T_{k+1}^{'}$ (acting on $\mathcal{H}_{k+1}^{'}$) of $T_k^{'}$. Using the maximality, it is easy to see that $\bigcap_{k=1}^{\infty} \mathcal{H}_k^{'} = \{o\}$ and hence, putting

$$\mathcal{H}_k = \mathcal{H}_k^{'} \ominus {\mathcal{H}_{k+1}}', \quad \mathcal{H}_1^{'} = \bigoplus_{k=1}^{\infty} \mathcal{H}_k,$$

we have the decomposition above stated.

A parallel consideration makes sense on an essentially completely non-hyponormal part, an essentially k-hyponormal but not $(k+1)$-hyponormal part, and an essentially subnormal part, of T.

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