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MATRIX MONOTONE FUNCTIONS ON C^*-ALGEBRAS

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1. Introduction

Operator monotone functions defined on an interval I of the real line (open, closed or half open etc.) are widely known and often used in functional calculus of operators. When we talk about those functions we implicitly mean that we are working on the algebra of all bounded linear operators, $B(\mathcal{H})$, on a (presumably infinite-dimensional) Hilbert space $\mathcal{H}$. On the other hand, when we work on the $n \times n$ matrix algebra M_n this class of functions is usually called as matrix monotone functions of order n. Denote by $P_\infty(I)$ the set of all operator monotone functions on I and by $P_n(I)$ the set of all matrix monotone functions of order n on I, respectively. We here naturally have the result:

$$P_\infty(I) = \bigcap_{n=1}^{\infty} P_n(I).$$

We see then a big difference between continuous numerical monotone functions and continuous matrix monotone functions. For instance, although both functions e^t and $\log(1+t)$ are typical monotone functions on the interval $I = [0, \infty)$ the exponential function e^t is not matrix monotone even on the matrix algebra M_2 (cf. [2]) whereas $\log(1+t)$ is operator monotone. By definition, obviously the family $\{P_n(I)\}$ is decreasing with respect to n but the existence of gaps between the classes $P_n(I)$'s, that is, $P_{n+1}(I) \subsetneqq P_n(I)$ for every n has been only recently assured in [5] (full details cf. [6]).

Now apparently these differences stem from the structure of $B(\mathcal{H})$ and M_n. We are however often working on some limited domains of operators, namely on certain C^*-algebras. In this situation then notions of operator theoretic monotonicity of functions on a C^*-algebra $\mathcal{A}$ may naturally change from those ones mentioned above. In fact, to the extreme extent if $\mathcal{A}$ is trivial, that is, consists of scalar multiples of the identity, there appears no difference between numerical and operator theoretic monotonicity of functions. Thus, it is quite natural to ask how the notion of monotonicity of functions changes according to the structure of the relevant C^*-algebra $\mathcal{A}$. We report here our present discussions, which are intermediate ones and are far behind our final goals.

2. Discussions and results

We may assume that the C^*-algebra $\mathcal{A}$ is unital. We call a continuous function $f(t)$ on I $\mathcal{A}$-monotone if for any pair of selfadjoint elements a, b in $\mathcal{A}$ such that $a \leq b$ with spectrums inside I we have that $f(a) \leq f(b)$. Denote by $P_{\mathcal{A}}(I)$ the set of all $\mathcal{A}$-monotone functions on I. The following result is then our starting point, which is a reformulation of Theorem 5 in [6] (or Theorem 4 in [5]) from our present point of view.

Theorem 1. *Suppose the C^*-algebra $\mathcal{A}$ has an n-dimensional irreducible representation (including $n = \infty$). Then for any natural number $k \leq n$ an $\mathcal{A}$-monotone function $f(t)$ on I becomes matrix monotone of order k.*

Conversely if dimensions of all irreducible representations of $\mathcal{A}$ are less than or equal to n, then every matrix monotone function on I of order n becomes $\mathcal{A}$-monotone.

As an immediate consequence we see

Corollary 2. *If $\mathcal{A}$ has irreducible representations with arbitrary high dimensions (including infinite-dimensional irreducible representations), then the class of $\mathcal{A}$-monotone functions coincides with the class of operator monotone functions for every interval I, that is, $P_{\mathcal{A}}(I) = P_\infty(I)$.*

Note that the above assumption for $\mathcal{A}$ does not necessarily mean that $\mathcal{A}$ has an infinite-dimensional irreducible representation. Most C^*-algebras however have often infinite-dimensional irreducible representations so that C^*-algebraic monotonicity often coincides with usual concept of operator monotonicity. There is however an important class of C^*-algebras without having infinite-dimensional irreducible representations, called subhomogeneous C^*-algebras. It is the C^*-algebra whose irreducible representations are all finite-dimensional and dimensions are bounded. If the highest dimension of irreducible representations of such a C^*-algebra $\mathcal{A}$ is equal to n, we say that $\mathcal{A}$ is of degree n. In particular, if the dimension of all irreducible representations of $\mathcal{A}$ is the fixed n, $\mathcal{A}$ is said to be n-homogeneous. The algebras M_n and $\mathcal{C}(X) \otimes M_n$ are the simplest examples of n-homogeneous C^*-algebras but in general structure of n-homogeneous C^*-algebras is not so simple.

On the other hand, there are many publications discussing when $\mathcal{A}$ becomes commutative with the requirements of operator theoretic monotonicity of certain monotone functions such as the cases of functions t^m $(m \geq 2)$, e^t etc. As illustrated in Ji and Tomiyama [2] (and [3]) these discussions are all concerned with the case $n = 1$ in the above theorem. In fact, in case of e^t, for instance, since it is not matrix monotone of order 2 the algebra $\mathcal{A}$ may not have an irreducible representation whose dimension is greater than 1 once we assume that e^t is $\mathcal{A}$-monotone. Hence $\mathcal{A}$ must have only one-dimensional irreducible representations, and it becomes commutative.

Next notice that a C^*-algebra $\mathcal{A}$ is nothing but a norm closed selfadjoint subalgebra of $B(\mathcal{H})$. Therefore, the main ingredient of our discussion is the relationship between monotonicity of $B(\mathcal{H})$ and that of its subalgebra $\mathcal{A}$, or M_n with its subalgebra as well. In this sense, the proof of Theorem 1 shows that with the assumption the algebra $\mathcal{A}$ contains a unital C^*-subalgebra $\mathcal{B}$ which has an n-dimensional irreducible representation. We have then deduced the conclusion. Unfortunately, we may not assert however that the highest dimension of irreducible representations of $\mathcal{B}$ is exactly n. Thus we have faced the following problem:

Problem. *Let $\mathcal{A}$ be a unital C^*-algebra having an n-dimensional irreducible representation. Then for any natural number $k \leq n$ does there always exist a subhomogeneous C^*-subalgebra of degree k in $\mathcal{A}$?*

Obviously the C^*-algebras $B(\mathcal{H})$ and M_n have this property. On the other hand, for $n = 1$ the question becomes trivial; we can simply consider any commutative C^*-subalgebra. As of now however we can only find the answer to this question for some limited classes of proper C^*-algebras. Namely we first have the following

Proposition 3. *Let $\mathcal{A}$ be a separable subhomogeneous C^*-algebra of degree n. Then for any natural number $k \leq n$ there exists a subhomogeneous C^*-subalgebra of $\mathcal{A}$ of degree k.*

In order to find this subalgebra we can take the ideal of the intersection of all kernels of irreducible representations of $\mathcal{A}$ whose dimensions are less than n. This ideal becomes a n-homogeneous C^*-subalgebra (non-unital), and with the help of the structure of homogeneous C^*-algebras we can deduce the conclusion.

Next, let $\sum = (X, \sigma)$ be a topological dynamical system on a compact Hausdorff space X with a simple homeomorphism σ. Let $\mathcal{A}(\sum)$ be the C^*-crossed product $\mathcal{C}(X) \rtimes_{\alpha} \mathbb{Z}$ with respect to the automorphism α on $\mathcal{C}(X)$ defined as $\alpha(f)(x) = f(\sigma^{-1}x)$. We can then further discuss about the above problem for this class of C^*-algebras to the extent of some details.

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