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# CHAOTIC ORDER AND FURUTA TYPE INEQUALITIES

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Abstract. Uchiyama gave a generalization of the grand Furuta inequality and Furuta discussed it based on his previous result. Motivated by such discussions around the Furuta inequality, we consider Furuta type operator inequalities, whose hidden key is the chaotic order, i.e., $\log A \geq \log B$ for positive invertible operators $A$ and $B$. Among others, Furuta's theorem is appeared as follows: For $A \geq B \geq C > 0$ and $0 \leq t \leq 1 \leq p$

(1)
$B \geq C \geq B^{\frac{t}{2}} A^{-t} B^{\frac{t}{2}} \sharp_{\frac{1}{\beta}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p) \geq B^{\frac{t}{2}} A^{-r} B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{\beta-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p)$

and

(2) $B \geq C \geq B^{\frac{t}{2}} A^{-r} B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{p-t+r}} C^p \geq B^{\frac{t}{2}} A^{-r} B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{\beta-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p)$

hold for $\beta \geq p$ and $r \geq t$.

## 1. Introduction

Throughout this note, $A$ and $B$ are positive operators on a Hilbert space. For convenience, we denote $A \geq 0$ (resp. $A > 0$) if $A$ is a positive (resp. invertible) operator. The $\alpha$-power mean of $A$ and $B$ introduced by Kubo-Ando [18] is given by

$$A \sharp_\alpha B = A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}} \quad \text{for } 0 \leq \alpha \leq 1.$$

The Furuta inequality [6] can be written by the form of $\alpha$-power mean as follows ([2],[3],[12],[13],[14]).

**Furuta inequality:** *If $A \geq B \geq 0$, then*

$$\text{(F)} \qquad A^u \sharp_{\frac{1-u}{p-u}} B^p \leq A \quad \text{and} \quad B \leq B^u \sharp_{\frac{1-u}{p-u}} A^p$$

*holds for $u \leq 0$ and $1 \leq p$.*

It is a marvelous extension of the Löwner-Heinz inequality:

$$\text{(LH)} \qquad If \quad A \geq B \geq 0, \quad \text{then} \quad A^\alpha \geq B^\alpha \quad \text{for} \quad 0 \leq \alpha \leq 1.$$

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As shown in [12](cf.[7]), we can arrange (F) in one line as a satellite theorem of the the Furuta inequality as follows:

*If $A \geq B \geq 0$, then*

$$A^u \sharp_{\frac{1-u}{p-u}} B^p \leq B \leq A \leq B^u \sharp_{\frac{1-u}{p-u}} A^p \tag{SF}$$

*holds for all $u \leq 0$ and $p \geq 1$.*

For $A,\ B > 0$, we denote by $A \gg B$ if $\log A \geq \log B$ and call it the chaotic order ([3],[16],[17]). The next characterization of the chaotic order we obtained in [3] is usefull and starting point of our following discussions, so we call it chaotic Furuta inequality.

*If $A \gg B$, then*

$$A^u \sharp_{\frac{-u}{p-u}} B^p \leq I \leq B^u \sharp_{\frac{-u}{p-u}} A^p \tag{CF}$$

*for any $p \geq 0$ and $u \leq 0$.*

A satellite theorem (SF) of the Furuta inequality (F) illustrates the chaotic order. As a matter of fact, we have the following ([16],[17]). In other words, it shows the difference between the usual order $A \geq B$ and the chaotic order $A \gg B$.

*If $A \gg B$, then*

$$A^u \sharp_{\frac{1-u}{p-u}} B^p \leq B \quad \text{and} \quad A \leq B^u \sharp_{\frac{1-u}{p-u}} A^p \tag{SCF}$$

*holds for any $p \geq 1$ and $u \leq 0$.*

We had generalized (CF) and (SCF) more as follows [16]:

**Theorem A.** *For $A,\ B > 0$, if $A \gg B$, then the following (1) and (2) hold.*

$$A^u \sharp_{\frac{\delta-u}{p-u}} B^p \leq B^\delta \quad \text{and} \quad A^\delta \leq B^u \sharp_{\frac{\delta-u}{p-u}} A^p \quad \text{for} \quad u \leq 0 \quad \text{and} \quad 0 \leq \delta \leq p \tag{1}$$

$$A^u \sharp_{\frac{\alpha-u}{p-u}} B^p \leq A^\alpha \text{ and } B^\alpha \leq B^u \sharp_{\frac{\alpha-u}{p-u}} A^p \quad \text{for} \quad u \leq \alpha \leq 0 \quad \text{and} \quad 0 \leq p. \tag{2}$$

## 2. Grand Furuta inequality

As a generalization of the Furuta inequality, Furuta [8] had given an inequality which we called the grand Furuta inequality in [4],[5] and [15]. It interpolates the Furuta inequality and the Ando-Hiai inequality [1] equivalent to the main result of log majorization. We here cite it in terms of operator mean:

**The grand Furuta inequality:** *If $A \geq B \geq 0$ and $A$ is invertible, then for each $1 \leq p$ and $0 \leq t \leq 1$,*

$$\text{(GF)} \qquad A^{-r+t} \sharp_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p) \leq A \quad \text{and} \quad B \leq B^{-r+t} \sharp_{\frac{1-t+r}{(p-t)s+r}} (B^t \natural_s A^p)$$

*holds for $t \leq r$ and $1 \leq s$.*

The best possibility of the power $\frac{1-t+r}{(p-t)s+r}$ is shown in [19]. Replacing $s$ in (GF) with $\frac{\beta-t}{p-t}$ for $1 \leq p \leq \beta$, we can state this theorem by the satellite form as follows [15]:

*If $A \geq B > 0$, then the following* (SGF) *holds for $0 \leq t \leq 1 \leq p \leq \beta$ and $u \leq 0$.*

$$A^u \sharp_{\frac{1-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \leq (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{1}{\beta}} \leq B \leq A \leq (B^t \natural_{\frac{\beta-t}{p-t}} A^p)^{\frac{1}{\beta}} \leq B^u \sharp_{\frac{1-u}{\beta-u}} (B^t \natural_{\frac{\beta-t}{p-t}} A^p).$$

The middle part in (SGF) had been obtained in [4], [5] as the key to (GF). We now present an improvement of this, which will be needed in the below:

**Theorem 1.** *Let $A \geq B > 0$ and $0 \leq t \leq 1 \leq p$. Then*

$$H(\beta) = (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{p}{\beta}}$$

*is a decreasing function with $\beta \geq p$ and in particular $H(\beta) \leq B^p$ for $\beta \geq p$.*

**Proof.** First of all, suppose that $1 \leq \frac{\beta-t}{p-t} \leq 2$. Then

$$A^t \natural_{\frac{\beta-t}{p-t}} B^p = B^p \natural_{\frac{p-\beta}{p-t}} A^t = B^p(B^{-p} \sharp_{\frac{\beta-p}{p-t}} A^{-t})B^p \leq B^p(B^{-p} \sharp_{\frac{\beta-p}{p-t}} B^{-t})B^p = B^\beta$$

By (LH), we have $(A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{p}{\beta}} \leq B^p$.

Next we assume that $H(\beta) \leq B^p$ for a given $\beta \geq p$. Since $p \geq 1$, we have $B_1 = (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{1}{\beta}} \leq B \leq A$. If we take $\beta_1$ with $1 \leq \frac{\beta_1-t}{\beta-t} \leq 2$, then the preceding argument ensures that

$$A^t \natural_{\frac{\beta_1-t}{p-t}} B^p = A^t \natural_{\frac{\beta_1-t}{\beta-t}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) = A^t \natural_{\frac{\beta_1-t}{\beta-t}} B_1^\beta \leq B_1^{\beta_1},$$

that is, $A^t \natural_{\frac{\beta_1-t}{p-t}} B^p \leq (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{\beta_1}{\beta}}$. So it follows from (LH) that

$$(A^t \natural_{\frac{\beta_1-t}{p-t}} B^p)^{\frac{p}{\beta_1}} \leq (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{p}{\beta}} \leq B^p,$$

which shows the monotonicity of $H(\beta)$.

## 3. Furuta's generalization of Uchiyama's theorem

Recently, Uchiyama [20] has shown the following inequality as an extension of (GF).

*If $A \geq B \geq C > 0$, then for each $0 \leq t \leq 1 \leq p$*

$$\text{(U)} \qquad A^{1-t} \geq A^{-r} \sharp_{\frac{1-t+r}{(p-t)s+r}} (B^{-\frac{t}{2}} C^p B^{-\frac{t}{2}})^s$$

*holds for* $r \geq t$ *and* $s \geq 1$.

Related to this, we proposed the following inequality in [5] because (U) seems to be a skewed form of (SGF).

*If* $A,\ B,\ C > 0$ *satisfy* $A \gg B$ *and* $B \geq C$*, then for each* $0 \leq t \leq 1$

$$B \geq C \geq (B^t \natural_s C^p)^{\frac{1}{(p-t)s+t}} \geq A^{-r+t} \sharp_{\frac{1+r-t}{(p-t)s+r}} (B^t \natural_s C^p)$$

*holds for all* $p \geq 1,\ s \geq 1$ *and* $r \geq t$.

In this inequality, if $A \geq B = C$, then we have (F) and if $A = B \geq C$, then (GF) is obtained.

Very recently, Furuta [9] improved (U), see [10] and [11], which will be seen as Theorem 4 in the below. We now investigate Furuta's theorem from the view point of the chaotic order.

**Theorem 2.** *For fixed* $A,\ B,\ C > 0$ *and* $0 \leq t \leq 1 \leq p$*, if* $A \gg D = (B^{-\frac{t}{2}} C^p B^{-\frac{t}{2}})^{\frac{1}{p-t}}$ *is satisfied, then* (1) *holds for* $\beta \geq p$ *and* $r \geq t$.

$$(1) \qquad B^{\frac{t}{2}} A^{-t} B^{\frac{t}{2}} \sharp_{\frac{1}{\beta}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p) \geq B^{\frac{t}{2}} A^{-r} B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{\beta-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p)$$

*Additionally, if* $A \geq B$*, then* (2) *holds.*

$$(2) \qquad B^{\frac{t}{2}} A^{-r} B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{p-t+r}} C^p \geq B^{\frac{t}{2}} A^{-r} B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{\beta-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p)$$

**Proof.** Since $A \gg D$, (CF) implies that

$$(\dagger) \qquad (A^{\frac{t}{2}} D^{\beta-t} A^{\frac{t}{2}})^{\frac{t}{\beta}} \leq A^t$$

and so $(A^{\frac{t}{2}} D^{\beta-t} A^{\frac{t}{2}})^{\frac{1}{\beta}} \ll A$. Therefore it follows from (SCF) that

$$A^{-r+t} \sharp_{\frac{1-(t-r)}{\beta-(t-r)}} \{(A^{\frac{t}{2}} D^{\beta-t} A^{\frac{t}{2}})^{\frac{1}{\beta}}\}^{\beta} \leq (A^{\frac{t}{2}} D^{\beta-t} A^{\frac{t}{2}})^{\frac{1}{\beta}},$$

namely

$$A^{-r} \sharp_{\frac{1-t+r}{\beta-t+r}} D^{\beta-t} \leq A^{-t} \sharp_{\frac{1}{\beta}} D^{\beta-t}.$$

Since $B^{\frac{t}{2}} D^{\beta-t} B^{\frac{t}{2}} = B^t \natural_{\frac{\beta-t}{p-t}} C^p$, we have (1) by multiplying $B^{\frac{t}{2}}$ on both sides.

(2) is also shown as follows: Since $A^t \gg (A^{\frac{t}{2}} D^{\beta-t} A^{\frac{t}{2}})^{\frac{t}{\beta}}$ as in above, Theorem A (1) implies that

$$(A^t)^{-\frac{r-t}{t}} \sharp_{\frac{\frac{p}{t}+\frac{r-t}{t}}{\frac{\beta}{t}+\frac{r-t}{t}}} (A^{\frac{t}{2}} D^{\beta-t} A^{\frac{t}{2}})^{\frac{t}{\beta}\frac{\beta}{t}} \leq (A^{\frac{t}{2}} D^{\beta-t} A^{\frac{t}{2}})^{\frac{t}{\beta}\frac{p}{t}},$$

that is,

$$A^{-r+t} \sharp_{\frac{p-t+r}{\beta-t+r}} A^{\frac{t}{2}} D^{\beta-t} A^{\frac{t}{2}} \leq (A^{\frac{t}{2}} D^{\beta-t} A^{\frac{t}{2}})^{\frac{p}{\beta}}.$$

Multiplying $A^{-\frac{t}{2}}$ from the both sides of the above, we have

$$A^{-r} \sharp_{\frac{p+r-t}{\beta+r-t}} D^{\beta-t} \leq A^{-t} \sharp_{\frac{p}{\beta}} D^{\beta-t} \leq B^{-t} \sharp_{\frac{p}{\beta}} D^{\beta-t} = B^{-\frac{t}{2}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p)^{\frac{p}{\beta}} B^{-\frac{t}{2}} \leq B^{-\frac{t}{2}} C^p B^{-\frac{t}{2}},$$

where the final inequality follows from Theorem 1. Again multiplying $B^{\frac{t}{2}}$ to each sides of this formula, we have

$$B^{\frac{t}{2}}A^{-r}B^{\frac{t}{2}} \sharp_{\frac{p-t+r}{\beta-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p) \leq C^p.$$

Hence it follows that

$$\begin{aligned} & B^{\frac{t}{2}}A^{-r}B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{\beta-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p) \\ = \;& B^{\frac{t}{2}}A^{-r}B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{p-t+r}} \{B^{\frac{t}{2}}A^{-r}B^{\frac{t}{2}} \sharp_{\frac{p-t+r}{\beta-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p)\} \\ \leq \;& B^{\frac{t}{2}}A^{-r}B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{p-t+r}} C^p. \end{aligned}$$

Next we show similar ineqalities as applications of Theorem A.

**Theorem 3.** *If $A,\ B,\ C > 0$ satisfy $A \gg D = (B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^{\frac{1}{p-t}}$ for some $0 \leq t \leq 1 \leq p$, the the following* (1) *and* (2) *hold for $\beta \geq p$ and $r \geq t$.*

(1) $B^t \sharp_{\frac{1-t}{p-t}} C^p \geq B^{\frac{t}{2}}A^{-t}B^{\frac{t}{2}} \sharp_{\frac{1}{\beta}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p) \geq B^{\frac{t}{2}}A^{-r}B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{\beta-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p)$

(2)
$B^t \sharp_{\frac{1-t}{p-t}} C^p \geq B^{\frac{t}{2}}A^{-t}B^{\frac{t}{2}} \sharp_{\frac{1}{p}} C^p \geq B^{\frac{t}{2}}A^{-r}B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{p-t+r}} C^p \geq B^{\frac{t}{2}}A^{-r}B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{\beta-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p)$

**Proof.** (1) follows from Theorem A (2) and (1). Actually we have

$$\begin{aligned} & A^{-r} \sharp_{\frac{1-t+r}{\beta-t+r}} D^{\beta-t} = D^{\beta-t} \sharp_{\frac{\beta-1}{\beta-t+r}} A^{-r} \\ = \;& D^{\beta-t} \sharp_{\frac{\beta-1}{\beta}} \{D^{\beta-t} \sharp_{\frac{\beta}{\beta-t+r}} A^{-r}\} \\ = \;& D^{\beta-t} \sharp_{\frac{\beta-1}{\beta}} \{A^{-r} \sharp_{\frac{-t+r}{\beta-t+r}} D^{\beta-t}\} \\ \leq \;& D^{\beta-t} \sharp_{\frac{\beta-1}{\beta}} A^{-t} = A^{-t} \sharp_{\frac{1}{\beta}} D^{\beta-t} \\ = \;& A^{-t} \sharp_{\frac{1-t+t}{\beta-t+t}} (B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^{\frac{\beta-t}{p-t}} \leq (B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^{\frac{1-t}{p-t}} \end{aligned}$$

So the conclusion is obtained by multiplying $B^{\frac{t}{2}}$ both sides of each term.

In addition, (2) except the final part is obtained by taking $\beta = p$ in (1). Moreover we have

$$A^{-r} \sharp_{\frac{1-t+r}{\beta-t+r}} D^{\beta-t} = A^{-r} \sharp_{\frac{1-t+r}{p-t+r}} \{A^{-r} \sharp_{\frac{p-t+r}{\beta-t+r}} D^{\beta-t}\} \leq A^{-r} \sharp_{\frac{1-t+r}{p-t+r}} D^{p-t}$$

by Theorem A (2). Multiplying $B^{\frac{t}{2}}$ to each term from both sides, we attain the conclusion.

The following theorem is due to Furuta [9], whose expression is presented by replacing $s$ with $\frac{\beta-t}{p-t}$ for $\beta \geq p$. Based on the discussion of Theorem 2 and 3, we give it a simplified proof a bit.

**Theorm 4 (Furuta).** *If $A \geq B \geq C > 0$ and $0 \leq t \leq 1 \leq p$, then*

(1)
$B \geq C \geq (B^t \natural_{\frac{\beta-t}{p-t}} C^p)^{\frac{1}{\beta}} \geq B^{\frac{t}{2}}A^{-t}B^{\frac{t}{2}} \sharp_{\frac{1}{\beta}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p) \geq B^{\frac{t}{2}}A^{-r}B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{\beta-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p)$

*and*

(2) $$B \geq C \geq B^{\frac{t}{2}} A^{-r} B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{p-t+r}} C^p \geq B^{\frac{t}{2}} A^{-r} B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{\beta-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p)$$

*hold for $\beta \geq p$ and $r \geq t$.*

**Proof.** First of all, the assumption $B \geq C > 0$ ensures $(B^t \natural_{\frac{\beta-t}{p-t}} C^p)^{\frac{1}{\beta}} \leq B$ by (SGF). As in the proof of Theorem 2, (†) is the essential point, which is shown as follows:

Let $D = (B^{-\frac{t}{2}} C^p B^{-\frac{t}{2}})^{\frac{1}{p-t}}$ be as in Theorem 2. Then

$$A^{-t} \sharp_{\frac{t}{\beta}} D^{\beta-t} \leq B^{-t} \sharp_{\frac{t}{\beta}} D^{\beta-t} = B^{-\frac{t}{2}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p)^{\frac{t}{\beta}} B^{-\frac{t}{2}} \leq B^{-\frac{t}{2}} B^t B^{-\frac{t}{2}} = I.$$

Since (†) is shown, (1) connects with Theorem 2 (1). Namely we have

$$\begin{aligned} & B^{\frac{t}{2}} A^{-r} B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{\beta-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p) \leq B^{\frac{t}{2}} A^{-t} B^{\frac{t}{2}} \sharp_{\frac{1}{\beta}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p) \\ \leq \quad & B^{\frac{t}{2}} B^{-t} B^{\frac{t}{2}} \sharp_{\frac{1}{\beta}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p) = (B^t \natural_{\frac{\beta-t}{p-t}} C^p)^{\frac{1}{\beta}} \leq C \leq B. \end{aligned}$$

Next we show (2). For this, we have only to check $B^{\frac{t}{2}} A^{-r} B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{p-t+r}} C^p \leq C \leq B$ by (†) and Theorem 2 (2). Fortunately, it is obtained by taking $\beta = p$ in the former (1).

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