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ESTIMATIONS OF REVERSE INEQUALITIES FOR CONVEX FUNCTIONS
– OPERATOR INEQUALITY DERIVED FROM QUASI-ARITHMETIC MEAN –

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Abstract. We show some order relations between a generalized quasi-arithmetic mean and the arithmetic mean. First we give an estimation $F(\lambda)$ of the following difference:

$$g(\langle f(A)x, x\rangle) - \lambda\langle Ax, x\rangle$$

for $\lambda > 0$ and $\| x \| = 1$, where $f(t)$, $g(t)$ are functions with some conditions and A is a self-adjoint operator. Next for self-adjoint operators A_i we show order relations:

$$K_{f,g}(p)\sum_{i=1}^{n}\lambda_i A_i + M_{f,g}(p) \leq g\left(\sum_{i=1}^{n}\lambda_i f(A_i)\right) \leq L_{f,g}(q)\sum_{i=1}^{n}\lambda_i A_i + N_{f,g}(q)$$

under some conditions. Moreover, we give ratio and difference inequalities from the above inequalities. As applications, we show that some constants e.g. Ky Fan-Furuta constant, logarithmic mean and Specht ratio play an important role in their estimations.

1. Introduction

In this note, an operator means a bounded linear operator acting on a Hilbert space H. For a convex (resp. concave) function $f(t)$, the Jensen inequality asserts

$$f(\langle Ax, x\rangle) \leq \langle f(A)x, x\rangle \quad (\text{resp. } f(\langle Ax, x\rangle) \geq \langle f(A)x, x\rangle)$$

for each unit vector x and all self-adjoint operators A. In [12], we presented reverse Jensen inequalities by Mond-Pečarić method [13], [14]. In this note, by its method we give an estimate $F(\lambda)$ of a difference $g(\langle f(A)x, x\rangle) - \lambda\langle Ax, x\rangle$ for all $\lambda > 0$ and unit vectors $x \in H$, where $f(t)$ is a real valued continuous strictly convex function, $g(t)$ is a real valued increasing function and A is a self-adjoint operator with $m \leq A \leq M$. Moreover putting $g(t) = f^{-1}(t)$, we give an explicit estimate $F_f(\lambda)$ of the following inequality:

$$f^{-1}(\langle f(A)x, x\rangle) - \lambda\langle Ax, x\rangle \leq F_f(\lambda) \tag{2.4}$$

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Next for $\lambda_1, \lambda_2, \ldots, \lambda_n \in [0,1]$ with $\sum_{i=1}^n \lambda_i = 1$ and $t_1, t_2, \ldots, t_n \in \mathbb{R}$. we recall that the quasi-arithmetic mean $f^{-1}\left(\sum_{i=1}^n \lambda_i f(t_i)\right)$ has the following inequality:

$$f^{-1}\left(\sum_{i=1}^n \lambda_i f(t_i)\right) \geq \sum_{i=1}^n \lambda_i t_i \quad \left(\text{resp. } f^{-1}\left(\sum_{i=1}^n \lambda_i f(t_i)\right) \leq \sum_{i=1}^n \lambda_i t_i\right), \tag{1.1}$$

where the function $f(t)$ is a strictly convex (resp. strictly concave) function on $\mathbb{R}$. We give an operator inequality representing estimations of $g\left(\sum_{i=1}^n \lambda_i f(A_i)\right)$ by $\sum_{i=1}^n \lambda_i A_i$ as an extension of (1.1)

$$K_{f,g}(p)\sum_{i=1}^n \lambda_i A_i + M_{f,g}(p) \leq g\left(\sum_{i=1}^n \lambda_i f(A_i)\right) \leq L_{f,g}(q)\sum_{i=1}^n \lambda_i A_i + N_{f,g}(q), \tag{2.7}$$

where self-adjoint operators A_i with $m \leq A_i \leq M$ and some conditions of Theorem 2.4 (for a convex function $f(t)$). Here constants $K_{f,g}(p)$, $L_{f,g}(q)$, $M_{f,g}(p)$ and $N_{f,g}(q)$ represent estimations. Moreover we consider a complementary quasi-arithmetic mean operator inequality by putting $g(t) = f^{-1}(t)$ in (2.7).

Furthermore we investigate the estimations of ratio and difference inequalities derived from (2.4) and (2.7).

As applications, we give explicit expressions of (2.4) and (2.7) for some typical functions (power, exponential, logarithmic or entropy function). Then obtained inequalities have well-known constants as values of estimations. In the case of the power function, we see that ratio and difference inequalities are estimated by the Ky Fan-Furuta constant $K(h,u)$ and the constant $C(m,M,u)$ defined as follows, respectively:

$$K(h,u) := \frac{h^u - h}{(u-1)(h-1)}\left(\frac{u-1}{u}\frac{h^u-1}{h^u-h}\right)^u \tag{1.2}$$

$$C(m,M,u) := \frac{Mm^u - mM^u}{M-m} + (u-1)\left(\frac{M^u - m^u}{u(M-m)}\right)^{\frac{u}{u-1}}. \tag{1.3}$$

These constants are related to reverse ratio and difference Hölder-McCarthy inequalities, respectively [1], [7], [10], [11], [16]. Moreover, in the case of exponential, logarithmic or entropy function, we see that ratio and difference inequalities are estimated by the logarithmic mean $L(m,M)$ [9] and the Specht ratio $S(h)$ [3], [15], [17] defined as follows:

$$L(m,M) := \frac{M-m}{\log M - \log m} \quad \text{and} \quad S(h) := \frac{h^{\frac{1}{h-1}}}{e\log h^{\frac{1}{h-1}}}, \tag{1.4}$$

where a generalized condition number $h = \frac{M}{m} (> 1)$ [20]. We generalize the Specht ratio to two valuable version in Corollary 3.3.

We remark that this note is based on preprints [18], [19].

2. Main Theorems

In this section, we first consider an estimation of $g(\langle f(A)x, x\rangle)$ by $\langle Ax, x\rangle$ under the condition of Theorem 2.1. To do it we define α_f and β_f for each real valued continuous function $f(t)$ on $[m, M]$ as follows:

$$\alpha_f := \frac{f(M) - f(m)}{M - m} \quad \text{and} \quad \beta_f := \frac{Mf(m) - mf(M)}{M - m}.$$

Theorem 2.1. *Let A be a self-adjoint operator on H with $m \le A \le M$. Let $f(t)$ be a real valued continuous convex function on $[m, M]$ and let $g(t)$ be a real valued increasing function on the range $f([m, M])$. Then for each $\lambda > 0$*

$$g(\langle f(A)x, x\rangle) - \lambda\langle Ax, x\rangle \le \max_{m\le t\le M}\{g(\alpha_f t + \beta_f) - \lambda t\} \tag{2.1}$$

holds for all unit vectors $x \in H$.

Remark 2.2. *Suppose the hypothesis of Theorem 2.1. We consider the Hilbert space $\mathbf{H} := H \oplus H \oplus \cdots \oplus H$ and the operator $\mathbf{A} := A_1 \oplus A_2 \oplus \cdots \oplus A_n$. Then we have for $\mathbf{x} := x_1 \oplus x_2 \oplus \cdots \oplus x_n$, $\| \mathbf{x} \|^2 = \sum_{i=1}^n \| x_i \|^2$, $\langle \mathbf{A}\mathbf{x}, \mathbf{x}\rangle = \sum_{i=1}^n \langle A_i x_i, x_i\rangle$ and $g(\langle f(\mathbf{A}\mathbf{x}, \mathbf{x}\rangle) = g(\sum_{i=1}^n \langle f(A_i x_i, x_i\rangle)$. Hence Theorem 2.1 implies that the following inequality holds*:

$$g\left(\sum_{i=1}^n \langle f(A_i)x_i, x_i\rangle\right) - \lambda\sum_{i=1}^n \langle A_i x_i, x_i\rangle \le \max_{m\le t\le M}\{g(\alpha_f t + \beta_f) - \lambda t\} \tag{2.2}$$

Here we define the function $F_f(\lambda)$ for $\lambda \in I_f$:

$$F_f(\lambda) := \mu_\lambda - \frac{f(\mu_\lambda) - \beta_f}{\alpha_f}\lambda, \tag{2.3}$$

where $\mu = \mu_\lambda \in [m, M]$ is a unique solution of the equation $f'(\mu) = \frac{\alpha_f}{\lambda}$ under the condition of Theorem 2.3. By the more consideration the equation $F_f(\lambda) = 0$ has a unique solution $\lambda = \lambda_f \in I_f$. The following theorem gives an estimation of the difference $f^{-1}(\langle f(A)x, x\rangle) - \lambda\langle Ax, x\rangle$:

Theorem 2.3. *Let A be a self-adjoint operator on H with $m \le A \le M$. Let $f(t)$ be a real valued continuous increasing strictly convex and twice differentiable function on $[m, M]$. Put $I_f := \left[\frac{\alpha_f}{f'(M)}, \frac{\alpha_f}{f'(m)}\right]$. Then for each $\lambda > 0$*

$$f^{-1}(\langle f(A)x, x\rangle) - \lambda\langle Ax, x\rangle \le \begin{cases} (1-\lambda)M & \text{if } 0 < \lambda < \frac{\alpha_f}{f'(M)} \\ F_f(\lambda) & \text{if } \lambda \in I_f \\ (1-\lambda)m & \text{if } \lambda > \frac{\alpha_f}{f'(m)} \end{cases} \tag{2.4}$$

holds for all unit vectors $x \in H$.

In particular,

(i) *The ratio inequality*

$$f^{-1}(\langle f(A)x, x\rangle) \le \lambda_f\langle Ax, x\rangle \tag{2.5}$$

holds, where $\lambda = \lambda_f (\in I_f)$ is a unique solution of the equation $F_f(\lambda) = 0$.

(ii) *The difference inequality*

$$f^{-1}(\langle f(A)x, x\rangle) - \langle Ax, x\rangle \le \mu_1 - \frac{f(\mu_1) - \beta_f}{\alpha_f} \tag{2.6}$$

holds, where $\mu = \mu_1 \in (m, M)$ is a unique solution of the equation $f'(\mu) = \alpha_f$.

Next we estimate lower and upper bounds of $g\left(\sum_{i=1}^n \lambda_i f(A_i)\right)$ by $\sum_{i=1}^n \lambda_i A_i$ under the condition of Theorem 2.4. Here if $f(t)$ is a real valued function on $[m, M]$ and $g(t)$ is a real valued function on $[f_{\min}, f_{\max}]$, we denote α_f, β_f, $\alpha_{f,g}$ and $\beta_{f,g}$ as follows:

$$\alpha_f := \frac{f(M)-f(m)}{M-m}, \qquad \beta_f := \frac{Mf(m)-mf(M)}{M-m},$$
$$\alpha_{f,g} := \frac{g(f_{\max})-g(f_{\min})}{f_{\max}-f_{\min}}, \qquad \beta_{f,g} := \frac{f_{\max}g(f_{\min})-f_{\min}g(f_{\max})}{f_{\max}-f_{\min}},$$

where $f_{\min} := \min_{m\le k\le M} f(k)$ and $f_{\max} := \max_{m\le k\le M} f(k)$.

Theorem 2.4. *Let A_i be self-adjoint operators on H with $m \le A_i \le M$ for $i = 1, 2, \ldots, n$. Let $f(t)$ be a real valued convex differentiable function on $\mathbb{R}$ and let $g(t)$ be a real valued increasing concave differentiable function on $[f_{\min}, f_{\max}]$. Suppose $p \in [m, M]$, $q \in \mathbb{R}$ with $f(q) \in [f_{\min}, f_{\max}]$. Then for each weight $\lambda_1, \lambda_2, \ldots, \lambda_n$, the inequality*

$$K_{f,g}(p)\sum_{i=1}^n \lambda_i A_i + M_{f,g}(p) \le g\left(\sum_{i=1}^n \lambda_i f(A_i)\right) \le L_{f,g}(q)\sum_{i=1}^n \lambda_i A_i + N_{f,g}(q) \tag{2.7}$$

holds, where $K_{f,g}(p)$, $L_{f,g}(q)$, $M_{f,g}(p)$ and $N_{f,g}(q)$ are defined as follows:

$$K_{f,g}(p) := \alpha_{f,g} f'(p), \qquad M_{f,g}(p) := \alpha_{f,g}(f(p)-pf'(p)) + \beta_{f,g},$$
$$L_{f,g}(q) := \alpha_f g'(f(q)), \qquad N_{f,g}(q) := g'(f(q))(\beta_f - f(q)) + g(f(q)).$$

Remark 2.5. *Let $p \notin [m, M]$ and $f(q) \notin [f_{\min}, f_{\max}]$ in Theorem 2.4. The inequality* (2.7) *holds, where $p = p_0, q = q_0 \in \mathbb{R}$ are given as follows*:

$$p_0 := \begin{cases} m & \text{if } p < m \\ M & \text{if } M < p \end{cases} \quad \text{and} \quad f(q_0) := \begin{cases} f_{\min} & \text{if } f(q) < f_{\min} \\ f_{\max} & \text{if } f_{\max} < f(q). \end{cases}$$

We estimate lower and upper bounds of the quasi-arithmetic mean $f^{-1}\left(\sum_{i=1}^n \lambda_i f(A_i)\right)$ by the arithmetic mean $\sum_{i=1}^n \lambda_i A_i$.

Theorem 2.6. *Let A_i be self-adjoint operators on H with $m \le A_i \le M$ for $i = 1, 2, \ldots, n$. Let $f(t)$ be a real valued increasing strictly convex differentiable function on $\mathbb{R}$. Suppose $p, q \in [m, M]$. Then for each weight $\lambda_1, \lambda_2, \ldots, \lambda_n$, the inequality*

$$K_f(p)\sum_{i=1}^n \lambda_i A_i + M_f(p) \le f^{-1}\left(\sum_{i=1}^n \lambda_i f(A_i)\right) \le L_f(q)\sum_{i=1}^n \lambda_i A_i + N_f(q) \tag{2.8}$$

holds, where $K_f(p)$, $L_f(q)$, $M_f(p)$ and $N_f(q)$ are defined as follows:

$$K_f(p) := \alpha_f^{-1} f'(p), \qquad M_f(p) := \alpha_f^{-1}(f(p)-pf'(p)) - \alpha_f^{-1}\beta_f,$$
$$L_f(q) := \alpha_f {f^{-1}}'(f(q)), \qquad N_f(q) := {f^{-1}}'(f(q))(\beta_f - f(q)) + q.$$

In particular,

(i) *The ratio inequality*

$$K_f(p_f)\sum_{i=1}^n \lambda_i A_i \le f^{-1}\left(\sum_{i=1}^n \lambda_i f(A_i)\right) \le L_f(q_f)\sum_{i=1}^n \lambda_i A_i \tag{2.9}$$

holds, where $p = p_f \in (m, M)$ (under the condition $m > 0$ or $M < 0$) and $q = q_f \in (m, M)$ are unique solutions of equations $M_f(p) = 0$ and $N_f(q) = 0$, respectively.
(ii) *The difference inequality*

$$M_f(p_f) \le f^{-1}\left(\sum_{i=1}^{n} \lambda_i f(A_i)\right) - \sum_{i=1}^{n} \lambda_i A_i \le N_f(q_f) \tag{2.10}$$

holds, where $p = p_f \in (m, M)$ and $q = q_f \in (m, M)$ are unique solutions of equations $K_f(p) = 1$ and $L_f(q) = 1$, respectively.

If $f(t)$ is a strictly concave function in theorems of this section, then we have each reverse order inequalities.

3. Applications to some typical functions

In this section, first as application of Theorems 2.1 and 2.3, we consider the explicit estimations for some typical functions (in the case $\lambda \in I_f$).

We give an upper bound of the power mean $\langle A^p x, x\rangle^{1/p}$ to the arithmetic mean $\langle Ax, x\rangle$ $(p > 1)$ as follows:

Theorem 3.1. *Let A be a positive operator on H with $0 < m \le A \le M$. Suppose $p, q > 1$ with $1/p + 1/q = 1$. Put $\alpha_{\text{pow}} := \frac{M^p - m^p}{M - m}$ and $I_{\text{pow}} := \left[\frac{\alpha_{\text{pow}}}{pM^{p-1}}, \frac{\alpha_{\text{pow}}}{pm^{p-1}}\right]$. Then for each $\lambda \in I_{\text{pow}}$, the inequality*

$$\langle A^p x, x\rangle^{1/p} - \lambda\langle Ax, x\rangle \le F_{\text{pow}}(\lambda) \tag{3.1}$$

holds for all unit vectors $x \in H$, where $F_{\text{pow}}(\lambda) := \frac{1}{q}\left(\frac{\alpha_{\text{pow}}}{p\lambda}\right)^{q-1} + \frac{Mm^p - mM^p}{M^p - m^p}\lambda$.

In particular,
(i) *The ratio inequality*

$$\langle A^p x, x\rangle^{1/p} \le \lambda_{\text{pow}}\langle Ax, x\rangle \tag{3.2}$$

holds, where $\lambda = \lambda_{\text{pow}} := \frac{h^p - 1}{p^{1/p} q^{1/q} (h-1)^{1/p} (h^p - h)^{1/q}} (\in I_{\text{pow}})$ is a solution of an equation $F_{\text{pow}}(\lambda) = 0$.
(ii) *The following difference inequality holds*

$$\langle A^p x, x\rangle^{1/p} - \langle Ax, x\rangle \le \frac{1}{q}\left(\frac{\alpha_{\text{pow}}}{p}\right)^{q-1} + \frac{Mm^p - mM^p}{M^p - m^p}. \tag{3.3}$$

Inequalities in Theorem 3.1 were obtained in our previous note [16]. In particular (3.3) is also given in [10]. The constant λ_{pow} coincides with the p-th root of the Ky Fan-Furuta constant $K(h, p)$ defined by (1.2), i.e., $\lambda_{\text{pow}} = K(h, p)^{1/p}$. Let $f(t) = t^p$ $(0 < p < 1,\ t > 0)$ in Theorem 3.1. Then we have the reverse order inequalities of obtained ones.

In the following theorem, we consider applications to the exponential function:

Theorem 3.2. *Let A be a self-adjoint operator on H with $m \le A \le M$. Put $\alpha_{\exp} := \frac{e^M - e^m}{M - m}$ and $I_{\exp} := \left[\frac{\alpha_{\exp}}{e^M}, \frac{\alpha_{\exp}}{e^m}\right]$. Then for each $\lambda \in I_{\exp}$, the inequality*

$$\log\langle e^A x, x\rangle - \lambda\langle Ax, x\rangle \le F_{\exp}(\lambda) \tag{3.4}$$

holds for all unit vectors $x \in H$, *where* $F_{\exp}(\lambda) := \log \frac{\alpha_{\exp}}{\lambda e} + \frac{Me^m - me^M}{e^M - e^m}\lambda$.

In particular,

(i) *The ratio inequality*

$$\log\langle e^A x, x\rangle \le \lambda_{\exp}\langle Ax, x\rangle \tag{3.5}$$

holds, where $\lambda = \lambda_{\exp}(\in I_{\exp})$ *is a solution of an equation* $F_{\exp}(\lambda) = 0$.

(ii) *The following difference inequality holds*

$$\log\langle e^A x, x\rangle - \langle Ax, x\rangle \le \log S(e^{M-m}). \tag{3.6}$$

In the following corollary we give reverse inequalities of the inequality $\log\langle Ax, x\rangle \ge \langle(\log A)x, x\rangle$ derived from Theorem 3.2:

Corollary 3.3. *Let* A *be a positive invertible operator on* H *with* $0 < m \le A \le M$, $h = \frac{M}{m} > 1$. *Put* $I_{\log} := \left[L(\frac{1}{h}, 1), L(1, h)\right]$. *Then for each* $\lambda \in I_{\log}$, *the inequality*

$$\log\langle Ax, x\rangle - \lambda\langle(\log A)x, x\rangle \le \log\left\{m^{1-\lambda}S(h, \lambda)\right\} \tag{3.7}$$

holds for all unit vector $x \in H$, *where a two valuable Specht ratio* $S(h, \lambda) = \frac{h^{\frac{\lambda}{h-1}}}{e \log h^{\frac{\lambda}{h-1}}}$.

In particular,

(i) *The ratio inequality*

$$\log\langle Ax, x\rangle \le \lambda_{log}\langle(\log A)x, x\rangle \tag{3.8}$$

holds, where $\lambda = \lambda_{\log}(\in I_{\log})$ *is a solution of an equation* $m^{1-\lambda}S(h, \lambda) = 1$.

(ii) *The following difference inequality holds*

$$\log\langle Ax, x\rangle - \langle(\log A)x, x\rangle \le \log S(h, 1). \tag{3.9}$$

Taking $m = 1$ in (3.8), we have $\lambda_{log} = \frac{M-1}{\log M}$ as a solution of the equation $S(h, \lambda)(= S(M, \lambda)) = 0$. Hence it follows that $\log\langle Ax, x\rangle \le \frac{M-1}{\log M}\langle(\log A)x, x\rangle$.

Here we recall that *determinant* [2], [3] defined as $\exp\langle(\log A)x, x\rangle$ has the inequality

$$\exp\langle(\log A)x, x\rangle \le \langle Ax, x\rangle. \tag{3.10}$$

The following theorem gives a lower bound of $\exp\langle(\log A)x, x\rangle$ by $\langle Ax, x\rangle$ as a reverse inequality of (3.10):

Theorem 3.4. *Let* A *be a positive invertible operator on* H *with* $0 < m \le A \le M$, $h = \frac{M}{m} > 1$. *Put* $J_{\log} := \left[\frac{m}{L(m,M)}, \frac{M}{L(m,M)}\right]$. *Then for each* $\lambda \in J_{\log}$, *the inequality*

$$\exp\langle(\log A)x, x\rangle - \lambda\langle Ax, x\rangle \ge F_{\log}(\lambda) \tag{3.11}$$

holds for all unit vectors $x \in H$, *where* $F_{\log}(\lambda) := -\lambda L(m, M)\log(\lambda S(h))$.

In particular,

(i) *The following ratio inequality holds*

$$\exp\langle(\log A)x, x\rangle \ge S(h)^{-1}\langle Ax, x\rangle. \tag{3.12}$$

(ii) *The following difference inequality holds*

$$\exp\langle(\log A)x, x\rangle - \langle Ax, x\rangle \ge -L(m, M)\log S(h). \tag{3.13}$$

We remark that (3.12) is related to [3]. The inequality (3.13) is given in [2].

Furthermore, we consider the operator convex function $f(t) = t\log t$ which plays an important role in the information and entropy theories. The following theorem represents an upper bound of a difference $\langle (A\log A)x, x\rangle - \lambda\langle Ax, x\rangle \log\langle Ax, x\rangle$ for a given $\lambda > 0$:

Theorem 3.5. *Let A be a positive operator on H with $1 < m \le A \le M$, $h = \frac{M}{m} > 1$. Put $I_{\mathrm{ent}} := \left[\frac{\log\left(Mh^{\frac{1}{h-1}}\right)}{\log(Me)}, \frac{\log\left(Mh^{\frac{1}{h-1}}\right)}{\log(me)}\right]$. Then for each $\lambda \in I_{\mathrm{ent}}$, the inequality*

$$\langle (A\log A)x, x\rangle - \lambda\langle Ax, x\rangle \log\langle Ax, x\rangle \le F_{\mathrm{ent}}(\lambda) \tag{3.14}$$

holds for all unit vectors $x \in H$, where $F_{\mathrm{ent}}(\lambda) := \frac{\lambda}{e}\left(Mh^{\frac{1}{h-1}}\right)^{1/\lambda} - \frac{M}{L(1,h)}$.

In particular,

(i) *The ratio inequality*

$$\langle (A\log A)x, x\rangle \le \lambda_{\mathrm{ent}}\langle Ax, x\rangle \log\langle Ax, x\rangle \tag{3.15}$$

holds, where $\lambda = \lambda_{\mathrm{ent}} (\in I_{\mathrm{ent}})$ is a solution of an equation $F_{\mathrm{ent}}(\lambda) = 0$.

(ii) *The following difference inequality holds*:

$$\langle (A\log A)x, x\rangle - \langle Ax, x\rangle \log\langle Ax, x\rangle \le \frac{M}{L(1,h)}(S(h) - 1). \tag{3.16}$$

Next we consider applications of Theorem 2.4 (and Theorem 2.6) for typical functions.

We estimate lower and upper bounds of $\left(\sum_{i=1}^n \lambda_i {A_i}^u\right)^v$ by $\sum_{i=1}^n \lambda_i A_i$ under the condition of Theorem 3.6. We denote α_u, β_u, $\alpha_{u,v}$ and $\beta_{u,v}$ as follows:

$$\alpha_u := \frac{M^u - m^u}{M - m},\ \beta_u := \frac{Mm^u - mM^u}{M - m},\ \alpha_{u,v} := \frac{M^{uv} - m^{uv}}{M^u - m^u},\ \beta_{u,v} := \frac{M^u m^{uv} - m^u M^{uv}}{M^u - m^u}.$$

Theorem 3.6. *Let A_i be positive operators on H with $0 < m \le A_i \le M$ for $i = 1, 2, \ldots, n$. Let $u \ge 1$ and $0 < v \le 1$. Suppose $p, q \in [m, M]$. Then for each weight $\lambda_1, \lambda_2, \ldots, \lambda_n$, the inequality*

$$K_{u,v}(p)\sum_{i=1}^n \lambda_i A_i + M_{u,v}(p) \le \left(\sum_{i=1}^n \lambda_i {A_i}^u\right)^v \le L_{u,v}(q)\sum_{i=1}^n \lambda_i A_i + N_{u,v}(q) \tag{3.17}$$

holds, where $K_{u,v}(p)$, $L_{u,v}(q)$, $M_{u,v}(p)$ and $N_{u,v}(q)$ are defined as follows:

$$K_{u,v}(p) := u\alpha_{u,v}p^{u-1}, \qquad M_{u,v}(p) := \alpha_{u,v}(p^u - up^u) + \beta_{u,v},$$
$$L_{u,v}(q) := v\alpha_u q^{u(v-1)}, \qquad N_{u,v}(q) := vq^{u(v-1)}(\beta_u - q^u) + q^{uv}.$$

Moreover, we estimate lower and upper bounds of an power mean $\left(\sum_{i=1}^n \lambda_i A_i^u\right)^{1/u}$ by the arithmetic mean $\sum_{i=1}^n \lambda_i A_i$ by Theorem 3.6:

Theorem 3.7. *Let A_i be positive operators on H with $0 < m \le A_i \le M$ for $i = 1, 2, \ldots, n$. Let $u \ge 1$. Suppose $p, q \in [m, M]$. Then for each weight $\lambda_1, \lambda_2, \ldots, \lambda_n$,* *the inequality*

$$K_u(p)\sum_{i=1}^n \lambda_i A_i + M_u(p) \le \left(\sum_{i=1}^n \lambda_i A_i^u\right)^{1/u} \le L_u(q)\sum_{i=1}^n \lambda_i A_i + N_u(q) \tag{3.18}$$

holds, where $K_u(p)$, $L_u(q)$, $M_u(p)$ *and* $N_u(q)$ *are defined as follows*:

$$K_u(p) := u\alpha_u^{-1}p^{u-1}, \qquad M_u(p) := -\alpha_u^{-1}(up^u + \beta_u - p^u),$$

$$L_u(q) := u^{-1}\alpha_u q^{1-u} \ (= K_u(q)^{-1}), \qquad N_u(q) := u^{-1}q^{1-u}(uq^u + \beta_u - q^u).$$

In particular,

(i) *The ratio inequality*

$$K_u\sum_{i=1}^n \lambda_i A_i \le \left(\sum_{i=1}^n \lambda_i A_i^u\right)^{1/u} \le K_u^{-1}\sum_{i=1}^n \lambda_i A_i \tag{3.19}$$

holds, where $K_u := \frac{u}{\alpha_u}\left(\frac{\beta_u}{1-u}\right)^{1-\frac{1}{u}}$.

(ii) *The difference inequality*

$$M_u \le \left(\sum_{i=1}^n \lambda_i A_i^u\right)^{1/u} - \sum_{i=1}^n \lambda_i A_i \le -M_u \tag{3.20}$$

holds, where $M_u := \frac{1-u}{\alpha_u}\left(\frac{\alpha_u}{u}\right)^{\frac{u}{u-1}} - \frac{\beta_u}{\alpha_u}$.

The constant K_u relates to the Ky Fan-Furuta constant $K(h,u)$ defined by (1.2) as follows:

$$K_u = K(h,u)^{-\frac{1}{u}} = K(h^u, \frac{1}{u}) \quad \text{for } h = \frac{M}{m}.$$

Moreover the constant M_u relates to $C(m,M,p)$ defined by (1.3) as follows:

$$M_u = \begin{cases} -\alpha_u^{-1}C(m,M,u) & \text{if } u > 1 \\ C(m^u, M^u, \frac{1}{u}) & \text{if } 0 < u < 1. \end{cases}$$

In Theorems 3.6 and 3.7, let $0 < u \le 1$ and $1 \le v$. Then we have the reverse order inequalities of obtained ones.

We consider applications to the exponential function. We obtain lower and upper bounds of $\log\left(\sum_{i=1}^n \lambda_i e^{A_i}\right)$ by $\sum_{i=1}^n \lambda_i A_i$ under the condition of Theorem 3.8. We denote $\alpha_{\exp}$ and $\beta_{\exp}$ as follows:

$$\alpha_{\exp} := \frac{e^M - e^m}{M-m} \quad \text{and} \quad \beta_{\exp} := \frac{Me^m - me^M}{M-m}.$$

Theorem 3.8. *Let* A_i *be self-adjoint operators on* H *with* $m \le A_i \le M$ *for* $i = 1, 2, \dots, n$. *Suppose* $p, q \in [m, M]$. *Then for each weight* $\lambda_1, \lambda_2, \dots, \lambda_n$, *the inequality*

(3.21)

$$K_{\exp}(p)\sum_{i=1}^n \lambda_i A_i + M_{\exp}(p) \le \log\left(\sum_{i=1}^n \lambda_i e^{A_i}\right) \le L_{\exp}(q)\sum_{i=1}^n \lambda_i A_i + N_{\exp}(q)$$

holds, where $K_{\exp}(p)$, $L_{\exp}(q)$, $M_{\exp}(p)$ *and* $N_{\exp}(q)$ *are defined as follows*:

$$K_{\exp}(p) := \alpha_{\exp}^{-1}e^p, \qquad M_{\exp}(p) := -\alpha_{\exp}^{-1}(pe^p + \beta_{\exp} - e^p),$$

$$L_{\exp}(q) := \alpha_{\exp} e^{-q} \ (= K_{\exp}(q)^{-1}), \qquad N_{\exp}(q) := e^{-q}(qe^q + \beta_{\exp} - e^q).$$

In particular,
(i) *The ratio inequality*

$$\sum_{i=1}^n \lambda_i A_i \le \log\left(\sum_{i=1}^n \lambda_i e^{A_i}\right) \le L_{\exp}(q_{\exp}) \sum_{i=1}^n \lambda_i A_i. \tag{3.22}$$

holds, where $q_{\exp}$ *is a solution* $q = q_{\exp} \in (m, M)$ *of an equation* $N_{\exp}(q) = 0$.
(ii) *The following difference inequality holds*

$$0 \le \log\left(\sum_{i=1}^n \lambda_i e^{A_i}\right) - \sum_{i=1}^n \lambda_i A_i \le \log S(e^{M-m}). \tag{3.23}$$

Taking $m = 0$ in (3.22), we have $q = 0$ as a solution of the equation $N_{\exp}(q) = 0$. Hence the following inequality holds:

$$\log\left(\sum_{i=1}^n \lambda_i e^{A_i}\right) \le \frac{e^M - 1}{M} \sum_{i=1}^n \lambda_i A_i.$$

We consider applications to a logarithmic function. We estimate lower and upper bounds of a chaotically geometric mean $\exp\left(\sum_{i=1}^n \lambda_i \log A_i\right)$ by the arithmetic mean $\sum_{i=1}^n \lambda_i A_i$ under the condition of Theorem 3.9. We denote $\alpha_{\log}$ and $\beta_{\log}$ as follows:

$$\alpha_{\log} := \frac{\log M - \log m}{M - m} \left(= \frac{\log h^{\frac{1}{h-1}}}{m}\right), \quad \beta_{\log} := \frac{M \log m - m \log M}{M - m} \left(= \log \frac{m}{h^{\frac{1}{h-1}}}\right).$$

Theorem 3.9. *Let* A_i *be positive invertible operators on* H *with* $0 < m \le A_i \le M$ *and* $h = \frac{M}{m} > 1$ *for* $i = 1, 2, \ldots, n$. *Suppose* $p, q \in [m, M]$. *Then for each weight* $\lambda_1, \lambda_2, \ldots, \lambda_n$, *the inequality*
(3.24)

$$K_{\log}(p) \sum_{i=1}^n \lambda_i A_i + M_{\log}(p) \ge \exp\left(\sum_{i=1}^n \lambda_i \log A_i\right) \ge L_{\log}(q) \sum_{i=1}^n \lambda_i A_i + N_{\log}(q)$$

holds, where $K_{\log}(p)$, $L_{\log}(q)$, $M_{\log}(p)$ *and* $N_{\log}(q)$ *are defined as follows*:

$$K_{\log}(p) := (p\alpha_{\log})^{-1}, \qquad M_{\log}(p) := \alpha_{\log}^{-1}(\log p - 1 - \beta_{\log}),$$
$$L_{\log}(q) := q\alpha_{\log} \ (= K_{\log}(q)^{-1}), \qquad N_{\log}(q) := -q(\log q - 1 - \beta_{\log}).$$

In particular,
(i) *The following ratio inequality holds*

$$S(h) \sum_{i=1}^n \lambda_i A_i \ge \exp\left(\sum_{i=1}^n \lambda_i \log A_i\right) \ge S(h)^{-1} \sum_{i=1}^n \lambda_i A_i. \tag{3.25}$$

(ii) *The following difference inequality holds*

$$L(m, M) \log S(h) \ge \exp\left(\sum_{i=1}^n \lambda_i \log A_i\right) - \sum_{i=1}^n \lambda_i A_i \ge -L(m, M) \log S(h). \tag{3.26}$$

The inequalities (3.25) and (3.26) for $n = 2$ is showed in [4] and [6], respectively.

Remark 3.10. *In* [5] *M.Fujii and R.Nakamoto represents connection between the power mean and a chaotically geometric mean as follows*:

$$s\text{-}\lim_{u\to 0}\left(\sum_{i=1}^{n}\lambda_i A_i^u\right)^{1/u} = \exp\left(\sum_{i=1}^{n}\lambda_i \log A_i\right) \tag{3.27}$$

under the conditions of Theorem 3.7. We see that Theorem 3.7 implies Theorem 3.9 from (3.27). *Indeed, we have results* $K_u(p) \to (p\alpha_{\log})^{-1}$ *from* $\frac{t^u-1}{u} \to \log t$ $(u \to 0)$ *etc.*

We consider applications to the entropy function. We obtain lower and upper bounds of $\sum_{i=1}^{n}\lambda_i A_i \log A_i$ by $(\sum_{i=1}^{n}\lambda_i A_i)\log(\sum_{i=1}^{n}\lambda_i A_i)$. In particular, a difference inequality (3.30) is given in [8].

Theorem 3.11. *Let* A_i *be positive invertible operators on* H *with* $1 < m \le A_i \le M$ *and* $h = \frac{M}{m} > 1$ *for* $i = 1, 2, \dots, n$. *Suppose* $q \in [m, M]$. *Then for each weight* $\lambda_1, \lambda_2, \dots, \lambda_n$, *the inequality*

$$\sum_{i=1}^{n}\lambda_i A_i \log A_i \ge \left(\sum_{i=1}^{n}\lambda_i A_i\right)\log\left(\sum_{i=1}^{n}\lambda_i A_i\right) \ge L_{\text{ent}}(q)\sum_{i=1}^{n}\lambda_i A_i \log A_i + N_{\text{ent}}(q) \tag{3.28}$$

holds, where $L_{\text{ent}}(q)$ *and* $N_{\text{ent}}(q)$ *are defined as follows*:

$$L_{\text{ent}}(q) := \frac{\log(eq)}{\log(Mh^{\frac{1}{h-1}})} \quad \text{and} \quad N_{\text{ent}}(q) := \frac{M\log h}{\log(\frac{M^h}{m})}\log(eq) - q.$$

In particular,

(i) *The ratio inequality*

$$\sum_{i=1}^{n}\lambda_i A_i \log A_i \ge \left(\sum_{i=1}^{n}\lambda_i A_i\right)\log\left(\sum_{i=1}^{n}\lambda_i A_i\right) \ge L(q_{\text{ent}})\sum_{i=1}^{n}\lambda_i A_i \log A_i \tag{3.29}$$

holds, where $q = q_{\text{ent}} \in (m, M)$ *is a solution of equations* $N_{\text{ent}}(q) = 0$.

(ii) *The following difference inequality holds*

$$0 \ge \left(\sum_{i=1}^{n}\lambda_i A_i\right)\log\left(\sum_{i=1}^{n}\lambda_i A_i\right) - \sum_{i=1}^{n}\lambda_i A_i \log A_i \ge -\frac{M}{L(1,h)}(S(h)-1). \tag{3.30}$$

We see that $L(m, M)$ and $S(h)$ play an important role in estimations of $g(\langle f(A)x, x\rangle)$ by $\langle Ax, x\rangle$ and $g(\sum_{i=1}^{n}\lambda_i f(A_i))$ by $\sum_{i=1}^{n}\lambda_i A_i$ for the exponential, logarithmic or entropy function.

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