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# ORDER PRESERVING INEQUALITIES INDUCED BY SOME OPERATOR FUNCTIONS AND ITS APPLICATIONS

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Abstract. This paper is a resume based on our talk at KOTAC 2003, and also this is an early announcement of [8].

As an application of [10, Theorem 1], we show a simple proof of the following result:

*If $A \geq B \geq C \geq 0$ with $B > 0$, then for each $t \in [0,1]$ and $p \geq t$, the following* (i) *and* (ii) *hold for a fixed real number $q$, and they are mutually equivalent:*

(i) *If $q \geq 0$, then*

$$G_{p,q,t}(A,B,C,r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(B^{\frac{-t}{2}}C^{p}B^{\frac{-t}{2}})^{s}A^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

*is a decreasing function for $r \geq t$ and $s \geq 1$ such that $(p-t)s \geq q-t$.*

(ii) *If $p \geq q$, then*

$$G_{p,q,t}(A,B,C,r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(B^{\frac{-t}{2}}C^{p}B^{\frac{-t}{2}})^{s}A^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

*is a decreasing function for $s \geq 1$ and $r \geq \max\{t, t-q\}$.*

This result is an extension of [10, Theorem 2]. On the other hand, M.Uchiyama [16] shows the following interesting result:

(iii) *If $A \geq B \geq C \geq 0$ with $B > 0$, then for each $t \in [0,1]$ and $p \geq 1$,*

$$A^{1-t+r} \geq \{A^{\frac{r}{2}}(B^{\frac{-t}{2}}C^{p}B^{\frac{-t}{2}})^{s}A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$$

*holds for $r \geq t$ and $s \geq 1$.*

We show that (i) is equivalent to (iii), that is, they follow from each other. And also, as an application of [10, Theorem 1], we give a simple proof of M.Uchiyama's result [16, Theorem 3.4].

## 1. Introduction

A capital letter means a bounded linear operator on a Hilbert space.

**Theorem L-H** (Löwner-Heinz inequality [12][14]). *$A \geq B \geq 0$ ensures $A^{\alpha} \geq B^{\alpha}$ for all $\alpha \in [0,1]$.*

Theorem L-H is very useful, but the condition "$\alpha \in [0,1]$" is too restrictive to be applied. The following result has been obtained from this point of view.

**Theorem F** (Furuta inequality [4]).

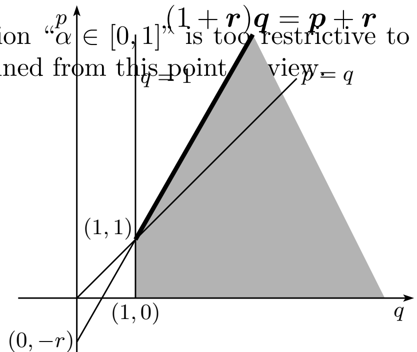
*If $A \geq B \geq 0$, then for each $r \geq 0$,*

(i) $$(B^{\frac{r}{2}}A^{p}B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^{p}B^{\frac{r}{2}})^{\frac{1}{q}}$$

*and*

(ii) $$(A^{\frac{r}{2}}A^{p}A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}})^{\frac{1}{q}}$$

*hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.*




Figure

Alternative proofs are in [13][1] and a one-page proof in [5]. It is proved in [15] that the domain drawn for $p$, $q$ and $r$ in Figure is the best possible for Theorem F. The following Theorem G is an extension of Theorem F.

**Theorem G** ([6][2]). *If $A \geq B \geq 0$ with $A > 0$, then for $t \in [0,1]$ and $p \geq 1$,*

$$A^{1-t+r} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$$

*holds for $s \geq 1$ and $r \geq t$.*

Very recently, M.Uchiyama shows the following interesting extension of Theorem G.

**Theorem U** ([16]). *If $A \geq B \geq C \geq 0$ with $B > 0$, then for $t \in [0,1]$ and $p \geq 1$,*

$$A^{1-t+r} \geq \{A^{\frac{r}{2}}(B^{\frac{-t}{2}}C^pB^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$$

*holds for $s \geq 1$ and $r \geq t$.*

We show that Theorem U is equivalent to (i) of Theorem 1 under below, that is, they follow from each other. And also, as an application of [10, Theorem 1], we give a simple proof of M.Uchiyama's result [16, Theorem 3.4].

## 2. Operator functions implying Theorem U

**Theorem 1.** *If $A \geq B \geq C \geq 0$ with $B > 0$, then for each $t \in [0,1]$ and $p \geq t$, the following* (i) *and* (ii) *hold for a fixed real number $q$, and they are mutually equivalent:*

(i) *If $q \geq 0$, then*

$$G_{p,q,t}(A,B,C,r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(B^{\frac{-t}{2}}C^pB^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

*is a decreasing function for $r \geq t$ and $s \geq 1$ such that $(p-t)s \geq q-t$.*

(ii) *If $p \geq q$, then*

$$G_{p,q,t}(A,B,C,r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(B^{\frac{-t}{2}}C^pB^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

*is a decreasing function for $s \geq 1$ and $r \geq \max\{t, t-q\}$.*

We need the following results to prove Theorem 1.

**Theorem A** ([10, Theorem 1]). *Let $A$ and $B$ be positive invertible operators satisfying*

$$A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{\beta_0}{\alpha_0+\beta_0}} \quad \text{for fixed } \alpha_0 \geq 0 \text{ and } \beta_0 \geq 0 \text{ with } \alpha_0+\beta_0 > 0.$$

*Then the following* (i) *and* (ii) *hold and they are mutually equivalent:*

(i) *For any fixed $\delta \geq -\beta_0$,*

$$f(\lambda,\mu) = A^{\frac{-\mu}{2}}(A^{\frac{\mu}{2}}B^\lambda A^{\frac{\mu}{2}})^{\frac{\delta+\beta_0\mu}{\alpha_0\lambda+\beta_0\mu}}A^{\frac{-\mu}{2}}$$

*is a decreasing function for $\mu \geq 1$ and $\lambda \geq 1$ such that $\alpha_0\lambda \geq \delta$.*

(ii) *For any fixed $\delta \leq \alpha_0$,*

$$f(\lambda,\mu) = A^{\frac{-\mu}{2}}(A^{\frac{\mu}{2}}B^\lambda A^{\frac{\mu}{2}})^{\frac{\delta+\beta_0\mu}{\alpha_0\lambda+\beta_0\mu}}A^{\frac{-\mu}{2}}$$

*is a decreasing function for $\lambda \geq 1$ and $\mu \geq 1$ such that $\beta_0\mu \geq -\delta$.*

**Lemma B** ([6])**.** *Let $X$ be a positive invertible operator and $Y$ be an invertible operator. For any real number $\lambda$,*

$$(YXY^*)^\lambda = YX^{\frac{1}{2}}(X^{\frac{1}{2}}Y^*YX^{\frac{1}{2}})^{\lambda-1}X^{\frac{1}{2}}Y^*.$$

## 3. Equivalence relation associated with Theorem 1

We show the following equivalence relation between Theorem 1 and related operator inequalities.

**Theorem 2.** *The following* (i), (ii), (iii) *and* (iv) *hold and follow from each other:*

(i) *If $A \geq B \geq C \geq 0$ with $B > 0$, then for each $t \in [0,1]$ and $p \geq 1$,*

$$A^{1-t+r} \geq \{A^{\frac{r}{2}}(B^{\frac{-t}{2}}C^pB^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$$

*holds for $r \geq t$ and $s \geq 1$.*

(ii) *If $A \geq B \geq C \geq 0$ with $B > 0$, then for each $1 \geq q \geq t \geq 0$ and $p \geq q$,*

$$A^{q-t+r} \geq \{A^{\frac{r}{2}}(B^{\frac{-t}{2}}C^pB^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}$$

*holds for $r \geq t$ and $s \geq 1$.*

(iii) *If $A \geq B \geq C \geq 0$ with $B > 0$, then for each $t \in [0,1]$ and $p \geq 1$,*

$$F_{p,t}(A,B,C,r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(B^{\frac{-t}{2}}C^pB^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

*is a decreasing function for $r \geq t$ and $s \geq 1$ .*

(iv) *If $A \geq B \geq C \geq 0$ with $B > 0$, then for each $t \in [0,1]$, $q \geq 0$ and $p \geq t$,*

$$G_{p,q,t}(A,B,C,r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(B^{\frac{-t}{2}}C^pB^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

*is a decreasing function for $r \geq t$ and $s \geq 1$ such that $(p-t)s \geq q-t$.*

We remark that Theorem 2 is an extension of [9, Theorem 1], a proof of (i) is in [16, Proposition 4.1], a one-page proof of (i) by using Theorem G itself is in [7], and a mean-theoretic proof of (i) is in [3].

## 4. Satellite inequalities

As simple applications of Theorem 1 and Theorem 2, we show the following inequalities.

**Theorem 3.** *If $A \geq B \geq C > 0$, then the following inequalities* (i) *and* (ii) *hold for each $t \in [0,1]$, $p \geq 1$, $r \geq t$ and $s \geq 1$:*

(i)
$$\begin{aligned}
&B^{\frac{t}{2}}C^{\frac{-r}{2}}\{C^{\frac{r}{2}}(B^{\frac{-t}{2}}A^pB^{\frac{-t}{2}})^sC^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}}C^{\frac{-r}{2}}B^{\frac{t}{2}}\\
&\geq B^{\frac{t}{2}}C^{\frac{-t}{2}}\{C^{\frac{t}{2}}(B^{\frac{-t}{2}}A^pB^{\frac{-t}{2}})^sC^{\frac{t}{2}}\}^{\frac{1}{(p-t)s+t}}C^{\frac{-t}{2}}B^{\frac{t}{2}}\\
&\geq A \geq B \geq C\\
&\geq B^{\frac{t}{2}}A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(B^{\frac{-t}{2}}C^pB^{\frac{-t}{2}})^sA^{\frac{t}{2}}\}^{\frac{1}{(p-t)s+t}}A^{\frac{-t}{2}}B^{\frac{t}{2}}\\
&\geq B^{\frac{t}{2}}A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(B^{\frac{-t}{2}}C^pB^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}}A^{\frac{-r}{2}}B^{\frac{t}{2}}.
\end{aligned}$$

$$\text{(ii)} \quad B^{\frac{t}{2}}C^{\frac{-r}{2}}\{C^{\frac{r}{2}}(B^{\frac{-t}{2}}A^pB^{\frac{-t}{2}})^sC^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}}C^{\frac{-r}{2}}B^{\frac{t}{2}}$$
$$\geq B^{\frac{t}{2}}C^{\frac{-r}{2}}(C^{\frac{r}{2}}B^{\frac{-t}{2}}A^pB^{\frac{-t}{2}}C^{\frac{r}{2}})^{\frac{1+r-t}{p+r-t}}C^{\frac{-r}{2}}B^{\frac{t}{2}}$$
$$\geq A \geq B \geq C$$
$$\geq B^{\frac{t}{2}}A^{\frac{-r}{2}}(A^{\frac{r}{2}}B^{\frac{-t}{2}}C^pB^{\frac{-t}{2}}A^{\frac{r}{2}})^{\frac{1+r-t}{p+r-t}}A^{\frac{-r}{2}}B^{\frac{t}{2}}$$
$$\geq B^{\frac{t}{2}}A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(B^{\frac{-t}{2}}C^pB^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}}A^{\frac{-r}{2}}B^{\frac{t}{2}}.$$

**Corollary 4.** *If $A \geq B > 0$, then the following inequalities* (i) *and* (ii) *hold for each $t \in [0,1]$, $p \geq 1$, $r \geq t$ and $s \geq 1$:*

$$\text{(i)} \quad B^{\frac{-(r-t)}{2}}\{B^{\frac{r}{2}}(B^{\frac{-t}{2}}A^pB^{\frac{-t}{2}})^sB^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}}B^{\frac{-(r-t)}{2}}$$
$$\geq \{B^{\frac{t}{2}}(B^{\frac{-t}{2}}A^pB^{\frac{-t}{2}})^sB^{\frac{t}{2}}\}^{\frac{1}{(p-t)s+t}}$$
$$\geq A \geq B$$
$$\geq \{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{t}{2}}\}^{\frac{1}{(p-t)s+t}}$$
$$\geq A^{\frac{-(r-t)}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}}A^{\frac{-(r-t)}{2}}.$$
$$\text{(ii)} \quad B^{\frac{-(r-t)}{2}}\{B^{\frac{r}{2}}(B^{\frac{-t}{2}}A^pB^{\frac{-t}{2}})^sB^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}}B^{\frac{-(r-t)}{2}}$$
$$\geq B^{\frac{-(r-t)}{2}}(B^{\frac{r-t}{2}}A^pB^{\frac{r-t}{2}})^{\frac{1+r-t}{p+r-t}}B^{\frac{-(r-t)}{2}}$$
$$\geq A \geq B$$
$$\geq A^{\frac{-(r-t)}{2}}(A^{\frac{r-t}{2}}B^pA^{\frac{r-t}{2}})^{\frac{1+r-t}{p+r-t}}A^{\frac{-(r-t)}{2}}$$
$$\geq A^{\frac{-(r-t)}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}}A^{\frac{-(r-t)}{2}}.$$

**Theorem C** ([13])**.** *If $A \geq B > 0$, then the following inequality holds for $p \geq 1$ and $r \geq 0$:*

$$B^{\frac{-r}{2}}(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1+r}{p+r}}B^{\frac{-r}{2}} \geq A \geq B \geq A^{\frac{-r}{2}}(A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}}A^{\frac{-r}{2}}.$$

## 5. M.Uchiyama's result via Theorem A

The following result is contained in [16, Theorem 3.4].

**Theorem V** ([16])**.** *Let $A$ and $B$ be both positive invertible operators. Also let $a$, $b$ and $c$ be positive real numbers and $d$ a real number. Define $F(r,s)$ and $G(r,s)$ by*

$$F(r,s) = A^{\frac{r}{2}}(A^{\frac{-r}{2}}B^sA^{\frac{-r}{2}})^{\frac{r}{r+sc}}A^{\frac{r}{2}} \quad \textit{for } r > 0 \textit{ and } s > 0$$

*and*

$$G(r,s) = A^{\frac{r}{2}}(A^{\frac{-r}{2}}B^sA^{\frac{-r}{2}})^{\frac{r+d}{r+sc}}A^{\frac{r}{2}} \quad \textit{for } r > 0 \textit{ and } s > 0 \textit{ with } 0 \leq \tfrac{r+d}{r+sc} \leq 1.$$

*Let $a > 0$, $b > 0$ and $-a \leq d \leq bc$. Then for $r_2 \geq r_1 \geq a$ and $s_2 \geq s_1 \geq b$, the following hold:*

(a) *If $F(a,b) \leq 1$, then $G(r_2,s_2) \leq G(r_1,s_1)$.*
(b) *If $F(a,b) \geq 1$, then $G(r_2,s_2) \geq G(r_1,s_1)$.*

On the other hand, replacing $A$ with $A^{\beta_0}$ and $B$ with $B^{\alpha_0}$ in Theorem A, then we have the following result in [11].

**Corollary D** ([11]). *Let $A$ and $B$ be positive invertible operators satisfying*

$$A^{\beta_0} \geq (A^{\frac{\beta_0}{2}} B^{\alpha_0} A^{\frac{\beta_0}{2}})^{\frac{\beta_0}{\alpha_0+\beta_0}} \quad \text{for fixed } \alpha_0 > 0 \text{ and } \beta_0 > 0.$$

*Then for any fixed $\delta \geq -\beta_0$,*

$$f(\alpha,\beta) = A^{\frac{-\beta}{2}} (A^{\frac{\beta}{2}} B^{\alpha} A^{\frac{\beta}{2}})^{\frac{\delta+\beta}{\alpha+\beta}} A^{\frac{-\beta}{2}}$$

*is a decreasing function for $\alpha \geq \max\{\delta, \alpha_0\}$ and $\beta \geq \beta_0$.*

We give a proof of Theorem V via Corollary D.

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