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II_1-FACTORS AND TRANSITIVE ALGEBRA QUESTION

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ABSTRACT. This survey is concerned with the transitive algebra question and the invariant subspace problem relative to II_1-factors. We mainly consider the hyperfinite II_1-factor $\mathcal{R}$, the free group factor $L(\mathbb{F}_\infty)$ on infinitely many generators and the Thompson group factor $L(F)$. For the hyperfinite II_1-factor $\mathcal{R}$ the strong operator closure of the algebra generated by two unitaries $\{U, V\}$ and the commutant $\mathcal{R}'$ is $\mathcal{B}(\mathcal{H})$. In the free group factor case, the algebra generated by two unitaries $\{L_a, L_b\}$ and the commutant $L(\mathbb{F}_\infty)'$ is strong-operator dense in $\mathcal{B}(\mathcal{H})$. Similarly, the algebra generated by two unitaries $\{L_{x_0}, L_{x_1}\}$ and $L(F)'$ is also strong-operator dense in $\mathcal{B}(\mathcal{H})$ where x_0, x_1 are two generators of F. We find non-trivial invariant projections of some operators in the Cuntz algebra $\mathcal{O}_n$ using a concrete isomorphism.

1. Introduction and Preliminaries

This survey is concerned with the transitive algebra question and the invariant subspace problem relative to II_1-factors. In particular, the purpose of this survey gives some techniques for finding a transitive set in a II_1-factor and a non-trivial transitive algebra. The invariant subspace problem has been studied by many people since it was asked by von Neumann in 1930s and Kadison asked the transitive algebra question with connection to the invariant subspace problem in 1950s [RR1]. There are many theorems and open problems concerning transitive algebras and invariant subspaces and works of the invariant subspaces of particular operators have interesting connections to the classical analysis. "Does every bounded operator on a separable Hilbert space have a non-trivial invariant subspace?" This is the famous (classical) invariant subspace problem on a separable Hilbert space. On Banach spaces, however, there have been several extraordinary constructions of counterexamples [En, Re1, Re2]. If $\mathcal{B}(\mathcal{H})$ is the algebra of all bounded linear operators on $\mathcal{H}$, then the invariant subspace problem can be restated as follows: *For any T in $\mathcal{B}(\mathcal{H})$, is there a non-trivial orthogonal projection P in $\mathcal{B}(\mathcal{H})$ which is invariant under T, that is, $PTP = TP$ and $P \neq 0, I$?* The finite dimensional Hilbert space case immediately follows from the fact that every matrix in a finite dimensional vector space is unitarily equivalent to an upper triangular matrix. The Jordan canonical form theorem for matrices can be regarded as exhibiting matrices as direct sums of their restrictions to certain invariant subspaces. But the infinite dimensional case is still open although there are many partial positive answers to it.

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Kadison [Ka1] asked the following non-commutative invariant subspace problem which is known as the "transitive algebra" question: *If $\mathcal{B}$ is a subalgebra (not necessarily $*$-subalgebra) of $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}$ has no non-trivial common invariant subspace in $\mathcal{H}$, then is $\mathcal{B}$ strong-operator dense in $\mathcal{B}(\mathcal{H})$?* In other words, if $\mathcal{B}$ is a strong-operator closed (non-commutative) subalgebra of $\mathcal{B}(\mathcal{H})$ and $\mathcal{B} \neq \mathcal{B}(\mathcal{H})$, does $\mathcal{B}$ always have a non-trivial invariant subspace in $\mathcal{H}$? If $\mathcal{B}$ doesn't have any non-trivial invariant subspace in $\mathcal{H}$, we call such an algebra $\mathcal{B}$ transitive. If $\mathcal{H}$ is finite dimensional, it follows from the Burnside's theorem [Ja] that the only transitive algebra is the full matrix algebra of all linear transformations. In the infinite dimensional case, the von Neumann's double commutant theorem shows that every selfadjoint transitive algebra is strong-operator dense in $\mathcal{B}(\mathcal{H})$. In fact, the invariant subspace problem asks if even a singly generated algebra acting on a separable Hilbert space can be transitive.

Let $W^*(T)$ be the strong-operator closed unital algebra generated by T and T^*. Then $W^*(T)$ is a von Neumann algebra (a strong-operator closed selfadjoint subalgebra of $\mathcal{B}(\mathcal{H})$). We can ask the following the invariant subspace problem relative to a von Neumann algebra: "Does every operator T in $\mathcal{B}(\mathcal{H})$ have a non-trivial invariant projection in $W^*(T)$?" Since, in case where $W^*(T)$ has a non-trivial center, the above question is trivial, we are interested in the case where the center of $W^*(T)$ is trivial and concentrate on the question of whether, for any operator T in a II_1-factor $\mathcal{M}$, there is a non-trivial projection $P \in \mathcal{M}$ such that $PTP = TP$. As we investigated above, several existence problems still remain open; the existence of invariant subspaces for all operators on a Hilbert space and the existence of non-trivial transitive algebras. For a general discussion of the invariant subspace problem, the transitive algebra question and related topics, see the monograph [RR1, RR2].

In this survey, we will focus on the transitive algebra question and the invariant subspace problem relative to some factors of type II_1 (see [He1, He2, He3] for details). This survey is organized in the following fashion. In the second section, we study the invariant subspace problem in the hyperfinite II_1-factor $\mathcal{R}$ and we show that some operator in $\mathcal{R}$ has infinitely many invariant subspaces affiliated with $\mathcal{R}$ and the strong operator closure of the algebra generated by two unitaries $\{U, V\}$ and the commutant $\mathcal{R}'$ is $\mathcal{B}(L^2(\mathcal{R}, \tau))$. In the third section, we consider the free group factor and the Thompson group cases. We prove that the algebra generated by two unitaries $\{L_a, L_b\}$ and the commutant $L(\mathbb{F}_\infty)'$ is strong-operator dense in $\mathcal{B}(\ell^2(\mathbb{F}_\infty))$. Similarly, the algebra generated by two unitaries $\{L_{x_0}, L_{x_1}\}$ and the commutant $L(F)'$ is also strong-operator dense in $\mathcal{B}(\ell^2(F))$ where x_0, x_1 are two generators of F. In the fourth section, we find non-trivial invariant projections of some operators in the Cuntz algebra $\mathcal{O}_n$ using a concrete isomorphism. The fifth section contains some remarks, comments and open questions. The list of references does not cover the whole subject, and it is important to follow up the references in the papers and in [RR1, RR2].

The remaining of this section is concerned with some results and motivations about the transitive algebra question. When Kadison posed the transitive algebra question, he didn't think that it was a hard problem. He had an idea that some selfadjoint maximal abelian subalgebra of $\mathcal{B}(\mathcal{H})$ and some elements not in the subalgebra might generate a non-trivial (strong-operator closed) transitive algebra. Arveson proved in [Ar] that Kadison's original idea doesn't work. That is, if $\mathcal{A}$ is a transitive subalgebra of $\mathcal{B}(\mathcal{H})$ which contains a selfadjoint maximal abelian von

Neumann algebra, then $\mathcal{A}$ is strong-operator dense in $\mathcal{B}(\mathcal{H})$. Here our idea is to start with a factor of type II_1 and take some elements from the commutant of the factor. Then we investigate if they together provide some examples of non-trivial transitive algebras. For example, if T in a factor $\mathcal{M}$ doesn't have any non-trivial invariant projection in $\mathcal{M}$, then T and $\mathcal{M}'$ will generate a strong-operator closed transitive algebra (even with a non-trivial commutant).

Definition 1.1. [RR2] We call a subset (or a subalgebra) $\mathcal{X}$ of a II_1-factor $\mathcal{M}$ *transitive with respect to* $\mathcal{M}$ if $\mathcal{X}$ has no non-trivial invariant projections in $\mathcal{M}$. In this case, we will simply say that $\mathcal{X}$ is *transitive in* $\mathcal{M}$.

This definition is similar to the original definition of transitivity (in the factor of type I_∞). Similar definitions can be carried over to factors of type II_∞ or III. In a factor of type II_1, there are many non-trivial strong-operator closed transitive subalgebras. Furthermore, the transitive algebra question could also be considered for algebras generated by special kinds of operators. If $\mathfrak{A}$ is a transitive algebra generated by selfadjoint operators, then $\mathfrak{A}$ is a von Neumann algebra and must equal to $\mathcal{B}(\mathcal{H})$. What is the situation if $\mathfrak{A}$ is generated by isometries or normal operators? In spite of a great deal of interest in this question, no transitive algebras other than $\mathcal{B}(\mathcal{H})$ have yet known. In this survey we are concerned if the transitive subset in $\mathcal{M}$ together with $\mathcal{M}'$ generate a non-trivial strong-operator closed transitive algebra in $\mathcal{B}(\mathcal{H})$. The following theorem follows from the above observation.

Proposition 1.2. [He1] *If $\mathcal{X}$ is a selfadjoint subset of a factor $\mathcal{M}$ with a trivial relative commutant, then $\mathcal{X}$ together with $\mathcal{M}'$ generates $\mathcal{B}(\mathcal{H})$ in the strong-operator topology.*

2. The hyperfinite II_1-factor

In this section we study the invariant subspace problem relative to the hyperfinite II_1-factor. Let $\mathcal{H}$ be the Hilbert space $\ell^2(\mathbb{Z}\times\mathbb{Z})$ and let U, V be two unitary operators on $\mathcal{H}$ such that

$$UV=\omega VU,\quad U(e_{m,n})=e_{m+1,n},\quad V(e_{m,n})=\omega^{-m}e_{m,n+1}$$

where $\omega=e^{2\pi i\theta}$ with θ an irrational number and $\{e_{m,n}:(m,n)\in\mathbb{Z}\times\mathbb{Z}\}$ is an orthonormal basis for $\mathcal{H}$ defined naturally from the group $\mathbb{Z}\times\mathbb{Z}$ ($e_{m,n}$ is 1 at (m,n) and 0 every where else). Then the von Neumann algebra generated by U and V is the hyperfinite II_1-factor.

Let τ be the (unique) normalized trace on the hyperfinite II_1-factor $\mathcal{R}$. Then we have $\tau(U^mV^n)=0$ for all integers m and n except when $m=n=0$. If we define the inner product on $\mathcal{R}$ by $\langle A,B\rangle=\tau(B^*A)$ for $A,B\in\mathcal{R}$, the completion of $\mathcal{R}$ under this inner product is a Hilbert space and denoted by $L^2(\mathcal{R},\tau)$. The action of $\mathcal{R}$ on $L^2(\mathcal{R},\tau)$ is induced by the left multiplication of $\mathcal{R}$ on $\mathcal{R}$ (the second $\mathcal{R}$ is considered as a dense subspace of $L^2(\mathcal{R},\tau)$).

We note the following result [Ha]: every operator with a spectrum more than one point in the hyperfinite II_1-factor $\mathcal{R}$ has a non-trivial invariant projection in $\mathcal{R}$. Thus $U+V$ has a non-trivial invariant projection in $\mathcal{R}$. For a non-zero invariant projection $P\in\mathcal{R}$ under $U+V$, $P\mathcal{R}P$ is again a hyperfinite factor (with the identity element P). By Haagerup's result [Ha], $P(U+V)P$ has a non-trivial invariant projection in $P\mathcal{R}P$, so that there exists a non-zero projection $Q \in P\mathcal{R}P$ such that $Q \neq P$ and $QP(U+V)PQ = (U+V)PQ$. Considering the above equation in $\mathcal{R}$, we obtain that $Q(U+V)Q = (U+V)Q$. This implies that there is no smallest non-zero invariant projection for $U+V$ in $\mathcal{R}$. Thus, this gives the following lemma.

Lemma 2.1. [He3] *There are* (*countably*) *infinitely many non-zero projections* P_j *in* $\mathrm{Lat}_{\mathcal{R}}(U+V)$ *such that* $P_j > P_{j+1}$ *and* $\tau(P_j)$ *tends to zero when* j *goes to infinity.*

Now we introduce a group action α of the abelian group $\mathbb{R}$, all real numbers, on $\mathcal{R}$. For any $r \in \mathbb{R}$, we define α_r by

$$\alpha_r(U) = e^{2\pi i r}U \quad \text{and} \quad \alpha_r(V) = e^{2\pi i r}V.$$

Then we have $\alpha_r(U^m V^n) = e^{2\pi i(m+n)r}U^m V^n$, so that α_r induces a $*$-automorphism of $\mathcal{R}$. Since α has the kernel $\mathbb{Z}$, it gives a faithful representation of $\mathbb{R}/\mathbb{Z}$ into the $*$-automorphism group of $\mathcal{R}$. Thus we can also regard α as an action of $\mathbb{R}/\mathbb{Z}$ on $\mathcal{R}$.

Theorem 2.2. [He3] *Given any* $\lambda \in (0,1)$, *there are* (*uncountably*) *infinitely many projections* $P \in \mathcal{R}$ *with* $\tau(P) = \lambda$ *such that* $P(U+V)P = (U+V)P$.

Let $\mathcal{A}_\theta$ be the irrational rotation C^*-algebra, that is, the C^*-algebra generated by two unitaries U, V with the relation $UV = e^{2\pi i\theta}VU$, $\theta \in [0,1] \setminus \mathbb{Q}$. We identify the unit circle $\mathbb{T}$, $\mathbb{R}/\mathbb{Z}$ and the unit interval $[0,1]$ (the points 0 and 1 are identified). By the function calculus, we identify U and the function $e^{2\pi i t}$ of t on [0,1]. Then for a continuous function f on [0,1], we get the relation $VfV^{-1}(t) = f(t-\theta)$. By a direct computation, we see that $U+V$ has no non-trivial invariant projections in $\mathcal{A}_\theta$ [He3]. Furthermore, since $\{U, V\}$ is a transitive subset in the hyperfinite II_1 factor $\mathcal{R}$, we get the following theorem.

Theorem 2.3. [He3] *The strong operator closure of the algebra generated by* $\{U, V\}$ *and the commutant* $\mathcal{R}'$ *is* $\mathcal{B}(\mathcal{H})$ *where* $\mathcal{H}$ *is the Hilbert space* $L^2(\mathcal{R}, \tau)$.

Note that the adjoint $U^* = U^{-1}$ is in the strong operator closure of the algebra generated by U and the commutant $\mathcal{R}'$ and that the adjoint $V^* = V^{-1}$ is also in the strong operator closure of the algebra generated by V and $\mathcal{R}'$.

3. Free group factor and Thompson group factor

Let G be a discrete group with the identity e and $\mathcal{H}$ the Hilbert space $\ell^2(G)$ with the usual inner product. We shall assume that G is countable, so that $\mathcal{H}$ is separable. For each $g \in G$, let L_g denote the left translation of functions in $\mathcal{H}$ by g^{-1}. Then the map $g \mapsto L_g$ is a faithful unitary representation of G on the Hilbert space $\mathcal{H}$. Let $L(G)$ be the Neumann algebra generated by $\{L_g : g \in G\}$. Similarly, let R_g be the right translation by g on $\mathcal{H}$ and $R(G)$ the Neumann algebra generated by $\{R_g : g \in G\}$. Then $L(G)' = R(G)$ and $R(G)' = L(G)$. The function χ_g that is 1 at g and 0 elsewhere is a cyclic trace vector for $L(G)$ (and $R(G)$). In general, $L(G)$ and $R(G)$ are finite von Neumann algebras. They are factors (of type II_1) precisely when each conjugacy class in G (other than that of e) is infinite. In this case we say that G is an infinite conjugacy class (i.c.c.) group [MvN].

Let $\mathbb{F}_n$ be the (non-abelian) free group with n generators ($n \geq 2$). When $n = \infty$, $\mathbb{F}_\infty$ means the free group with countably infinitely many generators. $L(\mathbb{F}_n)$ is called the *free group factor* (on n generators). It is proved in [Ge] by using the free entropy that free group factors $L(\mathbb{F}_n)$ ($n \geq 2$) are prime, that is, are non-isomorphic to the tensor product of any two factors of type II$_1$. However, there are many well-known open problems about free group factors. For examples, we don't know *if* $L(\mathbb{F}_2)$ *is* $*$*-isomorphic to* $L(\mathbb{F}_3)$ and *if* $L(\mathbb{F}_3)$ *is generated by a single operator.* See [VDN] and its references for some new development on free group factors.

Let $\mathbb{F}_\infty$ be the free group with countably infinitely generators a_j ($j = 1, 2, \dots$). Throughout this section, we assume that $\mathcal{H}$ is the Hilbert space $\ell^2(\mathbb{F}_\infty)$ with the basis $\{\chi_g : g \in \mathbb{F}_\infty\}$ where χ_g is the vector which takes value 1 at g and 0 elsewhere. Let $L(\mathbb{F}_\infty)$ ($R(\mathbb{F}_\infty)$, respectively) be the free group factor generated by $\{L_g : g \in \mathbb{F}_\infty\}$ ($\{R_g : g \in \mathbb{F}_\infty\}$, respectively).

Theorem 3.1. [He1] *The algebra generated by* $\{L_a, L_b\}$ *and* $L(\mathbb{F}_\infty)'$ *is strong-operator dense in* $\mathcal{B}(\mathcal{H})$.

Like the argument following Theorem 2.4, we see that the adjoint $L_a^* = L_{a^{-1}}$ is in the strong operator closure of the algebra generated by L_a and $L(\mathbb{F}_\infty)' = R(\mathbb{F}_\infty)$ where a is a generator of $\mathbb{F}_\infty$. Now we consider two sums $L_{a_1} + L_{a_2}, L_{a_3} + L_{a_4}$ of pairs of unitary generators in the free group von Neumann algebras $L(\mathbb{F}_\infty)$ where a_1, a_2, a_3 and a_4 are four of generators of $\mathbb{F}_\infty$.

Theorem 3.2. [He1] *The set* $\{L_{a_1}+L_{a_2}, L_{a_3}+L_{a_4}\}$ *together with* $L(\mathbb{F}_\infty)'$ *generates* $\mathcal{B}(\mathcal{H})$.

Replacing $L(\mathbb{F}_\infty)'$ with $L(\mathbb{F}_4)'$ in Theorem 3.2, the same result also holds, that is, the algebra generated by $\{L_{a_1} + L_{a_2}, L_{a_3} + L_{a_4}\}$ and the commutant $L(\mathbb{F}_4)'$ is strong-operator dense in $\mathcal{B}(\mathcal{H})$ where a_i's are generators of $\mathbb{F}_4$ and $\mathcal{H} = \ell^2(\mathbb{F}_4)$.

Let F be the group of piecewise linear homeomorphisms of $[0, 1]$ which, except at finitely many dyadic rational numbers, are differentiable with derivatives equal to powers of 2. F is called the Thompson group. It has the presentation as follows:

$$F = \langle x_0, x_1, \dots \mid x_i^{-1} x_n x_i = x_{n+1},\ 0 \leq i < n\rangle.$$

¿From this relation $x_i^{-1} x_n x_i = x_{n+1}$ ($0 \leq i < n$), we have $x_{n+1} = x_0^{-n} x_1 x_0^n$ for $n \geq 1$, so that F is generated by x_0 and x_1. It is known that the Thompson group F doesn't contain a non-abelian free subgroup, but many questions about Thompson group are still open. In particular, it is unknown whether or not F is amenable. This question is of considerable interest since F is expected to be a counterexample to the von Neumann's conjecture for finitely presented groups. See the expository note [CFP] for a good introduction, more details and historical remarks to the Thompson group.

Jolissant [Jo] has recently showed that the pairs of factors $L([F, F]) \subset L(D)$ and $L(D) \subset L(F)$ have relative McDuff property where $[F, F]$ is the commutator subgroup and D is the subgroup consisting in all elements of F which are trivial in a neighborhood of 1. Though F doesn't contain a non-abelian free group $\mathbb{F}_2$, we don't know if $L(F)$ contains a free group factor $L(\mathbb{F}_2)$. Concerning the transitive algebra question, we also get similar result as Theorem 2.4, Theorem 3.1 and Theorem 3.2.

Theorem 3.3. [He2] *The algebra generated by* $\{L_{x_0}, L_{x_1}\}$ *and the commutant* $L(F)'$ *is strong-operator dense in* $\mathcal{B}(\mathcal{H})$ *where* x_0 *and* x_1 *are generators of the Thompson group* F *and* $\mathcal{H} = \ell^2(F)$.

4. On Cuntz algebra and Choi algebra

Let $\mathcal{O}_n$ $(n \geq 2)$ be the C^*-algebra generated by isometries $S_1, \dots, S_n$ such that $S_1 S_1^* + \dots + S_n S_n^* = 1$. Paschke and Salinas [PS] proved that $\mathcal{O}_n$ and $M_k(\mathcal{O}_n)$ are non-isomorphic if k and $n-1$ are not relatively prime by computing Ext groups. Moreover, they showed that $M_k(\mathcal{O}_n)$ are isomorphic to $\mathcal{O}_n$ whenever $k \equiv 1 \mod n-1$. It is shown in [Ro] that $\mathcal{O}_{2n} \simeq M_k(\mathcal{O}_{2n})$ if and only if $(k, 2n-1) = 1$ by K-theory. In this section, we will use the concrete isomorphism from $\mathcal{O}_n$ onto $M_k(\mathcal{O}_n)$ to find a non-trivial invariant projection of the sum $S_1 + \cdots + S_k$ $(k \leq n)$ of isometries in $\mathcal{O}_n$. Now let's find a concrete isomorphism in the case when $k \equiv 1 \mod n-1$. By induction, one can see that if $k = l(n-1)+1$ $(n \geq 2)$ for $l = 0, 1, 2, \dots$, then there exist projections $p_1, \dots, p_k$ in $\mathcal{O}_n$ which are equivalent to 1 such that $p_1 + \cdots + p_k = 1$.

Using this fact, it is not hard to construct an isomorphism $\phi : \mathcal{O}_n \to M_k(\mathcal{O}_n)$ if $k \equiv 1 \mod n-1$. In fact, let $p_1, \dots, p_k$ be orthogonal projections such that

$$p_j = v_j v_j^*, \quad 1 = v_j^* v_j \quad \text{and} \quad p_1 + \cdots + p_k = 1.$$

Put $v = (v_1, \dots, v_k) \in M_{1,k}(\mathcal{O}_n)$. The map $\phi : \mathcal{O}_n \to M_k(\mathcal{O}_n)$ defined by $\phi(a) = v^* a v$ $(a \in \mathcal{O}_n)$ gives a $*$-isomorphism. Moreover, we can easily find the inverse map $\psi : M_k(\mathcal{O}_n) \to \mathcal{O}_n$ given by $\psi(A) = vAv^*$ $(A \in M_k(\mathcal{O}_n))$.

In particular, if we put $k = n$, then the isomorphism $\phi : \mathcal{O}_n \to M_n(\mathcal{O}_n)$ is given by $\phi(a) = v^* a v$, $a \in \mathcal{O}_n$ where $v = (S_1, \dots, S_n) \in M_{1,n}(\mathcal{O}_n)$. First, consider the operator $S_1 + S_2$. The image of $S_1 + S_2$ under ϕ is

$$v^*(S_1 + S_2)v = \begin{pmatrix} S_1 & S_2 & \dots & S_n \\ S_1 & S_2 & \dots & S_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

Then we can easily find a non-trivial invariant projection in $M_n(\mathcal{O}_n)$, that is,

$$\tilde{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \dots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \in M_n(\mathcal{O}_n)$$

is an invariant projection of $\phi(S_1 + S_2)$. The image $\psi(\tilde{P})$ in $\mathcal{O}_n$ under ψ is

$$P = v\tilde{P}v^* = \frac{1}{2}(S_1 + S_2)(S_1^* + S_2^*),$$

which becomes an invariant projection of $S_1 + S_2$. By the same method, we can see that a non-trivial invariant projection of $S_1 + \cdots + S_k$ $(1 \le k \le n)$ is

$$P = \frac{1}{k}(S_1 + \cdots + S_k)(S_1^* + \cdots + S_k^*).$$

Using these isomorphisms, one can find various invariant projections in $\mathcal{O}_n$.

Let $C^*(u, v)$ be the C^*-algebra generated by two unitary operators u, v on an infinite dimensional Hilbert space such that

$$u^2 = v^3 = 1, \quad f + u^* f u = 1 \quad \text{and} \quad f + v^* f v + v f v^* = 1.$$

$C^*(u, v)$ has no such a projection f satisfying above equations. It was showed that $C^*(u, v)$ is $*$-isomorphic to the reduced C^*-algebra of a discrete group G on two generators u, v with $u^2 = v^3 = e$, that is, $C^*(u, v) \simeq C_r^*(\mathbb{Z}_2 * \mathbb{Z}_3)$. Further, $C^*(u, v)$ can be regarded as a C^*-subalgebra of $\mathcal{O}_2$ generated by $u = S_1 S_2^* + S_2 S_1^*$ and $v = S_1 S_2^{*2} + S_2 S_1 S_1^* + S_2^2 S_1 S_2^*$. These relations are easily found by above isomorphism. It is well-known that the unique normalized trace τ of $C^*(u, v)$ is a non-hyperfinite II$_1$-factor state.

Consider invariant projections in $L(\mathbb{Z}_2 * \mathbb{Z}_3) = \pi_\tau(C^*(u, v))''$ which is isomorphic to $L(\mathbb{F}_r)$ with $r = \frac{7}{6}$ [Dy]. It is not hard to see that u and v are non-trivial invariant projections in $L(\mathbb{Z}_2 * \mathbb{Z}_3)$, in fact, in $C^*(u, v)$. Indeed, $p = \frac{1}{2}(1 + u)$ and $p_k = \frac{1}{3}(1 + e^{\frac{2\pi(k-1)i}{3}} v + e^{\frac{4\pi(k-1)i}{3}} v^*)$ $(k = 1, 2)$ are invariant projections of u and v, respectively. Moreover, p and p_k $(k = 1, 2)$ are contained in $C^*(u, v)$. But there is no common non-trivial invariant projection in $L(\mathbb{Z}_2 * \mathbb{Z}_3)$ of u and v since there is no projection commuting with u and v simultaneously. We see from a direct computation that $u + v$ is a generator of $L(\mathbb{Z}_2 * \mathbb{Z}_3)$.

5. Some remarks and questions

Let $(\mathfrak{A}, \phi)$ be a non-commutative probability space, and let $(a_i)_{i \in I}$ be a family of random variables in $(\mathfrak{A}, \phi)$. Let $\mathbb{C}\langle\{X_i : i \in I\}\rangle$ be a free algebra with a unit over $\mathbb{C}$ and generators X_i $(i \in I)$ and $h : \mathbb{C}\langle\{X_i : i \in I\}\rangle \to \mathfrak{A}$ a homomorphism such that $h(X_i) = a_i$ $(i \in I)$. The joint distribution of $(a_i)_{i \in I}$ is a functional $\mu : \langle\{X_i : i \in I\}\rangle \to \mathbb{C}$ defined by $\mu = \phi \circ h$. Voiculescu introduced a non-commutative probability theory whose basic objects inherit the asymptotic properties of families of random matrices. The basic idea in applications of the free probability theory to von Neumann algebras is to model some elements, especially generators of a von Neumann algebra, by large random matrices with entries from a classical probability space. He proved that the von Neumann algebra generated by a free semicircular family $(X_j)_{j \in J}$ is isomorphic to $L(\mathbb{F}_{|J|})$ associated to a free group with $|J|$ generators [Vo]. From Voiculescu's free probability theory, we see that the joint distribution of a circular system can be approximated by the joint distributions of corresponding systems of independent Gaussian random matrices.

Dykema and Haagerup [DH] found that Voiculescu's matrix model leads to upper triangular models for the circular operator. Using upper triangular realizations of the circular free Poisson element, they prove that the circular operator and each circular free Poisson operator which arise naturally in the free probability theory has a continuous family of invariant subspaces relative to the von Neumann algebra which it generates. This result is motivated from the following observation: any operator T in $\mathcal{B}(\mathcal{H})$ can have a non-trivial invariant subspace $\mathcal{H}_0$ such that the projection from $\mathcal{H}$ onto $\mathcal{H}_0$ lies in the von Neumann algebra generated by T if an upper triangular form of T with respect to a suitable decomposition of $\mathcal{H}$ satisfies a condition on the spectra of the elements in the upper triangular form.

Proposition 5.1. (cf. [DH]) *Let $Y = (Y_{ij})_{1\le i,j\le n}$ be a complex Gaussian random matrix. Then the random matrix Y' given by*

$$Y' = \begin{pmatrix} Y_{11} & \dots & \dots & \dots & Y_{1n} \\ Z_1 & \ddots & & & \vdots \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & Z_{n-1} & Y_{nn} \end{pmatrix}$$

is similar to Y, (in fact unitarily equivalent) where

$$Z_1 = \sqrt{Y_{21}^* Y_{21} + \cdots + Y_{n1}^* Y_{n1}}, \dots , Z_{n-1} = \sqrt{Y_{n,n-1}^* Y_{n,n-1}}.$$

Especially, Y' has the same $*$-distribution as Y.

Theorem 5.2. [He1] *Let $T = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ 0 & T_{22} & T_{23} \\ 0 & 0 & T_{33} \end{pmatrix}$ act on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$. Suppose that T_{11} is invertible and that $\|T_{33}\| < \|T_{11}^{-1}\|^{-1}$. Then there is an invariant subspace $\mathcal{K}$ for T such that $\mathcal{H}_1 \subseteq \mathcal{K} \subseteq \mathcal{H}_1 \oplus \mathcal{H}_2$ and the projection $P_{\mathcal{K}}$ from $\mathcal{H}$ onto $\mathcal{K}$ is in $\{T, T^*\}''$.*

In this survey we mainly investigated the existence of an invariant subspace for a pair of unitaries and the sum of two unitaries in a factor of type II_1. The motivation follows from the observation: if $\mathcal{M}$ is a finite von Neumann algebra, then any element T in $\mathcal{M}$ can be expressed by a sum of only two unitaries in $\mathcal{M}$. More generally, we get a similar result for any von Neumann algebra.

Proposition 5.3. [He1] *Let $\mathcal{M}$ be a von Neumann algebra. If T is any operator in $\mathcal{M}$, then there are two unitary operators U and V in $\mathcal{M}$ such that $T = 4\|T\|(U+V-\frac{1}{2}I)$. Moreover, we can choose U and V in the von Neumann subalgebra generated by T, T^* and I.*

From the above proposition, we see that the summation of two unitary operators and the identity operator can give any operator (up to a multiple of constant) as wanted. But the relations between two unitary operators can be very complicated. With the assumption that the von Neumann algebra $W^*(T)$ generated by T is a factor, we know that U and V generate a factor. The irrational rotation relation and the freeness between U and V are natural conditions to make them generate factors. When U and V are free elements, one can ask more for them and still can produce factors. For example, one can ask if U and V are taken as both finite order unitary operators (see [Dy] for a detailed discussion of free products of finite dimensional abelian algebras with respect to some states on them).

These are some more interesting examples one can work on. However we would more want to know the answers to the following questions: In the second and the third sections, we can see the following: If $\mathcal{M}$ is the hyperfinite II_1-factor, the Thompson group factor, or a free group factor, then the adjoint $U^* = U^{-1}$ is in the strong operator closure of the (non-selfadjoint) algebra generated by some unitary element U and the commutant $\mathcal{M}'$. More generally, we can ask the following question.

Question 5.4. *For any unitary element U in a factor $\mathcal{M}$ of type II_1, is the adjoint $U^* = U^{-1}$ always in the strong operator closure of the (non-selfadjoint) algebra generated by U and the commutant $\mathcal{M}'$?*

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