

**CONTINUITY OF GENERALIZED
INTERTWINING LINEAR OPERATORS
FOR OPERATORS WITH PROPERTY
(δ) AND HYPNORMAL OPERATORS**

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ABSTRACT. In this paper, we show that for a hyponormal operator the analytic spectral subspace coincides with the algebraic spectral subspace. Using this result, we have the following result: Let T be a operator with property (δ) on a Banach space X and let S be a hyponormal operator on a Hilbert space H . Then every generalized intertwining linear operator $\theta : X \rightarrow H$ for (S, T) is continuous if and only if the pair (S, T) has no critical eigenvalue.

1. INTRODUCTION

Let X and Y be Banach spaces and consider a linear operator $\theta : X \rightarrow Y$. The basic automatic continuity problem is to derive the continuity of θ from some prescribed algebraic conditions. For example, let T and S be bounded linear operators on Banach spaces X and Y , respectively, if $\theta : X \rightarrow Y$ is a linear operator intertwining for (S, T) , that is $\theta T = S\theta$, one may look for conditions on T and S which force θ to be continuous.

The study of continuity of a linear operator θ intertwining for (S, T) was initiated by Johnson and Sinclair [3]. In this paper, necessary conditions on T and S for the continuity of θ were obtained for the operator S with countable spectrum. After Johnson and Sinclair's paper, Vrbová presented an automatic continuity result concerning an intertwining operator with operators having suitable spectral decomposition properties [12].

In 1986 Laursen and Neumann introduced super-decomposable operators in order to consider necessary conditions for automatic continuity of intertwining operators: this class of operators contains most of interesting examples of decomposable operators. After this study, the study of automatic continuity of intertwining linear operators has been closely related to the classification of decomposable operators.

In this paper, we show that for a hyponormal operator the analytic spectral subspace coincides with the algebraic spectral subspace. Using this result, we have the following result: Let T be a operator with property (δ) on a Banach space X and let S be a hyponormal operator on a Hilbert space H . Then every generalized intertwining linear operator $\theta : X \rightarrow H$ for (S, T) is continuous if and only if the pair (S, T) has no critical eigenvalues.

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2. PRELIMINARIES

Throughout this paper we shall use the standard notions and some basic results on the local spectral theory and automatic continuity theory. Let X be a Banach space over the complex plane \mathbb{C} . And let $\mathcal{L}(X)$ denote the Banach algebra of all bounded linear operators on a Banach space X . Given an operator $T \in \mathcal{L}(X)$, $\text{Lat}(T)$ denotes the collection of all closed T -invariant linear subspaces of X , and for an $Y \in \text{Lat}(T)$ $T|_Y$ denotes the restriction of T on Y .

Definition 1. Let $T : X \rightarrow X$ be a linear operator on a Banach space X . Let F be a subset of the complex plane \mathbb{C} . Consider the class of all linear subspaces Y of X which satisfy $(T - \lambda)Y = Y$ for all $\lambda \notin F$ and let $E_T(F)$ denote the span of all such subspaces Y of X . $E_T(F)$ is called an *algebraic spectral subspace* of T .

In the next Remark, we collect a number of results on algebraic spectral subspaces. These results are found in [6].

Remark 2. (1) It is clear that $(T - \lambda)E_T(F) = E_T(F)$ for all $\lambda \notin F$ as well so that it is the largest linear subspace with this property.

(2) By the definition of the algebraic spectral subspace, it is clear that

$$E_T(F_1) \subseteq E_T(F_2) \quad \text{for } F_1 \subseteq F_2.$$

(3) Let A be a bounded linear operator on a Banach space X with $AT = TA$. For a given subset F of \mathbb{C} and $\lambda \notin F$, we obtain

$$(T - \lambda)AE_T(F) = A(T - \lambda)E_T(F) = AE_T(F).$$

By the maximality of $E_T(F)$ we have

$$AE_T(F) \subseteq E_T(F).$$

That is, the space $E_T(F)$ is a hyper-invariant subspace of T .

(4) It is well known that if $\{F_\alpha\}$ is a family of subsets of \mathbb{C} , then

$$E_T\left(\bigcap_{\alpha} F_{\alpha}\right) = \bigcap_{\alpha} E_T(F_{\alpha}).$$

A linear subspace Z of X is called a *T -divisible subspace* if

$$(T - \lambda)Z = Z \quad \text{for all } \lambda \in \mathbb{C}.$$

Hence $E_T(\emptyset)$ is precisely the largest T -divisible subspace. There is an operator which has non-trivial divisible subspaces. Indeed, the Volterra operator has a non-trivial divisible subspace.

For a given $T \in \mathcal{L}(X)$, let $\sigma(T)$ and $\rho(T)$ denote the spectrum and the resolvent set of T , respectively. The *local resolvent set* $\rho_T(x)$ of T at the point $x \in X$ is

defined as the union of all open subsets U of \mathbb{C} for which there is an analytic function $f : U \rightarrow X$ which satisfies

$$(T - \lambda)f(\lambda) = x \quad \text{for all } \lambda \in U.$$

The *local spectrum* $\sigma_T(x)$ of T at x is then defined as

$$\sigma_T(x) = \mathbb{C} \setminus \rho_T(x).$$

Clearly, the local resolvent set $\rho_T(x)$ is open, and the local spectrum $\sigma_T(x)$ is closed. For each $x \in X$, the function $f(\lambda) : \rho(T) \rightarrow X$ defined by $f(\lambda) = (T - \lambda)^{-1}x$ is analytic on $\rho(T)$ and satisfies

$$(T - \lambda)f(\lambda) = x \quad \text{for all } \lambda \in \rho(T).$$

Hence the resolvent set $\rho(T)$ is always subset of $\rho_T(x)$ and hence $\sigma_T(x)$ is always subset of $\sigma(T)$. The analytic solutions occurring in the definition of the local resolvent set may be thought of as local extensions of the function $(T - \lambda)^{-1}x$. There is no uniqueness implied. If for each $x \in X$ there is the unique analytic extension of $(T - \lambda)^{-1}x$, then T is said to have the *single-valued extension property*, abbreviated SVEP. Hence if T has the SVEP, then there is the maximal analytic extension of $(T - \lambda)^{-1}x$ from $\rho(T)$ to $\rho_T(x)$.

Given an arbitrary operator $T \in L(X)$ and for any set $F \subseteq \mathbb{C}$, we define the *analytic spectral subspace* of T by

$$X_T(F) = \{x \in X \mid \sigma_T(x) \subseteq F\}.$$

In the next Remark, we collect a number of results on analytic spectral subspaces. These results are found in [6].

Remark 3. (1) By the definition of the analytic spectral subspace, it is clear that

$$X_T(F_1) \subseteq X_T(F_2) \quad \text{for } F_1 \subseteq F_2.$$

(2) It is well known that the space $X_T(F)$ is a hyper-invariant subspace of T .

(3) It is easy to see that

$$X_T(F) = X_T(F \cap \sigma(T)).$$

(4) For all $\lambda \in \mathbb{C} \setminus F$,

$$(T - \lambda)X_T(F) = X_T(F)$$

This implies that

$$X_T(F) \subseteq E_T(F) \quad \text{for all } F \subseteq \mathbb{C}.$$

(5) If $\{F_\alpha\}$ is a family of subsets of \mathbb{C} , then

$$X_T\left(\bigcap_{\alpha} F_\alpha\right) = \bigcap_{\alpha} X_T(F_\alpha).$$

(6) It is well known that T has the SVEP if and only if $X_T(\emptyset) = \{0\}$.

An operator $T \in \mathcal{L}(X)$ is called *decomposable* if for every open covering $\{U, V\}$ of the complex plane \mathbb{C} , there exist $Y, Z \in \text{Lat}(T)$ such that

$$\sigma(T|Y) \subseteq U, \quad \sigma(T|Z) \subseteq V \quad \text{and} \quad Y + Z = X.$$

Decomposable operators are rich. For example, normal operators, spectral operators in the sense of Dunford, operators with totally disconnected spectrums and hence compact operators are decomposable.

An operator $T \in \mathcal{L}(X)$ is said to have *property* (δ) if for every open covering $\{U, V\}$ of the complex plane \mathbb{C} , there exist a pair of analytic functions $f : \mathbb{C} \setminus \overline{U} \rightarrow X$, $g : \mathbb{C} \setminus \overline{V} \rightarrow X$ such that

$$\begin{aligned} (T - \lambda)f(\lambda) &= u, & \text{for all } \lambda \in \mathbb{C} \setminus \overline{U}, \\ (T - \lambda)g(\lambda) &= v, & \text{for all } \lambda \in \mathbb{C} \setminus \overline{V}, \end{aligned}$$

and

$$x = u + v.$$

If T has the SVEP, then property (δ) simply means that

$$X = X_T(\overline{U}) + X_T(\overline{V})$$

for every open covering $\{U, V\}$ of the complex plane \mathbb{C} . Albrecht and Eschmeier showed that T satisfies (δ) if and only if T is similar to a quotient of a decomposable operator. That is, if $T : X \rightarrow Y$ has property (δ) then there exists a Banach space \widehat{X} and a continuous linear surjection $q : \widehat{X} \rightarrow X$ and a decomposable operator $\widehat{T} \in \mathcal{L}(\widehat{X})$ with $Tq = q\widehat{T}$ [1].

Let $\mathcal{F}(\mathbb{C})$ denote the family of all closed subsets of \mathbb{C} and let $\mathcal{S}(X)$ denote the family of all closed linear subspaces of X .

Definition 4. (1) A map $\mathcal{E}(\cdot) : \mathcal{F}(\mathbb{C}) \rightarrow \mathcal{S}(X)$ is called *stable* if it satisfies the following two conditions:

- (i) $\mathcal{E}(\emptyset) = \{0\}$, $\mathcal{E}(\mathbb{C}) = X$.
- (ii) $\mathcal{E}(\bigcap_{n=1}^{\infty} F_n) = \bigcap_{n=1}^{\infty} \mathcal{E}(F_n)$ for any sequence $\{F_n\}$ in $\mathcal{F}(\mathbb{C})$.

(2) A map $\mathcal{E}(\cdot) : \mathcal{F}(\mathbb{C}) \rightarrow \mathcal{S}(X)$ is called a *spectral capacity* if $\mathcal{E}(\cdot)$ is stable and satisfies the following condition:

- (iii) $X = \sum_j \mathcal{E}(\overline{G}_j)$ for every finite open cover $\{G_j\}$ of \mathbb{C} .

We say that $\mathcal{E}(\cdot)$ is *order preserving* if it preserves the inclusion order. Clearly a stable map is order preserving. It is well known that T is decomposable if and only if there exists a spectral capacity $\mathcal{E}(\cdot)$ such that $\mathcal{E}(F) \in \text{Lat}(T)$ and $\sigma(T|\mathcal{E}(F)) \subseteq F$ for each closed set $F \subseteq \mathbb{C}$. In this case the spectral capacity of a closed subset F of \mathbb{C} is uniquely determined and it is the analytic spectral subspace $X_T(F)$.

Let θ be a linear operator from a Banach space X into a Banach space Y . The space

$$\mathfrak{S}(\theta) = \{y \in Y : \text{there is a sequence } x_n \rightarrow 0 \text{ in } X \text{ and } \theta x_n \rightarrow y \text{ in } Y\}$$

is called the *separating space* of θ . It is easy to see that $\mathfrak{S}(\theta)$ is a closed linear subspace of Y . By the closed graph theorem, θ is continuous if and only if $\mathfrak{S}(\theta) = \{0\}$. The following lemma is found in [10].

Lemma 5. *Let X and Y be Banach spaces. If R is a continuous linear operator from Y to a Banach space Z , and if $\theta : X \rightarrow Y$ is a linear operator, then $(R\mathfrak{S}(\theta))^- = \mathfrak{S}(R\theta)$. In particular, $R\theta$ is continuous if and only if $R\mathfrak{S}(\theta) = \{0\}$.*

The next lemma states that a certain descending sequence of separating space which obtained from θ via a countable family of continuous linear operators is eventually constant. This lemma is proved in [11].

Stability Lemma. *Let $\langle T_n : n = 1, 2, \dots \rangle$ be a sequence of bounded linear operators on a Banach space X and let $\langle S_n : n = 1, 2, \dots \rangle$ be a sequence of bounded linear operators on a Banach space Y . Suppose that $\theta : X \rightarrow Y$ be a linear operator with separating space $\mathfrak{S}(\theta)$, satisfying $\theta T_n - S_n \theta$ is continuous for all $n = 1, 2, \dots$. Then there is an $n_0 \in \mathbb{N}$ for which*

$$(S_1 S_2 \cdots S_n \mathfrak{S}(\theta))^- = (S_1 S_2 \cdots S_{n_0} \mathfrak{S}(\theta))^- \quad \text{for all } n \geq n_0.$$

The following lemma, known as *localization of the singularities*, is adopted from [4].

Lemma 6. *Let X and Y be Banach spaces. Suppose that $\mathcal{E}_X : \mathcal{F}(\mathbb{C}) \rightarrow \mathcal{S}(X)$ is an order preserving map such that $X = \mathcal{E}_X(\overline{U}) + \mathcal{E}_X(\overline{V})$ whenever $\{U, V\}$ is an open cover of \mathbb{C} . And suppose that $\mathcal{E}_Y : \mathcal{F}(\mathbb{C}) \rightarrow \mathcal{S}(Y)$ is a stable map. If $\theta : X \rightarrow Y$ is a linear operator for which*

$$\mathfrak{S}(\theta|\mathcal{E}_X(F)) \subseteq \mathcal{E}_Y(F) \quad \text{for every } F \in \mathcal{F}(\mathbb{C}),$$

then there is a finite set $\Lambda \subseteq \mathbb{C}$ for which $\mathfrak{S}(\theta) \subseteq \mathcal{E}_Y(\Lambda)$.

This lemma tells us that under appropriate assumptions on a linear operator which have a large lattice of closed invariant subspaces the separating space will be contained eventually in a small closed invariant subspace.

We need the next theorem, known as Mittag-Leffler Theorem of Bourbaki, which is found in [10].

Mittag-Leffler Theorem. *Let $\langle R_n : n = 1, 2, \dots \rangle$ be a countable commuting sequence of bounded linear operators on a Banach space Y . Let Y_∞ be the maximal linear subspace of Y such that $R_n Y_\infty = Y_\infty$ for all $n = 1, 2, \dots$, and let Y^∞ be the maximal closed linear subspaces of Y such that $(R_n Y^\infty)^- = Y^\infty$ for all $n = 1, 2, \dots$. Then we have*

$$Y^\infty = (Y_\infty)^-.$$

3. CONTINUITY OF GENERALIZED INTERTWINING LINEAR OPERATOR FOR OPERATORS WITH PROPERTY (δ) AND HYPONORMAL OPERATORS

Let H be a Hilbert space over the complex plane \mathbb{C} with the inner product (\cdot, \cdot) and let $\mathcal{L}(H)$ denote the Banach algebra of bounded linear operators on H .

An operator $T \in \mathcal{L}(H)$ is said to be *hyponormal* if its self commutator $[T^*, T] = T^*T - TT^*$ is positive, that is

$$((T^*T - TT^*)\xi, \xi) \geq 0,$$

or equivalently,

$$\|T^*\xi\| \leq \|T\xi\|$$

for every $\xi \in H$.

It is well known that every hyponormal operator T has the single valued extension property and for any closed set F in \mathbb{C} , the analytic spectral subspace $H_T(F)$ is closed [7].

The following proposition is found in [9].

Proposition 7. *Let T be a hyponormal operator on a Hilbert space H . Then*

$$H_T(F) = \bigcap_{\lambda \notin F} (T - \lambda)H,$$

for any closed subset $F \subseteq \mathbb{C}$.

For a hyponormal operator, the following proposition allows us to combine the analytic tools associated with the space $H_T(F)$ with the algebraic tools associated with the space $E_T(F)$.

Proposition 8. *Let T be a hyponormal operator on a Hilbert space H . Then for any closed set F of \mathbb{C} , $H_T(F) = E_T(F)$.*

Proof. Let F be a closed subset of \mathbb{C} . From the definition of the algebraic spectral subspace, it is clear that

$$E_T(F) \subseteq \bigcap_{\lambda \notin F, n \in \mathbb{N}} (T - \lambda)^n H.$$

By Proposition 7, we have

$$E_T(F) \subseteq \bigcap_{\lambda \notin F, n \in \mathbb{N}} (T - \lambda)^n H \subseteq \bigcap_{\lambda \notin F} (T - \lambda)H = H_T(F).$$

Therefore we have,

$$H_T(F) = E_T(F)$$

for any closed subset F of \mathbb{C} . □

By the above proposition, hyponormal operators do not have non-trivial divisible subspaces.

Let T and S be bounded linear operators on Banach spaces X and Y , respectively. A linear operator $\theta : X \rightarrow Y$ is said to be an *intertwining linear operator* for the pair (S, T) if $S\theta = \theta T$. Let $C(S, T)$ denote the commutator,

$C(S, T)\theta = S\theta - \theta T$. For a natural number n , define $C(S, T)^n$ to be the n -th composition. That is,

$$C(S, T)^n\theta = C(S, T)^{n-1}(S\theta - \theta T) = \sum_{k=0}^n \binom{n}{k} S^k\theta(-T)^{n-k}.$$

Then we shall say that θ is a *generalized intertwining linear operator* for (S, T) if

$$\|C(S, T)^n\theta\|^{\frac{1}{n}} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

For this to make sense, continuity of $C(S, T)^n\theta$ for one, hence for all sufficiently large n , is assumed.

The following lemma is found in [8].

Lemma 9. *Let T and S be bounded linear operators on Banach spaces X and Y , respectively. And let $\theta : X \rightarrow Y$ be a linear operator. If $F \subseteq \mathbb{C}$ satisfies $C(S, T)^n\theta E_T(F) \subseteq E_S(F)$ for some $n \in \mathbb{N}$, then actually we have*

$$\theta E_T(F) \subseteq E_S(F).$$

Proposition 10. *Suppose that T be a decomposable operator on a Banach space X and let S be a hyponormal operator on a Hilbert space H . Then every generalized intertwining linear operator $\theta : X \rightarrow H$ for (S, T) necessarily satisfies the following:*

$$\theta X_T(F) \subseteq H_S(F) \quad \text{for all closed subsets } F \text{ of } \mathbb{C}.$$

Proof. Since θ is a generalized intertwining linear operator for (S, T) , there is a $k \in \mathbb{N}$ with $C(S, T)^k\theta$ is continuous. By the assumption we have

$$\|C(S, T)^n C(S, T)^k\theta\|^{\frac{1}{n}} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Thus we may apply the proof of [2, Theorem 2.3.3] (this theorem remains valid if the operator S on the range space H is only assumed to have the single valued extension property and closed space $H_S(F)$ for all closed $F \subseteq \mathbb{C}$, this condition is certainly fulfilled in the case of hyponormal operator). Then for a given closed set $F \subseteq \mathbb{C}$ we have

$$C(S, T)^k\theta X_T(F) \subseteq H_S(F).$$

Let $R = T|_{X_T(F)}$ and consider $E_R(F)$. Since $R - \lambda$ is surjective for any $\lambda \notin F$,

$$X_T(F) = E_R(F).$$

Hence by the assumption,

$$C(S, T)^k\theta E_R(F) = C(S, T)^k\theta|_{X_T(F)}(E_R(F)) \subseteq H_S(F) \subseteq E_S(F),$$

so that by the above lemma we have

$$\theta|_{X_T(F)}(E_R(F)) \subseteq E_S(F).$$

Since S is hyponormal, $H_S(F) = E_S(F)$, by Proposition 8. That is

$$\theta X_T(F) \subseteq H_S(F).$$

This completes the proof. \square

Let $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$. A complex number $\lambda \in \mathbb{C}$ is said to be a *critical eigenvalue* for the pair (S, T) if $(T - \lambda)X$ is of infinite codimension in X and λ is an eigenvalue of S .

The following proposition is well known and tells us to existence of discontinuous intertwining linear operators for (S, T) . We include the proof of the proposition for convenience.

Proposition 11. *Let T and S be bounded linear operators on Banach spaces X and Y , respectively. If (S, T) has a critical eigenvalue, then there is a discontinuous linear operator $\theta : X \rightarrow Y$ with $S\theta = \theta T$.*

Proof. Let μ be a critical eigenvalue for the pair (S, T) . Since $X/(T - \mu)X$ is of infinite dimension, we can find a discontinuous linear functional f on X such that

$$f((T - \mu)X) = \{0\}.$$

Let $y \neq 0$ be a μ -eigenvector of S in Y , and let $\theta : X \rightarrow Y$ be defined by $\theta(x) = f(x)y$ for all $x \in X$. Then θ is discontinuous and

$$\theta(T - \mu) = (S - \mu)\theta = 0$$

and so $S\theta = \theta T$. \square

Theorem 12. *Let T be a bounded linear operator on a Banach space X with property (δ) and let S be a hyponormal operator on a Hilbert space H . Then the following statements are equivalent:*

(a) *Every generalized intertwining linear operator $\theta : X \rightarrow H$ for (S, T) is necessarily continuous.*

(b) *The pair (S, T) has no critical eigenvalues.*

Proof. (a) \Rightarrow (b) By Proposition 11, it is clear.

(b) \Rightarrow (a) Assume that the condition (b) is fulfilled, and consider an arbitrary generalized intertwining linear operator $\theta : X \rightarrow H$ for (S, T) . To prove the continuity of θ , it suffices to construct a non-trivial polynomial p such that $p(S)\mathfrak{S}(\theta) = \{0\}$. Indeed if we do so, all injective factors $S - \lambda$ of $p(S)$ may be removed from $p(S)$; what is left still annihilate $\mathfrak{S}(\theta)$. Thus we have obtained a polynomial p , all of whose roots are eigenvalues of S . Since θ is a generalized intertwining linear operator for (S, T) , there is a natural number n such that $C(S, T)^n \theta$ is continuous.

(case 1.) $n = 1$

Suppose that $C(S, T)\theta$ is continuous. And suppose that there is a non trivial polynomial p such that $p(S)\mathfrak{S}(\theta) = \{0\}$ and all roots of p are eigenvalues of S . Since $S\theta - \theta T$ is continuous, $p(S)\theta - \theta p(T)$ is also continuous. By assumption $p(S)\theta$ is continuous, so we have $\theta p(T)$ is continuous. Let λ be a root of p . Since (S, T) has no critical eigenvalues, $(T - \lambda)X$ is of finite codimension in X . This means that

$p(T)X$ is of finite codimension in X . Hence the open mapping theorem implies that $p(T)X$ is closed and that $p(T)$ is an open mapping from X onto $p(T)X$. Therefore, θ is continuous.

(case 2.) $n > 1$

Suppose that $C(S, T)^n \theta$ is continuous for $n > 1$. And suppose that there is a non trivial polynomial p such that $p(S)\mathfrak{S}(\theta) = \{0\}$ and all roots of p are eigenvalues of S . We define

$$\theta_k = C(S, T)^{n-k} \theta$$

for $k = 0, 1, \dots, n$. Then $\theta_0 = C(S, T)^n \theta$ is continuous and $\theta_n = \theta$. Moreover, for all polynomial q and all $k = 1, \dots, n$, we have

$$\theta_{k-1} q(T) = C(S, T)(\theta_k q(T)) \quad \text{and} \quad p(S)\mathfrak{S}(\theta_k q(T)) = \{0\},$$

since the continuity of $p(S)\theta$ obviously forces $p(S)\theta_k q(T)$ to be continuous as well. Hence we may successively apply the proof of (case 1) of this case to obtain polynomials p_1, p_2, \dots, p_n whose roots are all eigenvalues of S and $\theta_k p_1(T) \cdots p_k(T)$ is continuous for $k = 1, \dots, n$. Let $r = p_1 \cdots p_n$. Then all roots of r are eigenvalues of S . Hence $\theta r(T)$ is continuous. Since the pair (S, T) has no critical eigenvalues, $r(T)X$ is of finite codimension in X . By the open mapping theorem $r(T)X$ is closed in X and $r(T)$ is an open mapping from X onto $r(T)X$. Therefore, θ is continuous.

In any case, if we have a non trivial polynomial p such that $p(S)\mathfrak{S}(\theta) = \{0\}$, then the continuity of θ is ensured. Now, we will construct a non trivial polynomial p such that $p(S)\mathfrak{S}(\theta) = \{0\}$. Since T has property (δ) , there is a Banach space \widehat{X} and a continuous linear surjection $q : \widehat{X} \rightarrow X$ and a decomposable operator $\widehat{T} \in \mathcal{L}(\widehat{X})$ with $Tq = q\widehat{T}$. Hence it is clear that

$$(C(S, T)^n \theta)q = C(S, \widehat{T})^n (\theta q).$$

Since $\theta : X \rightarrow Y$ is a generalized intertwining linear operator for (S, T) , $\theta q : \widehat{X} \rightarrow Y$ is a generalized intertwining linear operator for (S, \widehat{T}) . Let $\widehat{\theta} = \theta q$. Then we observe that it suffices to consider the case that $C(S, \widehat{T})\widehat{\theta}$ is continuous: indeed, the general case can be easily deduced by this special case and the argument of the proof of (case 2). Since \widehat{T} is decomposable and S is hyponormal, from Proposition 10, we infer that

$$\widehat{\theta}(\widehat{X}_{\widehat{T}}(F)) \subseteq H_S(F)$$

for all closed subsets F of \mathbb{C} . Since $\widehat{X}_{\widehat{T}}(F)$ is the spectral capacity and $H_S(F)$ is stable, by Lemma 6, there is a finite set Λ of \mathbb{C} such that $\mathfrak{S}(\widehat{\theta}) \subseteq H_S(\Lambda)$. An application of the Stability Lemma to the sequence $S - \lambda$, where $\lambda \in \Lambda$, yields a polynomial p for which

$$((S - \lambda)p(S)\mathfrak{S}(\widehat{\theta}))^- = (p(S)\mathfrak{S}(\widehat{\theta}))^- \quad \text{for every } \lambda \in \Lambda.$$

Applying Mittag-Leffler Theorem, there exists a dense subspace

$$W \subseteq (p(S)\mathfrak{S}(\widehat{\theta}))^-$$

for which $(S - \lambda)W = W$ for every $\lambda \in \Lambda$. This means that

$$W \subseteq E_S(\mathbb{C} \setminus \Lambda)$$

by the definition of algebraic spectral subspaces. From the continuity of $C(S, \widehat{T})\widehat{\theta}$ we deduce that $p(S)\mathfrak{S}(\widehat{\theta}) \subseteq \mathfrak{S}(\widehat{\theta})$. Hence $W \subseteq \mathfrak{S}(\widehat{\theta}) \subseteq E_S(\Lambda)$, and we obtain that

$$W \subseteq E_S(\Lambda) \cap E_S(\mathbb{C} \setminus \Lambda) = E_S(\emptyset).$$

Since S has no non trivial divisible subspace, we have $W = \{0\}$. Consequently, $p(S)\widehat{\theta} = \{0\}$. Hence $p(S)\widehat{\theta} = p(S)\theta q$ is continuous. Since q is a continuous linear surjection, by the open mapping theorem, $p(S)\theta$ is also continuous. Therefore, θ is continuous. This completes the proof. \square

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