Trends in Mathematics
Information Center for Mathematical Sciences
Volume 6, Number 2, December, 2003, Pages 41–49

GRAPH C^*-ALGEBRAS AND TOPOLOGICAL ENTROPY

JA A JEONG

ABSTRACT. Given a directed graph E, it is well known that there exists a universal C^*-algebra $C^*(E)$ generated by a Cuntz-Krieger E family. For example, Cuntz-Krieger algebras $\mathcal{O}_A$ are graph C^*-algebras associated with finite graphs. $\mathcal{O}_A$ was first introduced and studied as an invariant of the topological conjugacy class of the topological Markov shift (X_A, σ_A) ([CK]), and the map $\Phi_E : C^*(E) \to C^*(E)$ defined by

$$\Phi_E(x) = \sum_{e \in E^1} s_e x s_e{}^*$$

played an important role in that direction of research. Choda ([Ch1]) proved that for the Cunzt algebra $\mathcal{O}_n$ the Voiculescu's topological entropy ([Vo]) becomes $ht(\Phi_n) = \log n$, which has been extended to Cuntz-Krieger algebras $\mathcal{O}_A$ in [BG] and [PWY] when A is an (irreducible) finite matrix with no sinks. In this paper we discuss the Voiculescu's topological entropy $ht(\Phi_E)$ of the (completely positive) map Φ_E defined on a C^*-algebra $C^*(E)$ when E is an arbitrary directed graph ([JP2]). We also deal with some of other entropies like the (classical) topological entropy $h_{top}(X_E)$, Gurevic entropy, and a variant of the Salama's block entropy of the edge shift space of E.

1. INTRODUCTION

1.1. It is well known that for a directed graph E there exists a universal C^*-algebra $C^*(E)$ associated with E (see [KPR]). Recently the ideal structures, pure infiniteness conditions, and stability conditions of graph C^*-algebras have been obtained by many authors ([BHRS], [KPRR], [Hj], etc.). It is also known about when $C^*(E)$ has real rank zero in terms of loops in E ([JPS], [JP1]). More recently, the possible stable range for $C^*(E)$ has been obtained ([JPS], [DHS], [KPRR], [BHRS]).

This class of graph C^*-algebras includes the Cuntz-Krieger algebras $\mathcal{O}_A$. Briefly, if $A = (A_{ij})$ is an $n \times n$ $\{0, 1\}$-matrix the algebra $\mathcal{O}_A$ is defined to be the universal C^*-algebra generated by n partial isometries $\{s_i \mid 1 \le i \le n\}$ with orthogonal range projections satisfying the relations:

$$s_i^* s_i = \sum_{j=1}^{n} A_{ij}\, s_j s_j^*,$$

and if E is the finite graph with the incidence matrix A then $\mathcal{O}_A$ is the graph algebra $C^*(E)$. Besides the class of C^*-algebras arising from graphs, there have

2000 *Mathematics Subject Classification.* 46L05, 46L55.

been many generalizations of the Cuntz-Krieger algebras (for example, [EL], [Ma], [Pm] among others).

1.2. The Cuntz-Krieger algebras $\mathcal{O}_A$ were introduced in [CK] to study the shift space induced by the $n\times n$ $\{0,1\}$-matrix A. Such a matrix A is used as a transition matrix in symbolic dynamics. Let σ_A be the subshift acting on the compact one-sided shift space X_A. Then it is proved in [CK] that the Cuntz-Krieger algebra $\mathcal{O}_A$ is an invariant for the topological conjugacy class of σ_A. The positive linear map $\Phi_A : \mathcal{O}_A \to \mathcal{O}_A$ defined by

$$\Phi_A(x) = \sum_{i=1}^{n} s_i\, x\, s_i^*$$

played an important role in the study of the algebra $\mathcal{O}_A$. If D_A is the commutative C^*-subalgebra of $\mathcal{O}_A$ generated by $\{\Phi_A^k(s_i s_i^*) \mid 1 \le i \le n, k \ge 0\}$ then D_A is identified with $C(X_A)$ in such a way that $\Phi_A|_{D_A}$ corresponds to the endomorphism on $C(X_A)$ induced by the shift σ_A.

1.3. The map Φ_A on the algebra $\mathcal{O}_A$ is in fact a completely positive (cp) map which is not necessarily a $*$-homomorphism, in general. Voiculescu [Vo] introduced the topological entropy $ht(\alpha)$ for an automorphism of a unital nuclear C^*-algebra A, and later the notion was extended to automorphisms on exact C^*-algebras by Brown [Br]. But as described in [BG], even for a cp map Φ on an exact C^*-algebra A one has the same definition of the topological entropy $ht(\Phi)$ simply by replacing α by Φ.

Let $\mathcal{O}_n$ be the Cuntz algebra, that is, $\mathcal{O}_n = \mathcal{O}_A$ for the $n \times n$ matrix A with all entries equal to 1. Then $\mathcal{O}_n$ is a simple unital purely infinite nuclear algebra on which $\Phi_n (= \Phi_A)$ becomes a $*$-endomorphism. In [Ch1] Choda computed the topological entropy of Φ_n showing that

$$ht(\Phi_n) = \log n.$$

On the other hand, if E is a finite graph, then one can consider two entropies, that is, the block entropy $h(X_E) = \limsup_{n\to\infty} \frac{1}{n} \log |E^n|$ and the topological entropy $h_{top}(X_E)$ associated with the (compact) edge shift space (X_E, σ_E), and then from [Wt, Theorem 7.13] we see that the two entropies coincide each other. Moreover it is easy to see that $h(X_E) = \log n = \log r(A_E)$ if E is the graph consisting of n simple loops at a single vertex (so that $C^*(E) \cong \mathcal{O}_n$) and $r(A_E)$ is the spectral radius of the edge matrix A_E of E. Thus for the Cuntz algebras $\mathcal{O}_n$, Choda's result means that

$$ht(\Phi_n) = \log n = \log r(A_E) = h(X_E) = h_{top}(X_E),$$

and this has been extended by Boca and Goldstein [BG] to a Cuntz-Krieger algebra $\mathcal{O}_A$ when A is the edge matrix of a strongly connected finite graph E. More generally, the same equality $ht(\Phi_E) = \log r(A_E)$ is obtained for any finite graph E with no sinks in [PWY] where the full C^*-dynamical systems are deeply investigated.

If E is a finite graph possibly with sinks, the cp map $\Phi_E (= \Phi_{A_E})$ is not unital. We defined in [JP2] another cp map $\Psi_E : C^*(E) \to C^*(E)$ which is unital to prove that

$$ht(\Psi_E) = \log r(A_E),$$

and then using this fact we also proved that $ht(\Phi_E) = \log r(A_E)$.

1.4. Let E be an infinite graph and $C^*(E) = C^*\{p_v, s_e \mid v \in E^0, e \in E^1\}$ be its graph C^*-algebra. Then as in the case of finite graph one would think of the map $\Phi_E : C^*(E) \to C^*(E)$ given by

$$\Phi_E(x) = \sum_{e \in E^1} s_e x s_e^*$$

and its topological entropy $ht(\Phi_E)$. But we first need to prove that the sum converges and the map is a cp map. In [JP2], it is shown that if E is locally finite, that is, if every vertex receives and emits only finitely many edges, then Φ_E is a well defined contractive cp map. Thus the topological entropy $ht(\Phi_E)$ is defined. Another problem we meet in dealing with the edge shift space X_E is that it is not compact any more, and so $h_{top}(X_E)$ is not given. Nevertheless, if we use Gurevic's compactification $\overline{X}_E$ ([Gu]) the following inequality can be shown ([JP2]);

$$h_{top}(\overline{X}_E) \le ht(\Phi_E),$$

whenever E is a locally finite irreducible infinite graph. In fact, the left hand side is equal to $\sup_{E'} h_{top}(X_{E'})$, where the supremum is taken over the set of all finite subgraphs E' of E ([Gu]). The inequality is proved by finding a commutative C^*-subalgebra D_E of $C^*(E)$ such that $\Phi_E(D_E) \subseteq D_E$ and $h_{top}(\overline{X}_E) = ht(\Phi_E|_{D_E})$.

Finally we consider the topological entropy $ht(\Phi_E|_{\mathcal{A}})$ of the map Φ_E restricted to the AF subalgebra $\mathcal{A}_0$ which is invariant under Φ_E and $D_E \subset \mathcal{A}$. It is proved in [JP3] that

$$ht(\Phi_E|_{\mathcal{A}_0}) \le \limsup_{n\to\infty} \frac{1}{n} \log |E^n(v)|,$$

where $E^n(v)$ denotes the set of all finite paths with length n passing through the vertex v. Here the value $\limsup_{n\to\infty} \frac{1}{n} \log |E^n(v)|$ does not depend on the choice of a vertex v if E is irreducible.

2. Graphs and graph C^*-algebras

Let $E = (E^0, E^1, r, s)$ be a directed graph (or simply a graph) with a countable vertex and edge sets E^0 and E^1, where $r, s : E^1 \to E^0$ are the range and source maps. By $\mathcal{S}(E)$ we denote the set of all sinks v ($s^{-1}(v) = \emptyset$) of E. If a finite sequence $\alpha == \alpha_1\alpha_2 \cdots \alpha_n$ of edges satisfies $r(\alpha_i) = s(\alpha_{i+1})$ for each $i = 1, \ldots, n-1$, α is called a (finite) *path of length* $|\alpha| = n$ with the range $r(\alpha) = r(\alpha_n)$ and the source $s(\alpha) = s(\alpha_1)$. E^n will be the set of all finite paths of length n, and we put $E^* = \cup_{n\ge 0} E^n$ (regarding each vertex as a finite path of length zero). A path α ($|\alpha| > 0$) with $s(\alpha) = r(\alpha)$ is called a *loop*.

A graph E is said to have property (L) if every loop has an exit, and (K) if there are at least two distinct loops at v whenever v lies on a loop.

For a graph E, a family $\{s_e, p_v \mid e \in E^1,\ v \in E^0\}$ of partial isometries s_e and mutually orthogonal projections p_v is called a *Cuntz-Krieger E-family* if it satisfies the following relations:

$$s_e^*s_e = p_{r(e)}, \quad s_e s_e^* \le p_{s(e)},$$

$$p_v = \sum_{s(e)=v} s_e s_e^* \quad \text{if } \ 0 < |s^{-1}(v)| < \infty.$$

It is known (see [BHRS], [KPR] for example) that there exists a universal C^*-algebra $C^*(E)$ generated by a Cuntz-Krieger E-family $\{s_e, p_v\}$. We call $C^*(E)$ the *graph C^*-algebra associated with E*. By universality there is the *gauge action* $\gamma : \mathbb{T} \to Aut(C^*(E))$ given by

$$\gamma_z(p_v) = p_v \ \text{ and } \ \gamma_z(s_e) = zs_e.$$

If a row finite graph E satisfies (L) any two Cuntz-Krieger E-families of non-zero elements give rise to the isomorphic C^*-algebra ([KPR]), which is called the uniqueness theorem of graph algebras. Also if E is an arbitrary graph and $\{S_e, P_v \mid e \in E^1, v \in E^0\} \subset B(H)$ is a Cuntz-Krieger E-family with an action $\beta : \mathbb{T} \to Aut(C^*(S_e, P_v))$ such that $\beta_z \circ \pi = \pi \circ \gamma_z$ for $z \in \mathbb{T}$ ($\pi : C^*(E) \to C^*(S_e, P_v)$ is the $*$-homomorphism such that $s_e \mapsto S_e$ and $p_v \mapsto P_v$, such a homomorphism π exists by universality of the graph algebra) then π is in fact an isomorphism ([BHRS]).

¿From definition it follows immediately that all graph algebras are separable since we consider graphs E with countable vertices and edges (see (b) below). Moreover the following useful and interesting facts are known for graph algebras.

(a) $C^*(E)$ is unital if and only E has finitely many vertices.
(b) The linear span of the set $\{s_\alpha s_\beta^* \mid \alpha, \beta \in E^*\}$ is norm dense in $C^*(E)$, where $s_\alpha = s_{\alpha_1} \cdots s_{\alpha_k}$ for $\alpha = \alpha_1 \cdots \alpha_k \in E^k$, $k \ge 1$, and $s_\alpha = p_v$ for $\alpha = v \in E^0$.
(c) $C^*(E)$ is AF if and only if E has no loops ([KPR]).
(d) $C^*(E) \times_\gamma \mathbb{T}$ is always AF: The crossed product $C^*(E) \times_\gamma \mathbb{T}$ is stably isomorphic to a graph C^*-algebra associated with a graph which has no loops ([KP]).
(e) Graph C^*-algebras are nuclear: $C^*(E)$ is stably isomorphic to

$$(C^*(E) \times_\gamma \mathbb{T}) \times_{\hat\gamma} \hat{\mathbb{T}} \cong AF \times_{\hat\gamma} \mathbb{Z}$$

by the Takesaki-Takai duality ([Ku, p.193]).
(f) $C^*(E)$ is simple if and only if E is a cofinal graph with (L) (E is cofinal if $v \in E^0$ and $\alpha = \alpha_1\alpha_2 \cdots$ is an infinite path there exists a finite path β such that $s(\beta) = v$ and $r(\beta) = s(\alpha_i)$ for some i). Moreover, if $C^*(E)$ is simple then it is either AF (when there is no loops in E) or purely infinite (there are loops in E). We refer the reader to [KR] for the definition of purely infinite C^*-algebras.
(g) Let E be a locally finite directed graph with no sinks. Then $C^*(E)$ is purely infinite if and only if no quotient of $C^*(E)$ contains an AF ideal, nor does it contain a corner isomorphic to $M_n(C(\mathbb{T}))$ for some $n \in \mathbb{N}$ ([Hj]).

(h) The ideal structure of $C^*(E)$ can be read from the graph itself ([BHRS]). Particularly, if a locally finite graph E has (K), there is a lattice isomorphism between the ideals of $C^*(E)$ and the lattice of saturated hereditary subsets of E^0 ([KPRR]).

(i) A C^*-algebra is of *real rank zero* ($RR(A) = 0$) if every non-zero hereditary C^*-subalgebra of A contains an approximate identity of projections ([BP]). It is known that $RR(C^*(E)) = 0$ if and only if E satisfies (K) ([JP1]).

(j) The only possible values of the stable rank of $C^*(E)$ are 1, 2, or ∞ ([DHS]). In particular, $sr(C^*(E)) = 1$ if and only if no loops in E has an exit ([JPS], [DHS]).

3. Shift space and entropies

Let $\mathcal{A}$ be a finite set and let $X \subset \mathcal{A}^{\mathbb{N}}$ be a (one-sided) *shift space* with the shift map σ_X. Then the *entropy* $h(X)$ of X is defined by

$$h(X) = \lim_{n\to\infty} \frac{1}{n} \log |W_n(X)|,$$

where $W_n(X)$ is the set of all words of length n that appear in a sequence of X ([LM, Definition 4.1.1] or [Kt, p.23]). If $X \neq \emptyset$ we have $0 \leq h(X) < \log |\mathcal{A}| < \infty$ since $1 \leq |W_n(X)| \leq |\mathcal{A}|^n$. In particular, the full shift space $X_n = \mathcal{A}^{\mathbb{N}}$, $|\mathcal{A}| = n$, has $h(X_n) = \log n$. If $X = \emptyset$ then $h(X) = -\infty$ by definition.

Let $T : X \to X$ be a continuous map on a compact space X. If $\mathcal{U}$ is an open cover of X then so is $T^{-1}\mathcal{U}$. By $N(\mathcal{U})$ we denote the number of sets in a finite subcover of $\mathcal{U}$ with smallest cardinality. Then the *entropy of T relative* to $\mathcal{U}$ is given by

$$h_{top}(T,\mathcal{U}) := \lim_{n\to\infty} \frac{1}{n} \log(N(\vee_{i=0}^{n-1} T^{-i}\mathcal{U})),$$

where $\mathcal{U}\vee\mathcal{V}$ denotes the join of $\mathcal{U}$ and $\mathcal{V}$, and the *topological entropy* of T is defined by

$$h_{top}(T) = \sup_{\mathcal{U}} h_{top}(T,\mathcal{U}),$$

where the supremum is taken over all the open covers (or equivalently, over all the finite open covers) of X ([Wt, Chapter 7]).

Remark 3.1. Let E be a finite graph and X_E be the one-sided shift space associated with E.

(a) Let $X_E = \{\alpha = (\alpha_i) \in (E^1)^{\mathbb{N}} \mid r(\alpha_i) = s(\alpha_{i+1}) \text{ for all } i \in \mathbb{N}\}$ be the edge shift space with the shift map $\sigma = \sigma_E$ given by $\sigma(\alpha)_i = \alpha_{i+1}$ for each $i \in \mathbb{N}$. If there is an infinite path in E

$$h_{top}(X_E) = h(X_E).$$

Otherwise $X_E = \emptyset$ and so $h(X_E) = -\infty$ ([Wt, Theorem 7.13]).

(b) $h(X_E) = \log \lambda_E = \log r(A_E)$, where λ_E is the Perron value of the edge matrix A_E of E and $r(A_E)$ is the spectral radius of A_E (see [JP2], for example).

4. Voiculescu's topological entropy of cp maps Φ_E

Let $\pi : A \to B(H)$ be a faithful representation of a C^*-algebra A and let $Pf(A)$ be set of all finite subsets of A. Put

- $CPA(\pi, A) = \{(\phi, \psi, B) \mid \phi : A \to B, \psi : B \to B(H)$ are contractive completely positive maps and $\dim B < \infty\}$,
- $rcp(\pi, \omega, \delta) = \inf\{rank(B) \mid (\phi, \psi, B) \in CPA(\pi, A), \|\psi \circ \phi(x) - \pi(x)\| < \delta$ for all $x \in \omega\}$, where $rank(B)$ denotes the dimension of a maximal abelian subalgebra of B.

Since $C^*(E)$ is nuclear (hence exact) and every exact C^*-algebra is nuclearly embeddable ([Kr]), for each $\omega \in Pf(C^*(E))$ and $\delta > 0$ the value $rcp(\pi, \omega, \delta)$ is finite and moreover is independent of the choice of π ([Br], [BG]). Assuming that $C^*(E) \subset B(H)$ for a Hilbert space H we may write $rcp(\omega, \delta)$ for $rcp(\pi, \omega, \delta)$.

Definition 4.1. ([Br], [BG]) Let $A \subset B(H)$ be an exact C^*-algebra and $\Phi : A \to A$ be a cp (completely positive) map. Then we define

$$\begin{aligned} ht(\Phi, \omega, \delta) &= \limsup_{n\to\infty} \frac{1}{n} \log\big(rcp(\omega \cup \Phi(\omega) \cup \cdots \cup \Phi^{n-1}(\omega), \delta)\big), \\ ht(\Phi, \omega) &= \sup_{\delta>0} ht(\Phi, \omega, \delta), \\ ht(\Phi) &= \sup_{\omega\in Pf(A)} ht(\Phi, \omega). \end{aligned}$$

$ht(\Phi)$ is called the *topological entropy* of Φ.

Remark 4.1. (a) ([De]) Let $T : X \to X$ be a continuous map on a compact metric space X. Then $ht(T^*) = h_{top}(X, T)$, where $T^* : C(X) \to C(X)$ is the cp map given by $T^*(f) = f \circ T,\ f \in C(X)$.

(b) Let α be an automorphism of a unital C^*-algebra A. Choda ([Ch2]) introduced the C^*-dynamical entropy $ht_\phi(\alpha)$ with respect to an α-invariant state ϕ of A, and proved that $h_\phi(\alpha) \le ht_\phi(\alpha) \le ht(\alpha)$, where $h_\phi(\alpha)$ is the CNT-entropy.

(c) The following basic results are well known.

(i) If $\Phi : A \to A$ is a cp map and $\theta : A \to B$ is a C^*-isomorphism then $\theta\Phi\theta^{-1} : B \to B$ is also a cp map and

$$ht(\Phi) = ht(\theta\Phi\theta^{-1}).$$

(ii) Let $\tilde{A}$ be the unital C^*-algebra obtained by adjoining a unit. Let $\tilde{\Phi} : \tilde{A} \to \tilde{A}$ be the extension of Φ. Then

$$ht(\tilde{\Phi}) = ht(\Phi).$$

(iii) Let Φ be a cp map on A. If $A_0 \subset A$ is a C^*-subalgebra of A such that $\Phi(A_0) \subset A_0$. Then

$$ht(\Phi|_{A_0}) \le ht(\Phi).$$

Let E be a finite graph with the sinks $\mathcal{S}(E)$. Then the graph algebra $C^*(E) = C^*\{s_e, p_v\}$ is unital with the unit $\sum_{v\in E^0} p_v$. Consider the following cp maps $\Phi_E, \Psi_E : C^*(E) \to C^*(E)$ given by

$$\Phi_E(x) = \sum_{e\in E^1} s_e x {s_e}^*,$$
$$\Psi_E(x) = \Phi_E(x) + \sum_{v\in\mathcal{S}(E)} p_v x p_v.$$

The map Ψ_E is always unital while Φ_E may not. We call Φ_E the *canonical cp map* of $C^*(E)$.

Proposition 1. ([JP2]) *Let E be a finite graph with the edge matrix A_E. If E contains an infinite path then*

$$ht(\Psi_E) = \log r(A_E),$$

where $r(A_E)$ is the spectral radius of the edge matrix A_E of E.

The inequality $\log r(A_E) \leq ht(\Psi_E)$ follows from Remark 3.1.(b) by applying Remark 4.1.(c)(iii) to the Ψ_E-invariant commutative C^*-subalgebra D_E of $C^*(E)$ that is isomorphic to $C(X_E)$ and $\Phi_E|_{D_E}$ corresponds to the shift map σ_{X_E}.

We can prove the following theorem using the above proposition.

Theorem 2. ([Ch1], [BG], [PWY], [JP2]) *Let E be a finite graph. Then the topological entropy of the cp map Φ_E is*

$$ht(\Phi_E) = \log r(A_E) = h_{top}(X_E).$$

Now let E be a locally finite infinite graph. Then it is known in [JP2] that the map

$$\Phi_E : C^*(E) \to C^*(E), \quad \Phi_E(x) = \sum_{e\in E^1} s_e x {s_e}^*, \ x \in C^*(E)$$

is a well defined contractive cp map. Hence we can think of its topological entropy $ht(\Phi_E)$. As in the case of finite graph, one might expect that $h_{top}(X_E)$ is the lower bound for $ht(\Phi_E)$. But the locally compact shift space X_E may not be compact, hence $h_{top}(X_E)$ is not defined and we need to consider a compactification of X_E. So let $\overline{X}_E$ be the Gurevic's compactification of X_E ([Gu]).

Theorem 3. ([JP2]) *Let E be a locally finite irreducible infinite graph. Then*

$$h_{top}(\overline{X}_E) = \sup_{E'} h(X_{E'}) \ \leq \ ht(\Phi_E),$$

where the supremum is taken over all the finite subgraphs of E.

The first equality in the theorem is proved in [Gu]. One can consider the one point compactification $\widetilde{X}_E$ of the locally compact space X_E. It is easy to see that if E is locally finite then $\widetilde{X}_E$ is topologically conjugate to Gurevic's compactification

$\overline{X}_E$. Hence they have the same topological entropy, $h_{top}(\widetilde{X}_E) = h_{top}(\overline{X}_E)$. Also the topological entropy by Gurevic coincides with the growth rate of the number of loops of length n from any fixed vertex to itself if the graph is irreducible.

Now let E be an irreducible infinite graph, and let $\mathcal{A}_0$ be the AF subalgebra of $C^*(E)$ generated by $\{s_\alpha s_\beta^* \mid \alpha, \beta \in E^*, |\alpha| = |\beta|\}$. Then clearly $D_E \subset \mathcal{A}_0$ and $\mathcal{A}_0$ is Φ_E-invariant so that $ht(\Phi_E|_{\mathcal{A}_0}) \leq ht(\Phi_E)$. Also an AF subalgebra $\mathcal{A}_v = C^*\{s_\alpha s_\beta^* \mid \alpha, \beta \in E^*, |\alpha| = |\beta|, r(\alpha) = r(\beta) = v\}$ of $\mathcal{A}_0$ is Φ_E-invariant for each $v \in E^0$. Recall that the Salama's block entropy of the shift space (X_E, σ_E) is given by $\limsup_{n\to\infty} \frac{1}{n} \log |B_n(v)|$, where $B^n(v)$ is the set of all finite paths α with $|\alpha| = n$, $s(\alpha) = v$. The entropy value does not depend on the choice of a vertex v if E is irreducible. If $B'_n(v)$ is the set of all finite paths α with $r(\alpha) = v$, $|\alpha| = n$, then we can think of a variant of Salama's entropy, that is, $\limsup_{n\to\infty} \frac{1}{n} \log |B'_n(v)|$.

Theorem 4. ([JP3]) *Let E be a locally finite irreducible infinite graph and let $E^n(v)$ be the set of all finite paths α passing through v and $|\alpha| = n$. Then*

$$ht(\Phi_E|_{\mathcal{A}_0}) \leq \limsup_{n\to\infty} \frac{1}{n} \log |E^n(v)|,$$

and $\limsup_{n\to\infty} \frac{1}{n} \log |E^n(v)|$ *is independent of the choice of v. Also we have*

$$ht(\Phi_E|_{\mathcal{A}_v}) \leq \limsup_{n\to\infty} \frac{1}{n} \log |B'_n(v)|.$$

Remark 4.2. Theorem 4 implies that $ht(\Phi_E) = \infty$ in many cases. It would be important to find an upper bound for $ht(\Phi_E)$ or a better lower bound for $ht(\Phi_E|_{\mathcal{A}_0})$ than the Gurevic entropy. These efforts will be helpful for computing the entropy $ht(\Phi_E)$.

References

[AKM] R. L. Adler, A. G. Konheim and M. H. McAndrew, *Topological entropy*, Trans. Amer. Math. Soc. **114** (1965), 309–319.

[BG] Florin P. Boca and Paul Goldstein, *Topological entropy for the canonical endomorphism of Cuntz-Krieger algebras*, Bull. London Math. Soc. **32** (2000), 345–352.

[BHRS] T. Bates, J. H. Hong, I. Raeburn and W. Szymanski, *The ideal structure of the C^*-algebras of infinite graphs*, Illinois J. Math., to appear.

[Br] N. P. Brown, *Topological entropy in exact C^*-algebras*, Math. Annalen, **314** (1999), 347-367.

[BP] L. G. Brown and G. K. Pedersen, *C^*-algebras of real rank zero*, J. Funct. Anal. **99** (1991), no. 1, 131–149.

[Ch1] M. Choda, *Endomorphisms of shift type (entropy for endomorphisms of Cuntz algebras)*, Operator Algebras and Quantum Field Theory (Rome, 1996), 469–475, International Press, Cambridge, MA.

[Ch2] M. Choda, *Dynamical entropy for automorphisms of exact C^*-algebras*. J. Funct. Anal. **198** (2003), no. 2, 481–498.

[Cu] J. Cuntz, *Simple C^*-algebras generated by isometries*, Comm. Math. Phys. **57** (1977), 173–185.

[CK] J. Cuntz and W. Krieger, *A class of C^*-algebras and topological Markov chains*, Invent. Math. **56** (1980), 251–268.

[De] V. Deaconu, *Entropy estimates for some C^*-endomorphisms*, Proc. Amer. Math. Soc. **127** (1999), no. 12, 3653–3658.

[DHS] K. Deicke, J. H. Hong, and W. Szymanski, *Stable rank of graph algebras. Type I graph algebras and their limits*, arXiv:math.OA/0211144v1 Nov 2002.

[EL] R. Exel and M. Laca, *Cuntz-Krieger algebras for infinite matrices*, J. Reine Angew. Math. **512** (1999), 119–172.

[Gu] B. M. Gurevic, *Topological entropy of enumerable Markov chains*, Dokl. Akad. Nauk SSSR **187** (1969), 216–226. Soviet Math. Dokl. **10** (1969), 911–915.

[Hj] J. Hjelmborg, *Purely infinite and stable C^*-algebras of graphs and dynamical systems*. Ergodic Theory Dynam. Systems **21** (2001), no. 6, 1789–1808.

[JP1] J. A Jeong and G. H Park, *Graph C^*-algebras with real rank zero*, J. Func. Anal. **188** (2002), 216-226.

[JP2] J. A Jeong and G. H Park, *Topological entropy for the canonical conmpletely positive maps of graph C^*-algebras*, preprint.

[JP3] J. A Jeong and G. H Park, in preparation.

[JPS] J. A Jeong, G. H Park, and D. Y Shin, *Stable rank and real rank of graph C^*-algebras*, Pacific J. Math. **200** (2001), 331-343.

[Kr] E. Kirchberg, *On subalgebras of the CAR-algebra*, J. Funct. Anal. **129** (1995), no. 1, 35–63.

[KR] E. Kirchberg and M. Rordam *Non-simple purely infinite C^*-algebras*, Amer. J. Math. **122** (2000), no. 3, 637–666.

[Kt] B. P. Kitchens, *Symbolic Dynamics*, Springer 1998.

[Ku] A. Kumjian, *Notes on C^*-algebras of graphs*, Contemporary Math. 228, Operator Algebras and Operator Theory, 1998, AMS.

[KP] A. Kumjian and D. Pask, *C^*-algebras of directed graphs and group actions*, Ergodic Theory Dynam. Systems.

[KPR] A. Kumjian, D. Pask, and I. Raeburn, *Cuntz-Krieger algebras of directed graphs*, Pacific J. Math. **184** (1998), 161–174.

[KPRR] A. Kumjian, D. Pask, I. Raeburn, and J. Renault, *Graphs, Groupoids, and Cuntz-Krieger algebras*, J. Funct. Anal. **144** (1997), 505-541.

[LM] D. Lind and B. Marcus, *An introduction to symbolic dynamics and coding*, Cambridge University Press 1995.

[Ma] K. Matsumoto, *On C^*-algebras associated with subshifts*, Internat. J. Math. **8** (1997), 357-374.

[PWY] C. Pinzari, Y. Watatani, and K. Yonetani, *KMS states, entropy and the variational principle in full C^*-dynamical systems*, Commun. Math. Phys. **213** (2000), 331-379.

[Pm] M. Pimsner, *A class of C^*-algebras generalizing both Cuntz-Krieger algebras and crossed products by $\mathbb{Z}$*, in Free probability theory (D. Voiculescu, Ed.), 189-212, Fields Institute Communications **12**, Amer. Math. Soc.

[Vo] D. Voiculescu, *Dynamical approximation entropies and topological entropy in operator algebras*, Comm. Math. Phys. **170** (1995), 249–281.

[Wt] P. Walters, *An introduction to ergodic theory*, GTM 79, Springer 1982.

School of Mathematical Sciences, Seoul National University, Seoul, 151–742, Korea, email: jajeong@math.snu.ac.kr