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ALUTHGE TRANSFORMS OF OPERATORS

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ABSTRACT. The Aluthge transform $\widetilde{T}$ (defined below) of an operator T on Hilbert space has been studied extensively, most often in connection with p-hyponormal operators. In [19] one initiated a study of various relations between an arbitrary operator T and its associated $\widetilde{T}$, and this study was continued to the theories of spectral pictures, Aluthge and Duggal iterations, and numerical ranges. In this article we survey those recent works and give some related open problems.

1. INTRODUCTION

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. An arbitrary operator T in $\mathcal{L}(\mathcal{H})$ has a unique polar decomposition $T = U|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$ and U is the appropriate partial isometry (with ker $U = \ker T$ and ker $U^* = \ker T^*$). An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *p-hyponormal* if $(T^*T)^p - (TT^*)^p \geq 0$, $p \in (0, \infty)$ ([14]). If $p = 1$, T is hyponormal and if $p = \frac{1}{2}$, T is *semi-hyponormal* ([28]). The Lőwner-Heinz inequality ([17]) implies that p-hyponormal operators are q-hyponormal operators for $q \leq p$. In particular, T is said to be *∞-hyponormal* if T is p-hyponormal for every $p > 0$ ([24]). It is well known that every qusinormal operators are ∞-hyponormal. Associated with T there is a very useful related operator $\widetilde{T}_{r,t} = |T|^t U |T|^{r-t}$ for $r \geq t \geq 0$, called the *generalized Aluthge transform* of T ([16]). This transform $\widetilde{T}_{r,t}$ is said to be *(r,t)-Aluthge transform.* Then $(1, \frac{1}{2})$-Aluthge transform is refered as the Aluthge transform which is denoted by $\widetilde{T}$ ([1]). The $(1,1)$-Aluthge transform is refered as *Duggal transform of T*, which is denoted by $\widehat{T}$ (i.e., $\widehat{T} := |T|U$) ([13]). In many cases Aluthge and Duggal transforms are useful, and ones concentrate to discuss here their transforms. An operator T is *(r,t)-weakly hyponormal* if $|\widetilde{T}_{r,t}| \geq |T| \geq |\widetilde{T}^*_{r,t}|$ ([18]). The $(1, \frac{1}{2})$-weakly hyponormal operator is refered as w-hyponormal operator ([3], [4]). Recall that if $T = U|T|$ is a p-hyponormal for $r \geq t \geq 0$ and $p \leq 1$ then $(\widetilde{T}^*_{r,t}\widetilde{T}_{r,t})^q \geq |T|^{2rq} \geq (\widetilde{T}_{r,t}\widetilde{T}^*_{r,t})^q$, where $q = \min\{(p+t)/r, (p+r-t)/r, 1\}$ ([16]). Thus $|\widetilde{T}_{1,t}| \geq |T| \geq |\widetilde{T}^*_{1,t}|$, and so every p-hyponormal is w-hyponormal. In this note we survey spectral theory, spectral

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pictures, Aluthge iteration, backward iterations, and invariant subspace problem related to Aluthge and Duggal transforms from [13], [19], [20], [21], and [22].

2. Spectral theory

Our first theorem shows that various spectra of (an arbitrary operator) T coincide with those of $\widetilde{T}$. As usual, we write $\sigma(T)$, $\sigma_p(T)$, and $\sigma_{ap}(T)$ for the spectrum, point spectrum, and approximate point spectrum of T, respectively.

Theorem 2.1. *For every $T = U|T|$ (polar decomposition) in $\mathcal{L}(\mathcal{H})$, $\sigma(T) = \sigma(\widetilde{T})$, $\sigma_{ap}(T) = \sigma_{ap}(\widetilde{T})$, $\sigma_p(T) = \sigma_p(\widetilde{T})$, $\sigma_{ap}(T^*)\backslash(0) = \sigma_{ap}((\widetilde{T})^*)\backslash(0)$, and $\sigma_p(T^*)\backslash(0) = \sigma_p((\widetilde{T})^*)\backslash(0)$.*

Consider the Hilbert space $\mathcal{H} = L^2([0,1],\mu)$, where μ is Lebesgue measure, and let $\{e_n\}_{n=1}^{\infty}$ be any orthonormal basis for $\mathcal{H}$ such that e_1 is the constant function 1. Let $U \in \mathcal{L}(\mathcal{H})$ be defined by $Ue_n = e_{n+1}$, $n \in \mathbb{N}$, so U is a unilateral shift, and consider $T = U(M_x)^2$, where M_x is multiplication by the position function. Then T is clearly not a quasiaffinity, but an easy calculation shows that $\widetilde{T} = M_x U M_x$ is a quasiaffinity. Thus this example shows that all the spectral equalities in Theorem 2.1 are best possible.

For an operator $A \in \mathcal{L}(\mathcal{H})$, we write, as usual, $\sigma_e(A)$, $\sigma_{le}(A)$, and $\sigma_{re}(A)$ for the essential (Calkin), left essential, and right essential spectra of A, respectively.

Theorem 2.2. *For any $T \in \mathcal{L}(\mathcal{H})$ with associated Aluthge transform $\widetilde{T}$, we have $\sigma_e(T) = \sigma_e(\widetilde{T})$, $\sigma_{le}(T) = \sigma_{le}(\widetilde{T})$, and $\sigma_{re}(T)\backslash(0) = \sigma_{re}(\widetilde{T})\backslash(0)$.*

We turn now to the intimate connection between the invariant subspace lattices of an arbitrary operator T in $\mathcal{L}(\mathcal{H})$ and its associated $\widetilde{T}$. If $T \in \mathcal{L}(\mathcal{H})$ is not a quasiaffinity, then $0 \in \sigma_p(T) \cup \sigma_p(T^*)$, so trivially T has a nontrivial invariant subspace. Thus we investigate the relation between $\mathrm{Lat}(T)$ and $\mathrm{Lat}(\widetilde{T})$ only when T is a quasiaffinity. In this case, we know that T and $\widetilde{T}$ are quasisimilar. In general, one does not know that quasisimilar operators have nontrivial invariant subspace lattices together, but the beauty of the Aluthge transform is that here this does happen.

Theorem 2.3. *Let $T = U|T|$ (polar decomposition) be an arbitrary quasiaffinity in $\mathcal{L}(\mathcal{H})$. Then the mapping $\phi : \mathcal{N} \longrightarrow (|T|^{\frac{1}{2}}\mathcal{N})^-$, $\mathcal{N} \in \mathrm{Lat}(T)$, maps $\mathrm{Lat}(T)$ into $\mathrm{Lat}(\widetilde{T})$, and moreover if $(0) \neq \mathcal{N} \neq \mathcal{H}$, then $(0) \neq \phi(\mathcal{N}) = (|T|^{\frac{1}{2}}\mathcal{N})^- \neq \mathcal{H}$. Moreover the mapping $\psi : \mathcal{M} \longrightarrow (U|T|^{\frac{1}{2}}\mathcal{M})^-$, $\mathcal{M} \in \mathrm{Lat}(\widetilde{T})$, maps $\mathrm{Lat}(\widetilde{T})$ into $\mathrm{Lat}(T)$, and if $(0) \neq \mathcal{M} \neq \mathcal{H}$, then $(0) \neq \psi(\mathcal{M}) = (U|T|^{\frac{1}{2}}\mathcal{M})^- \neq \mathcal{H}$. Consequently, $\mathrm{Lat}(T)$ is nontrivial if and only if $\mathrm{Lat}(\widetilde{T})$ is nontrivial.*

Let $\mathcal{H}$=$L^2([0,1],\mu)$ where μ is Lebesque measure, and let M_x denote multiplication by the position function on $\mathcal{H}$. For given $0 \le \alpha \le \beta$, define U and $|T|$ in $\mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ by the matrices

$$U = \begin{pmatrix} 0 & \delta_\alpha 1_{\mathcal{H}} \\ 0 & 0 \end{pmatrix}, \qquad |T| = \begin{pmatrix} \beta^{\frac{1}{2}} M_x & 0 \\ 0 & \alpha\beta^{-\frac{1}{2}} 1_{\mathcal{H}} \end{pmatrix},$$

where $\delta_0 = 0$, and $\delta_\alpha = 1$ if $\alpha > 0$, and $\alpha\beta^{-\frac{1}{2}} := 0$ if $\alpha = \beta = 0$. Define $T \in \mathcal{L}(\mathcal{H})$ by $T = UP^2$ and $\widetilde{T} = PUP$. Elementary calculations show that UP^2 is the polar decomposition of T and hence that $\widetilde{T}$ is the Aluthge transform of T. Thus the result obtained in Theorem 2.3 is close to best possible. Finally, in this same example, $\mathrm{Hlat}(T)$ is nontrivial but $\mathrm{Hlat}(\widetilde{T}) = \{(0), \mathcal{H}\oplus\mathcal{H}\}$. Thus the result below in Theorem 2.5 is close to best possible.

Corollary 2.4. *If $T \in \mathcal{L}(\mathcal{H})$ and $\widetilde{T}$ has a nontrivial invariant subspace, then so does T. Moreover, there are examples in which $\mathrm{Lat}(T)$ and $\mathrm{Lat}(\widetilde{T})$ are not isomorphic via the above example.*

We write $\mathrm{Hlat}(A)$ for the lattice of hyperinvariant subspaces of an operator $A \in \mathcal{L}(\mathcal{H})$. If T is not a quasiaffinity, then $0 \in \sigma_p(T) \cup \sigma_p(T^*)$ and if $T \neq 0$, then $\mathrm{Hlat}(T) \neq \{(0),\mathcal{H}\}$ for trivial reasons. Then we investigate the relation between $\mathrm{Hlat}(T)$ and $\mathrm{Hlat}(\widetilde{T})$, only in the case in which T is a quasiaffinity.

Theorem 2.5. *Let $T \in \mathcal{L}(\mathcal{H})$ be an arbitrary nonzero quasiaffinity. Then T has a nontrivial hyperinvariant subspace if and only if its Aluthge transform $\widetilde{T}$ does. Thus if $T \in \mathcal{L}(\mathcal{H})$ and $\mathrm{Hlat}(\widetilde{T})$ is nontrivial, then so is $\mathrm{Hlat}(T)$. Moreover, $\mathrm{Hlat}(\widetilde{T})$ may be trivial but $\mathrm{Hlat}(T)$ nontrivial.*

If $\mathcal{U}$ is a bounded open set in the complex plane $\mathbb{C}$, recall that a subset $\Lambda \subset \mathcal{U}$ is said to be *dominating* for $\mathcal{U}$ if every function $h(z)$ holomorphic and bounded on $\mathcal{U}$ satisfies $\sup_{z\in\mathcal{U}} |h(z)| = \sup_{z\in\mathcal{U}\cap\Lambda} |h(z)|$.

Theorem 2.6. *Suppose T is an arbitrary w-hyponormal operator and suppose that there exists a nonempty open set $\mathcal{U}$ in $\mathbb{C}$ such that $\sigma(T)\cap\mathcal{U}$ is dominating for $\mathcal{U}$. Then T has a nontrivial invariant subspace.*

The following result generalizes a surprising theorem of Berger ([5]) for hyponormal operators to the context of p-hyponormal operators.

Theorem 2.7. *Let $T \in \mathcal{L}(\mathcal{H})$ be a w-hyponormal operator in $\mathcal{L}(\mathcal{H})$. Then there exists a positive integer K such that for all positive integers $k \geq K$, T^k has a nontrivial invariant subspace.*

3. Spectral pictures

We write $(\mathcal{SF})$ [resp., $(\mathcal{F})$] for the open set of all semi-Fredholm [resp., Fredholm] operators in $\mathcal{L}(\mathcal{H})$, and for $T \in (\mathcal{SF})$, we write $i(T)$ for the Fredholm index of T, i.e., $i(T) = \dim\ker T - \dim\ker T^*$. Recall that for $T \in \mathcal{L}(\mathcal{H})$ and $\lambda \in \mathbb{C}$, $T - \lambda = T - \lambda I_{\mathcal{H}}$ belongs to $(\mathcal{F})$ if and only if $\lambda \notin \sigma_e(T)$ and that $i : (\mathcal{SF}) \to \mathbb{Z} \cup \{+\infty, -\infty\}$ is a continuous function (and thus constant on connected open subsets of $(\mathcal{SF})$). For more information about Fredholm and semi-Fredholm operators, see [10] and [11]. If $T \in \mathcal{L}(\mathcal{H})$, a *hole* in $\sigma_e(T)$ is a bounded component of $\mathbb{C}\backslash\sigma_e(T)$ (and thus is an open set). A *pseudohole* in $\sigma_e(T)$ is a component of $\sigma_e(T) \setminus \sigma_{le}(T)$ $(= \sigma_e(T)^\circ \setminus \sigma_{le}(T))$ or $\sigma_e(T) \setminus \sigma_{re}(T) (= \sigma_e(T)^\circ \setminus \sigma_{re}(T))$, and thus is a subset of $\sigma_e(T)$ that is open in $\mathbb{C}$. The spectral picture of T (notation: $\mathrm{SP}(T)$), introduced in [27], is the structure consisting of the set $\sigma_e(T)$, the collection of holes and pseudoholes in $\sigma_e(T)$, and the Fredholm indices associated with those holes and pseudoholes. The concept of the spectral picture of an operator has been useful in operator theory (cf., for example, [6], [8]). In particular, several deep theorems in the subject can be easily stated in terms of spectral pictures (cf. [27]).

Theorem 3.1. *For all T in $\mathcal{L}(\mathcal{H})$ and all $\lambda \in \mathbb{C} \setminus \{0\}$,* dim ker$(T-\lambda)-$ dim ker$(T-\lambda)^* =$ dim ker$(\widetilde{T}-\lambda)-$ dim ker$(\widetilde{T}-\lambda)^*$ *as extended real numbers whenever at least one side is meaningful as an extended real number.*

Consider the Hilbert space $\mathcal{H} = L^2([0,1],\mu)$, where μ is Lebesgue measure, and let $\{e_n\}_{n=1}^{\infty}$ be any orthonormal basis for $\mathcal{H}$ such that e_1 is the constant function 1. Let $U \in \mathcal{L}(\mathcal{H})$ be defined by $Ue_n = e_{n+1}$, $n \in \mathbb{N}$, so U is a unilateral shift, and consider $T = U(M_x)^2$, where M_x is multiplication by the position function. Then an easy calculation shows that dim ker$T^* = 1$ and dim ker$\widetilde{T}^* = 0$. On the other hand, if $S = U^*$, then dim ker$S^* = 0$ and dim ker$\widetilde{S}^* = 1$. This shows that theorem 3.1 may fail for $\lambda = 0$. In particular, if T is as in that example, then dim ker$T-$ dim ker$T^* = 0-1 = -1$, while dim ker$\widetilde{T}-$ dim ker$\widetilde{T}^* = 0-0 = 0$. Nevertheless, we have the following.

Theorem 3.2. *For every T in $\mathcal{L}(\mathcal{H})$ and for every λ in $\mathbb{C}$, $T-\lambda \in (\mathcal{F})$ if and only if $\widetilde{T}-\lambda \in (\mathcal{F})$, and when $T-\lambda \in (\mathcal{F})$, we have $i(T-\lambda) = i(\widetilde{T}-\lambda)$.*

Now let us examine the situation of the pseudoholes in SP(T) and their relation to the pseudoholes in SP$(\widetilde{T})$. By definition, a pseudohole with index $-\infty$ in SP(T) is a connected component of the open set $\sigma_e(T)\setminus\sigma_{le}(T)$, and since by Theorem 2.2, $\sigma_e(T) = \sigma_e(\widetilde{T})$ and $\sigma_{le}(T) = \sigma_{le}(\widetilde{T})$, the pseudoholes whose associated Fredholm index is $-\infty$ are exactly the same in $\sigma_e(T)$ as in $\sigma_e(\widetilde{T})$. Next suppose that U is a pseudohole in $\sigma_e(T)$ of index $+\infty$. Then, by definition U is a connected component of the open set $\sigma_e(T)\setminus\sigma_{re}(T)$. Moreover, $\sigma_{re}(T)\setminus\{0\} = \sigma_{re}(\widetilde{T})\setminus\{0\}$ from Theorem 2.2. An elementary but tedious topological argument shows that there are only three possibilities in this case— either U, $U\cup\{0\}$, or $U\setminus\{0\}$ must be a pseudohole in $\sigma_e(\widetilde{T})$ (and examples below show that two of these possibilities may occur). Thus we have the following.

Theorem 3.3. *Let $T \in \mathcal{L}(\mathcal{H})$ and let $\widetilde{T}$ be its Aluthge transform. Then an open set $U \subset \sigma_e(T)\setminus\sigma_{le}(T)$ is a pseudohole in $\sigma_e(T)$ with index $-\infty$ if and only if U is a pseudohole in $\sigma_e(\widetilde{T})$ with index $-\infty$. Moreover, if an open set $V \subset \sigma_e(T)\setminus\sigma_{re}(T)$ is a pseudohole in $\sigma_e(T)$ with index $+\infty$, then either V, $V\cup\{0\}$, or $V\setminus\{0\}$ is a pseudohole in $\sigma_e(\widetilde{T})$ with index $+\infty$, and thus if an open set $W \subset \sigma_e(\widetilde{T})\setminus\sigma_{re}(\widetilde{T})$ is a pseudohole in $\sigma_e(\widetilde{T})$ with index $+\infty$, then either W, $W\cup\{0\}$, or $W\setminus\{0\}$ is a pseudohole in $\sigma_e(T)$ with index $+\infty$.*

Corollary 3.4. *For all $T \in \mathcal{L}(\mathcal{H})$ such that* SP(T) *(or* SP$(\widetilde{T})$*) has no pseudohole,* SP$(T) =$ SP$(\widetilde{T})$.

4. Some transforms

We will explore below various relations between T, $\widehat{T}$, and $\widetilde{T}$ by studying maps between the Riesz-Dunford algebras associated with these operators. We denote by $\mathrm{Hol}(\sigma(T))$ the algebra of all complex-valued functions which are analytic on some neighborhood of $\sigma(T)$, where linear combinations and products in $\mathrm{Hol}(\sigma(T))$ are defined (with varying domains) in the obvious way. Moreover, the (Riesz-Dunford) algebra $\mathcal{A}_T \subset \mathcal{L}(\mathcal{H})$ is defined as $\mathcal{A}_T = \{f(T) : f \in \mathrm{Hol}(\sigma(T))\}$, (where $f(T)$ is defined by the Riesz-Dunford functional calculus). Recall that *the complete contraction* means, by definition, that for every $n \in \mathbb{N}$ and for every $n \times n$ matrix (f_{ij}), where each $f_{ij} \in \mathrm{Hol}(\sigma(T))$, the inequality $\|(f_{ij}(\widehat{T}))\| \le \|(f_{ij}(T))\|$ is satisfied, where of course, the $n \times n$ operator matrices act on the Hilbert space $\mathcal{H}^{(n)}$, the direct sum of n copies of $\mathcal{H}$, and the norm indicated is the operator norm on $\mathcal{L}(\mathcal{H}^{(n)})$. As the following theorem shows, it is possible to obtain useful information about $\widetilde{T}$ and $\widehat{T}$ by studying maps between the algebras $\mathcal{A}_T$, $\mathcal{A}_{\widetilde{T}}$, and $\mathcal{A}_{\widehat{T}}$.

Theorem 4.1. *For every T in $\mathcal{L}(\mathcal{H})$, with $\widehat{T}$, $\widetilde{T}$, and $\mathrm{Hol}(\sigma(T))$ as defined above,*

a) *the maps $\widehat{\Phi} : \mathcal{A}_T \to \mathcal{A}_{\widehat{T}}$ and $\widetilde{\Phi} : \mathcal{A}_T \to \mathcal{A}_{\widetilde{T}}$ defined by $\widehat{\Phi}(f(T)) = f(\widehat{T})$, $\widetilde{\Phi}(f(T)) = f(\widetilde{T})$, $f \in \mathrm{Hol}(\sigma(T))$, are well-defined contractive algebra homomorphisms; in particular,* $\max\{\|f(\widehat{T})\|, \|f(\widetilde{T})\|\} \le \|f(T)\|$, $f \in \mathrm{Hol}(\sigma(T))$.

b) *More generally, the maps $\widehat{\Phi}$ and $\widetilde{\Phi}$ in* a) *are completely contractive, meaning that for every $n \in \mathbb{N}$ and every $n \times n$ matrix (f_{ij}) with entries from* $\mathrm{Hol}(\sigma(T))$,

$$\max\{\|(f_{ij}(\widehat{T}))\|, \|(f_{ij}(\widetilde{T}))\|\} \le \|(f_{ij}(T))\|.$$

c) *Every spectral set $[M$-spectral set (for fixed $M > 1)]$ for T is also a spectral set [respectively, M-spectral set] for both $\widehat{T}$ and $\widetilde{T}$.*

d) *If $W(S)$ denotes the numerical range of an operator S in $\mathcal{L}(\mathcal{H})$, then $W(f(\widehat{T}))^- \cup W(f(\widetilde{T}))^- \subset W(f(T))^-$, $f \in \mathrm{Hol}(\sigma(T))$, and, moreover, if T belongs to some class $\mathcal{C}_\rho$, then $\widehat{T}$ and $\widetilde{T}$ belong to $\mathcal{C}_\rho$ also (see* [25, p.45] *for the definition of these classes).*

One reason for establishing that the maps $\widetilde{\Phi}$ and $\widehat{\Phi}$ are completely contractive is that the extension theorems of Arveson and Stinespring can be applied to obtain the structure of such maps (cf., e.g., [26]), and thus we get the following.

Theorem 4.2. Let T be an arbitrary operator in $\mathcal{L}(\mathcal{H})$, and let $\widetilde{\Phi}$ and $\widehat{\Phi}$ be the maps defined in Theorem 4.1. Then there exist Hilbert spaces $\widetilde{\mathcal{K}} = \widetilde{\mathcal{K}}_T$ and $\widehat{\mathcal{K}} = \widehat{\mathcal{K}}_T$ containing $\mathcal{H}$, and C^*-homomorphisms $\widetilde{\Psi} : C^*(T) \to \mathcal{L}(\widetilde{\mathcal{K}})$ and $\widehat{\Psi} : C^*(T) \to \mathcal{L}(\widehat{\mathcal{K}})$ (where $C^*(T)$ is the smallest unital C^*-algebra containing $\mathcal{A}_T$) such that for every f in $\mathrm{Hol}(\sigma(T))$, $\widetilde{\Phi}(f(T)) = P_{\mathcal{H}}\widetilde{\Psi}(f(T))|_{\mathcal{H}}$ and $\widehat{\Phi}(f(T)) = P^{(2)}_{\mathcal{H}}\widehat{\Psi}(f(T))|_{\mathcal{H}}$, where $P^{(1)}_{\mathcal{H}}$ and $P^{(2)}_{\mathcal{H}}$ are the orthogonal projections of $\widetilde{\mathcal{K}}$ and $\widehat{\mathcal{K}}$, respectively, onto $\mathcal{H}$.

5. Aluthge iterates

For an arbitrary T in $\mathcal{L}(\mathcal{H})$, let $\widetilde{T}^{(0)} = T$ and $\widetilde{T}^{(n+1)} = (\widetilde{T}^{(n)})^\sim$ for $n \in \mathbb{N}$. In this section we discuss whether the sequence $\{\widetilde{T}^{(n)}\}_{n=1}^\infty$of iterated Aluthge transforms of T converges in an operator topology in $\mathcal{L}(\mathcal{H})$. This question is affimative under several assumptions and we introduce them below.

Proposition 5.1. *Suppose $T \in \mathcal{L}(\mathcal{H})$ is nilpotent of order $m \geq 2$. Then $(\widetilde{T})^{m-1} = 0$ and $\widetilde{T}^{(m-1)} = 0$. Thus the sequence $\{\widetilde{T}^{(n)}\}_{n=1}^{\infty}$ is norm-convergent to zero.*

Theorem 5.2. *Suppose $T \in \mathcal{L}(\mathcal{H})$ and is strongly quasinilpotent. Then the sequence $\{\widetilde{T}^{(n)}\}$ converges to zero in the norm topology.*

However, in general the above question is negative, for example we have the following counterexample.

Proposition 5.3. *Suppose a and b are any distinct positive real numbers. Then there is a unilateral weighted shift $T := W_\alpha$ with weight sequence α such that the sequence of the first weights of $\{\widetilde{T}^{(n)}\}_{n=1}^{\infty}$ have two subsequences converging to a and b, respectively, which implies that there exists an operator T such that the sequence $\{\widetilde{T}^{(n)}\}_{n=1}^{\infty}$ does not converge in the weak operator topology.*

In fact, we are unable to decide whether $\{\widetilde{T}^{(n)}\}_{n=1}^{\infty}$ converges in the strong operator topology (or the weak operator topology) when T is hyponormal.

Theorem 5.4. *Let $T \equiv W_\alpha$ be a hyponormal bilateral weighted shift on $\ell_2(\mathbb{Z})$ with a weight sequence $\alpha \equiv \{\alpha_n\}_{n\in\mathbb{Z}}$. Let $a := \inf\{\alpha_n\}_{n\in\mathbb{Z}}$ and $b := \sup\{\alpha_n\}_{n\in\mathbb{Z}}$. Then $\{\widetilde{T}^{(n)}\}_{n=1}^{\infty}$ converges to a quasinormal operator in the norm topology if and only if $a = b$.*

The following example shows the existence of an operator T such that $\{\widetilde{T}^{(n)}\}_{n=1}^{\infty}$ converge in the strong operator topology but not the norm topology.

Example 5.5. Let $T \equiv W_\alpha$ be a hyponormal bilateral weighted shift on $\ell_2(\mathbb{Z})$ with weight sequence $\alpha \equiv \{\alpha_n\}_{n\in\mathbb{Z}}$, where α_n is given by $\alpha_n = 1/2$ $(n < 0)$ and $\alpha_n = 1$ $(n \geq 0)$. By Theorem 5.4, $\{\widetilde{T}^{(n)}\}_{n=1}^{\infty}$ does not converge to a quasinormal operator in the norm topology. In fact SOT-$\lim_{n\to\infty} \widetilde{T}^{(n)} = B$ (where B is the bilateral unweighted shift). Indeed, we first observe that $\widetilde{T}^{(n)} e_k = (\prod_{j=0}^{n} \alpha_{j+k}^{\binom{n}{j}})^{\frac{1}{2^n}} e_{k+1}$, for all $n \in \mathbb{Z}$. So the weight sequence of $\widetilde{T}^{(n)}$ is composed of $\widetilde{\alpha}_k^{(n)} := (\prod_{j=0}^{n} \alpha_{j+k}^{\binom{n}{j}})^{\frac{1}{2^n}}$, $k \in \mathbb{Z}$. For $n > k$, we have $|\ln \widetilde{\alpha}_{-k}^{(n)}| = |1/2^n \cdot \sum_{j=0}^{n} \binom{n}{j} \ln \alpha_{j-k}| = |\ln 2 \cdot 1/2^n \sum_{j=0}^{k-1} n!/j!(n-j)!|$ and $\lim_{n\to\infty} 1/2^n \cdot \sum_{j=0}^{k-1} n!/j!(n-j)! = 0$, for a fixed $k \in \mathbb{N}$. Hence $\lim_{n\to\infty} \widetilde{\alpha}_{-k}^{(n)} = 1$ for each $k \in \mathbb{N}$. Since $\lim_{n\to\infty} \widetilde{\alpha}_k^{(n)} = 1$ for $k \geq 0$ obviously, SOT-$\lim_{n\to\infty} \widetilde{T}^{(n)} = B$.

If T is quasinormal, obviously $\{\widetilde{T}^{(n)}\}_{n=1}^{\infty}$ converges to T. If T is a hyponormal weighted shift with weight sequence $\{\alpha_n\}_{n\in\mathbb{Z}_+}$, which converges to α, then by the previous argument, $\widetilde{T}^{(n)}$ is a weighted shift with weight sequence $\{(\prod_{j=0}^{n} \alpha_{j+k}^{\binom{n}{j}})^{\frac{1}{2^n}}\}_{k=0}^{\infty}$ for each $n \in \mathbb{Z}_+$, whose k-th weight, by a straightforward calculation, converges to α for each $k = 0, 1, \ldots$. Consequently, $\{\widetilde{T}^{(n)}\}_{n=1}^{\infty}$ converges to αU (where U is the unilateral unweighted shift) in the norm topology. Note that αU is quasinormal. Thus we may suggest a conjecture as following.

Conjecture 5.6. If $T \in \mathcal{L}(\mathcal{H})$ is a p-hyponormal operator with $0 < p \leq \infty$, then $\{\widetilde{T}^{(n)}\}_{n=1}^{\infty}$ converges in the strong operator topology.

The ideal of Hilbert-Schmidt operators in $\mathcal{L}(\mathcal{H})$ will be denoted by $\mathcal{C}_2(\mathcal{H})$ and the inner product and norm on $\mathcal{C}_2(\mathcal{H})$ by $\langle \cdot\,, \cdot \rangle_2$ and $\|\ \|_2$, respectively. If $T \in \mathcal{C}_2(\mathcal{H})$, we denote by $\mathcal{S}(T)$ the set of all operators S in $\mathcal{C}_2(\mathcal{H})$ such that for every $m \in \mathbb{N}\backslash\{1\}$, $\operatorname{tr}(T^m) = \operatorname{tr}(S^m)$.

Proposition 5.7. *Suppose $T\in\mathcal{C}_2(\mathcal{H})$ and $\{\widetilde{T}^{(n_k)}\}_{k=1}^{\infty}$ is a subsequence of $\{\widetilde{T}^{(n)}\}$ that satisfies $\lim_k \|\widetilde{T}^{(n_k)} - L\|_2 = 0$ for some operator L (necessarily in $\mathcal{C}_2(\mathcal{H})$). Then $\|L\| = \inf_{n\in\mathbb{N}} \|\widetilde{T}^{(n)}\| = \lim_k \|\widetilde{T}^{(n_k)}\|$, $\|L\|_2 = \inf_{n\in\mathbb{N}} \|\widetilde{T}^{(n)}\|_2 = \lim_k \|\widetilde{T}^{(n_k)}\|_2$, and $L \in \mathcal{S}(T)$.*

Unfortunately, since the unit ball in $\mathcal{C}_2(\mathcal{H})$ is not $\|\ \|_2$-compact, we can not draw the same conclusions as were established in the above remark in the finite dimensional case. Further progress along these lines would be more likely if we could settle the following questions.

Problem 5.8. Is the map $\widetilde{\ }: \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ continuous at *every* $T \in \mathcal{L}(\mathcal{H})$ when both copies of $\mathcal{L}(\mathcal{H})$ are given the operator-norm topology (cf. [5])?

Problem 5.9. Is the map $\widetilde{\ }: \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ (SOT, WOT)-continuous at every T in $\mathcal{L}(\mathcal{H})$? (Here SOT [WOT] denotes the strong [respectively, weak] operator topology on $\mathcal{L}(\mathcal{H})$).

Problem 5.10. Is the map $\widetilde{\ }: \mathcal{C}_2(\mathcal{H}) \to \mathcal{C}_2(\mathcal{H})$ $(\|\ \|_2, \|\ \|_2)$-continuous?

6. Backward Aluthge iterates

Suppose $T \in \mathcal{L}(\mathcal{H})$ and there exist $T_0 \in \mathcal{L}(\mathcal{H})$ and a positive integer k such that $\widetilde{T}_0^{(k)} = T$. We then say that T_0 is a *backward Aluthge iterate of T of order k* and we write $\mathbb{A}_{-k}(T)$ for the set of all such T_0. Moreover, we write $\mathbb{A}_{-\infty}(T) = \bigcup_{k\in\mathbb{N}} \mathbb{A}_{-k}(T)$. The first remark to be noted concerning such a set $\mathbb{A}_{-\infty}(T)$ is that it may be empty. This will happen, of course, if and only if T is not in the range of the mapping $\widetilde{\ }: \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$. For example, it is immediate from [19, Th. 1.3 and Prop. 1.12] that if T satisfies $\ker T \neq (0)$ but $\ker T^* = (0)$, then T cannot be in the range of $\widetilde{\ }$, so $\mathbb{A}_{-\infty}(T) = \emptyset$. On the other hand, $\mathbb{A}_{-\infty}(0)$ contains the set of all nilpotent operators in $\mathcal{L}(\mathcal{H})$. For many operators T in $\mathcal{L}(\mathcal{H})$ it is of interest to try to identify $\mathbb{A}_{-\infty}(T)$ for the following reasons.

Theorem 6.1. *If $T \in \mathcal{L}(\mathcal{H})$, then T has a nontrivial invariant subspace if and only if every (equivalently, some) quasiaffinity in $\mathbb{A}_{-\infty}(T)$ has a nontrivial invariant subspace.*

The following result shows that finding (members of) $\mathbb{A}_{-\infty}(H)$ when H is a hyponormal operator might be useful.

Theorem 6.2. *If H is a hyponormal operator in $\mathcal{L}(\mathcal{H})$ and $T \in \mathbb{A}_{-\infty}(H)$, then for all sufficiently large $k \in \mathbb{N}$, T^k has a nontrivial invariant subspace. Moreover,* *if the spectrum $\sigma(T)$ of T has the property that $\sigma(T) \cap \mathcal{U}$ is dominating for some nonempty open set $\mathcal{U} \subset \mathbb{C}$, then T itself has a nontrivial invariant subspace.*

Problem 6.3. Let $\mathbb{H} = \mathbb{H}(\mathcal{H})$ denote the class of all hyponormal operators in $\mathcal{L}(\mathcal{H})$, and denote by $\mathbb{A}_{-k}(\mathbb{H})$ the union $\cup_{H\in\mathbb{H}}\mathbb{A}_{-k}(H)$. One knows from [1] that $\mathbb{A}_{-1}(\mathbb{H})$ contains all $\frac{1}{2}$-hyponormal operators and that $\mathbb{A}_{-2}(\mathbb{H})$ contains all log-hyponormal operators and all p-hyponormal operators for which $0 < p \le 1$. What can be said about the classes $\mathbb{A}_{-3}(\mathbb{H})$, etc.? (Note that by virtue of Theorems 6.1 and 6.2, every operator in $\mathbb{A}_{-k}(\mathbb{H})$, $k \in \mathbb{N}$, with spectrum dominating some open set has a nontrivial invariant subspace.)

Problem 6.4. Suppose we denote by $C(\mathcal{H})$ the set of all T in $\mathcal{L}(\mathcal{H})$ such that the sequence $\{\widehat{T}^{(n)}\}$ is norm-convergent. Is the map $\widehat{\ } : C(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ defined by $\widehat{T} = \lim_n \widetilde{T}^{(n)}$ norm-continuous on $C(\mathcal{H})$?

7. Invariant subspace problem

If $T \in \mathcal{H}$ is not a quasiaffinity then $0 \in \sigma_p(T) \cup \sigma_p(T^*)$, so trivially T has a nontrivial invariant subspace. Thus we will focus on the cases where T is a quasiaffinity, and so T and $\widehat{T}$ are unitarily equivalent. In the sequel we will assume that T is a quasiaffinity. As Aluthge interation, we define $\widehat{T}^{(1)} := \widehat{T}$ and $\widehat{T}^{(n+1)} := \widehat{(\widehat{T}^{(n)})}$ for every $n \in \mathbb{Z}_+$. Then the following question arises naturally: For every hyponormal operator $T \in H$, does the sequence $\{\widehat{T}^{(n)}\}$ of iterated transforms converge in the strong operator topology ? Here is a partial solution for this question.

Theorem 7.1. *Suppose $T \in \mathcal{H}$ is a hyponormal operator. Then the sequence $\{\widehat{T}^{(n)}\}$ converges in* SOT *to a limit $\widehat{T}_L$, which is a quasinormal operator such that $||\widehat{T}_L|| = ||T||$, or equivalently, $r(\widehat{T}_L) = r(T)$, where $r(\cdot)$ denotes the spectral radius.*

Note that T is pure (i.e., T has no reducing subspace $\mathcal{M}$ such that $T|_{\mathcal{M}}$ is normal) if and only if $\widehat{T}$ is pure because T and $\widehat{T}$ are unitarily equivalent. So if T is pure then $\{\widehat{T}^{(n)}\}$ is a sequence of pure operators. We need not, however, expect that $\widehat{T}_L$ is pure. For example, if $T \equiv W_\alpha$ is a hyponormal bilateral weighted shift with weight sequence $\alpha \equiv \{\alpha_n\}_{n\in\mathbb{Z}}$ then $\widehat{T}^{(n+1)} = B^{*n}|T|\, B^{n+1}$, where $|T| := \text{diag}\,(\alpha_n)_{n\in\mathbb{Z}}$ and B is the bilateral unweighted shift. Thus we can see that $\{\widehat{T}^{(n)}\}$ converges in SOT to $k\,B$, where $k := \sup\{\alpha_n\}$. Note that $k\,B$ is normal even though T is a pure hyponormal operator. We however have:

Theorem 7.2. *If $T \in \mathcal{H}$ is a hyponormal operator and $\widehat{T}_L$ is pure then T has a nontrivial invariant subspace.*

To solve the invariant subspace problem for hyponormal operators, in view of Theorem 7.2, we should answer it for the cases that $\widehat{T}_L$ is not pure. First of all we need to answer:

Problem 7.3. If $T \in \mathcal{H}$ is hyponormal and $\widehat{T}_L$ is normal, does T have a nontrivial invariant subspace ?

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