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QUASISIMILARITY OF NON-HYPONORMAL OPERATORS

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ABSTRACT. In this article we give informal accounts of some recent results on quasisimilarity of various non-hyponormal operators.

1. INTRODUCTION

Let H and K be infinite dimensional complex Hilbert spaces and let $L(H, K)$ denote the set of bounded linear operators from H to K. If $H = K$, we write $L(H)$ in place of $L(H, K)$.

Recall ([3],[6],[13],[14],[21],[28]) that an operator $X \in L(H)$ is called a *quasi-affinity* if X is injective and has dense range. For $T_1 \in L(H_1)$ and $T_2 \in L(H_2)$, if there exist quasiaffinities $X \in L(H_2, H_1)$ and $Y \in L(H_1, H_2)$ such that

$$T_1X = XT_2 \text{ and } YT_1 = T_2Y,$$

then we say that T_1 and T_2 are *quasisimilar*.

Quasisimilarity was first introduced by Sz. Nagy and Foias([20]) in their theory of infinite dimensional analogue of the Jordan form for certain classes of contractions as a means of studying their invariant subspace structures. It replaces the familiar notion of similarity which is the appropreate equivalence relation to use with finite dimensional Jordan forms. In finite dimensional spaces quasisimilarity is the same thing as similarity, but in infinite dimensional spaces it is a much weaker relation. It is well known that simlarity of operators preserve compactness, cyclicity, algebraicity, and the spectral picture(i.e., the spectrum, essential spectrum, and index function), and that similar operators have isomorphic lattices of invariant and hyperinvariant subspaces.

In general quasisimilarity preserve nothing mentioned above except the point spectrum and hyperinvariant subspaces. Actually, Hoover([13]) give an easy example that quasisimilarity need not preserve the spectrum and compactness. However, in special classes of operators quasisimilarity may preserve many things. Actually, it is well known that quasisimilar p-hyponormal operators have the same spectral picture([7],[30]), which is an extension of earlier well-known results(e.g., [3],[6],[21]).

On the other hand, Douglas ([5]) proved that quasisimilar normal operators are unitarily equivalent. Hoover([13]) proved that quasisimilar isometries are unitarily equivalent. Conway ([4]) proved that the normal parts of quasisimilar subnormal operators are unitarily equivalent and gave an example showing that the pure parts of quasisimilar subnormal operators need not be quasisimilar. This result was extended by Williams([28]) to the more general case of dominant operators. Recently, Conway's result also has been extended to another general interesting classes of non-hyponormal operators. The aim of this article is to survey these recent results.

2. Some classes of non-hyponormal operators

As an easy extension of normal operators, hyponormal operators have been studied by many mathematicians. Though there are many unsolved interesting problems for hyponormal operators(e.g., the invariant subspace problem), one of recent trends in operator theory is studying natural extensions of hyponormal operators. So we introduce some these non-hyponormal operators in this section.

Löwner-Heinz inequality(to be mentioned in the section 3) give an natural extension of hyponormal operators. Recall ([1],[6],[10],[14]) that an operator $T \in L(H)$ is called *p-hyponormal* if

$$(T^*T)^p - (TT^*)^p \geq 0 \quad \text{for } p \in (0,1].$$

We can notice that T is hyponormal if $p = 1$ and that a p-hyponormal operator is also q-hyponormal for every $0 < q \leq p$ from Löwner-Heinz inequality. Also, an operator T is said to be *log-hyponormal* if T is invertible and satisfies the following inequality

$$\log(T^*T) \geq \log(TT^*).$$

It is known that invertible p-hyponormal operators are log-hyponormal and the converse is not true ([23]). But it is very interesting that we may regard log-hyponormal operators as 0-hyponormal operators ([23]).

Also, recall ([17],[19],[25]) that $T \in L(H)$ is called *p-quasihyponormal* if

$$T^*\{(T^*T)^p - (TT^*)^p\}T \geq 0.$$

If $p = 1$, T is quasihyponormal ([2]). In [25], it is well known that if an operator $T \in L(H)$ is p-quasihyponormal for some $0 < p \leq 1$ then

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix} \quad \text{on} \quad H = \overline{R(T)} \oplus N(T^*),$$

where $T_1 = T|_{\overline{R(T)}}$ is p-hyponormal which satisfies $({T_1}^*T_1)^p \geq (T_1{T_1}^* + T_2{T_2}^*)^p$. Thus we easily see that if T is p-quasihyponormal with dense range then T is just p-hyponormal (for details, see [25]). So p-hyponormal operators are p-quasihyponormal.

An operator T is *paranormal* if

$$||Tx||^2 \leq ||T^2x||||x|| \quad \text{for all } x \in H.$$

If arrows below mean implications we can notice that for $p \in (0,1]$

$$\begin{array}{ccccc} & \nearrow & P - \text{hyponormal} \rightarrow P - \text{quasihyponormal} & \searrow & \\ \text{Hyponormal} & & & & \text{Paranormal.} \\ & \searrow & \text{Invertible } p - \text{hyponormal} \rightarrow \text{Log-hyponormal} & \nearrow & \end{array}$$

There have been well known another classes of operators properly containing hyponormal operators. Operators belonging to these classes are not paranormal unlike the former. Recall ([27], [28]) that an operator $T \in L(H)$ is said to be *dominant* if for each $\lambda \in \mathbb{C}$ there exists a positive number M_λ such that

$$(T-\lambda)(T-\lambda)^* \leq M_\lambda (T-\lambda)^*(T-\lambda).$$

If the constants M_λ are bounded by a positive number M, then T is said to be *M-hyponormal.* Also, we may note that if T is 1-hyponormal, then T is hyponormal. Evidently, we see that

$$\text{Hyponormal} \ \rightarrow \ M-\text{hyponormal} \ \rightarrow \ \text{Dominant}$$

3. Main tools

One of our major tools comes from *Aluthge transform* (cf. [1],[6],[7],[14],[15]) of a p-hyponormal operator. We completely introduce a refinement of Aluthge transform because this tool will be used in proof of lemmas listed in this article.

We decompose a p-hyponormal operator T into its normal and pure parts by $T = T_1 \oplus T_2$ with respect to a decomposition $H = H_1 \oplus H_2$. Then it is well known T_2 is also p-hyponormal. Letting T_2 have the polar decomposition $T_2 = U|T_2|$, we consider its Aluthge transform $\widehat{T_2} = |T_2|^{1/2}U|T_2|^{1/2}$. Again, let $\widehat{T_2} = V|\widehat{T_2}|$, and define $\widetilde{T_2} = |\widehat{T_2}|^{1/2}V|\widehat{T_2}|^{1/2}$. Using the Furuta's inequality([9]), Aluthge([1, Theorem 1,2]) showed the Aluthge transform $\widehat{T_2}$ of T_2 is semi-hyponormal and the second Aluthge transform $\widetilde{T_2}$ of T_2 is hyponormal. (Though it was proved in the special case in which the partial isometry in the polar decomposition of a p-hyponormal operator is unitary, the proof can be made to work in general case.) Letting $W = |\widehat{T_2}|^{1/2}|T_2|^{1/2}$, by Corollary 4 in [6] we can see that W is a quasiaffinity such that $\widetilde{T_2}W = WT_2$. Since $\widetilde{T_2}$ is hyponormal, if we set $X := I_{H_1} \oplus W$ and $\widetilde{T} := T_1 \oplus \widetilde{T_2}$, then X is a quasiaffinity such that $\widetilde{T}X = XT$ where $\widetilde{T}$ is also hyponormal. Thus we have

Proposition A. ([7]) *If T is p-hyponormal, then there exists a hyponormal operator A and a quasi-affinity X such that $AX = XT$.*

Our another tools are two famous operator inequalities.

Löwner-Heinz inequality ([10]). *If $B \geq A \geq 0$, then $B^\alpha \geq A^\alpha \geq 0$ for $\alpha \in (0,1]$.*

Hansen's inequality ([12]). *If $A,\ B \in L(H)$ satisfy $A \geq 0$ and $||B|| \leq 1$, then $(B^*AB)^\delta \geq B^*A^\delta B$ for all $\delta \in (0,1]$.*

These inequalities are essential tools in proofs of structural properties for p-hyponormal, p-quasihyponormal, and log-hyponormal operators.

4. Results

First, we consider some structural properties of operators.

A restriction invariant property. Let an operator $T \in L(H)$ satisfy an property P and let $\mathcal{M}$ be an invariant subspace for T. Then the restriction of T to $\mathcal{M}$, denoted by $T|_{\mathcal{M}}$, also satisfies P.

As in the case of hyponormal operators, it is well known that if $T \in L(H)$ is a p-hyponormal operator, then the restriction $T|_{\mathcal{M}}$ of T to an invariant subspace $\mathcal{M}$ is p-hyponormal ([24, Lemma 4]). This restriction invariant property also holds for p-quasihyponormal operators ([17]). In the case of log-hyponormal operators, we need additional invertibility assumption for the restriction of each log-hyponormal operator to its invariant subspace ([18]). For giving proofs of these facts, we must use inequalities mentioned above.

The following property is well known for hyponormal operators and more generally dominant operators ([22]).

A reducing property. Let $\mathcal{M}$ be an invariant subspace for an operator $T \in L(H)$. If the restriction of T to $\mathcal{M}$ is normal, then $\mathcal{M}$ reduces T.

This property holds for p-hyponormal operators ([15]). But we need additional injectivity assumptions for restrictions of p-quasihyponormal operators to their invariant subspaces ([17]). Actually, this assumption is essential because the restriction of a p-quasihyponormal operator to its kernel is trivially normal. But the kernel of p-quasihyponormal operator is not always a reducing subspace for it. On the other hand, for log-hyponormal operators, we have the following result enough to prove our main result.

Proposition B ([18]). *Let $T \in L(\mathcal{H})$ be log-hyponormal. Then $T = T_1 \oplus T_2$ on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ where T_1 is normal and T_2 is pure and log-hyponormal, i.e., T_2 has no invariant subspace $\mathcal{M}$ such that $T_2|_{\mathcal{M}}$ is normal.*

Let $\mathfrak{H}$ denote one of the classes of p-hyponormal, log-hyponormal, and injective p-quasihyponormal operators, respectively. Then we have the following result for $\mathfrak{H}$-operators (cf., [15],[17],[18]).

Proposition C. *Let $T_1 \in L(H_1)$ be a $\mathfrak{H}$-operator and let $T_2 \in L(H_2)$ be a normal operator. If there exists an operator $X \in L(H_2, H_1)$ with dense range such that $T_1X = XT_2$, then T_1 is normal.*

The following result proved in [28] is also essential to prove our main result.

Willams's lemma. *Let $N_i \in L(H_i)$ be normal for each $i = 1, 2$. If $X \in L(H_2, H_1)$ and $Y \in L(H_1, H_2)$ are injective such that $N_1X = XN_2$ and $YN_1 = N_2Y$, then N_1 and N_2 are unitarily equivalent.*

We are ready to write our main result (cf., [15],[17],[18]).

Theorem. *Let $T_i \in L(H_i)(i = 1, 2)$ be $\mathfrak{H}$-operators such that T_1 and T_2 are quasisimilar and let $T_i = N_i \oplus V_i$ on $H_i = H_{i1} \oplus H_{i2}$, where N_i and V_i are the* *normal and pure parts of T_i, respectively. Then N_1 and N_2 are unitarily equivalent and there exist $X_* \in L(H_{22}, H_{12})$ and $Y_* \in L(H_{12}, H_{22})$ with dense ranges such that $V_1 X_* = X_* V_2$ and $Y_* V_1 = V_2 Y_*$*

From this theorem we have very important result which has been open question after Clary' result([3]).

Corollary ([7],[17]). *Let $T_1 \in L(H_1)$ and $T_2 \in L(H_2)$ be $\mathfrak{H}$-operators. If T_1 and T_2 are quasisimilar then they have same spectra and essential spectra.*

Now we conclude this article with some problems. To do so, we introduce more general non-hyponormal operators than operators mentioned above.

Recently, Furuta-Ito-Yamazaki ([11]) have defined an interesting class of Hilbert space operators. An operator $A \in L(H)$ is said to belong to *Class A* if A satisfies an absolute value condition $|A^2| \geq |A|^2$. In the following we denote "Class A" by simply $\mathfrak{A}$. In [11], it is shown that $\mathfrak{A}$ stands in the middle of classes of p-hyponormal and paranormal operators. More explicitly, we have the following inclusions:

$P-$ quasihyponormal

↗ ↘

$P-$ hyponormal Paranormal.

↘ ↗

Log-hyponormal → $\mathfrak{A}$

Readers who see more informations for $\mathfrak{A}$-operators are refered to [10],[16],[26].

Problem. Does an analogue of Theorem for $\mathfrak{A}$-operators holds?

To approach this problem on the lines of arguments above, we may first consider supplementary problems;

1. Let $T \in \mathfrak{A}$ and $\mathcal{M}$ be an invariant subspace of T. Does $\mathcal{M}$ reduce T when $T|_{\mathcal{M}}$ is normal ?

2. Let $T_1 \in L(H_1)$ be a $\mathfrak{A}$-operator and let $T_2 \in L(H_2)$ be a normal operator. Is T_1 normal if there exists an injective operator $X \in L(H_2, H_1)$ with dense range such that $T_1 X = X T_2$?

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