Trends in Mathematics
Information Center for Mathematical Sciences
Volume 6, Number 2, December, 2003, Pages 29–32

UNIQUENESS PROPERTY OF C^*-ALGEBRAS LIKE THE TOEPLITZ ALGEBRA

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ABSTRACT. Reduced semigroup C^*-algebras are the C^*-algebras generated by the left regular isometric representations of left-cancellative semigroups. We prove that the reduced semigroup C^*-algebras of semigroups generating the integer group are isomorphic to the Toeplitz algebra. For a semigroup $P = \{0, 2, 3, \dots\}$ we show that the reduced semigroup C^*-algebra of P is not isomorphic to the full semigroup C^*-algebra of P.

1. INTRODUCTION

The Toeplitz algebra motivated to develop the theory of C^*-algebras generated by isometries. Among C^*-algebras generated by isometries, the C^*-algebra generated by the left regular isometric representation of a left cancellative semigroup can be considered as the appropriate analogue for the Toeplitz algebra in the non-commutative sense. The C^*-algebra generated by the left regular isometric representation of a left cancellative semigroup M has several names. We shall call it the reduced semigroup C^*-algebra, and denote it $C^*_{red}(M)$ in this paper [4].

Besides the reduced semigroup C^*-algebra, we will consider the semigroup C^*-algebra introduced by G. J. Murphy [7]. This is obtained by enveloping all isometric representations of M. It is called the full semigroup C^*-algebra and denoted by $C^*(M)$.

Ever since L. A. Coburn proved his well-known theorem, which asserts that the C^*-algebra generated by a non-unitary isometry on a separable infinite dimensional Hilbert space does not depend on the particular choice of the isometry [1], many authors have taken an interest in the generalization of Coburn's theorem. The uniqueness property of C^*-algebras generated by isometries describes when $C^*_{red}(M)$ and $C^*(M)$ are isomorphic or when $C^*_{red}(M)$ has a universal property for certain kinds of isometric representations of S [1,2,3,4,8, etc].

Concerning the uniqueness property, A. Nica introduced the guasi-lattice ordered group (G, M) and the amenabilty problem of quasi lattice ordered groups [8]. The quasi lattice ordered group (G, M) is amenable if the left regular isometric representation is faithful on the universal C^*-algebra whose representations are given by the covariant representations of M.

We show that if a subsemigroup M of the integer group $\mathbb{Z}$ generates $\mathbb{Z}$ and $M \cap (-M) = \{0\}$, then the reduced semigroup C^*-algebra $C^*_{red}(M)$ is isomorphic to the Toeplitz algebra. We give a very simple example of non quasi-lattice ordered semigroup $P = \{0, 2, 3, \dots\}$ And we show that the reduced semigroup C^*-algebra $C^*_{red}(P)$ is isomorphic to the Toeplitz algebra and also show that $C^*_{red}(P)$ is not isomorphic to $C^*(P)$.

2. Reduced Semigroup C^*-algebras

Let $\mathcal{B}$ be a unital C^*-algebra and M be a semigroup with unit e. A map $W : M \to \mathcal{B}, x \mapsto W_x$ is an *isometric homomorphism* from M to $\mathcal{B}$ if

(1) $W_e = I$
(2) W_x is an isometry ($W_x{}^* W_x = I$)
(3) $W_{xy} = W_x W_y$ for all $x, y \in M$.

If $\mathcal{B}$ is the algebra $\mathcal{B}(H)$ of all bounded linear operators on a non-zero Hilbert space H, we call (H, W) an *isometric representation* of M.

When M is a countable discrete left-cancellative semigroup, then the left regular isometric representation W of M on $l^2(M, H)$ is a map $\mathcal{L} : M \to l^2(M, H), x \mapsto W_x$ defined by the equation

$$(\mathcal{L}_x f)(z) = \begin{cases} f(y), & \text{if } z = xy \text{ for some } y \in M, \\ 0, & \text{if } z \notin xM. \end{cases}$$

For a given C^*-dynamical system $(\mathcal{A}, M, \alpha)$, G. J. Murphy constructed the full crossed product $\mathcal{A} \rtimes_\alpha M$ by the semigroup M under the action α with the universal property of the covariant homomorphisms of $(\mathcal{A}, M, \alpha)$. There exists the cannonical isometric representation $V : M \to \mathcal{A} \rtimes_\alpha M$.

When a C^*-dynamical system $(\mathcal{A}, M, \alpha)$ is trivial, $\mathbb{C} \rtimes_\alpha M$ is called the full semigroup C^*-algebra and denoted by $C^*(M)$. By the universal property of $\mathcal{A} \rtimes_\alpha M$, if $W : M \to B$ is a isometric homomorphism into a unital C^*-algebra B, there exists a unique *-homomorphism $\pi : C^*(M) \to B$ such that $\pi(V_x) = W_x$ for all $x \in M$ where $V : M \to C^*(M)$ is the canonical isometric representation.

Let (π_u, H_u) be the universal representation of $\mathcal{A}$ and $\mathcal{L}$ be the left regular isometric representation of M on $l^2(M, H_u)$. Then π_u induces a covariant representation $(\bar{\pi}_u, W)$ of $(\mathcal{A}, M, \alpha)$ on $l^2(M, H_u)$ defined by

$$\bar{\pi}_u((a)f)(x) = \pi_u(\alpha_x^{-1}(a))(f(x))$$

for $a \in \mathcal{A}$, $f \in l^2(M, H_u)$, and $x \in M$. Since $\mathcal{A} \rtimes_\alpha M$ has the universal property of the covariant homomorphisms, there exists a unique *-homomorphism $\bar{\pi}_u \times \mathcal{L} : \mathcal{A} \rtimes_\alpha M \to \mathcal{B}(l^2(M, H_u))$ such that $\bar{\pi}_u \times \mathcal{L}(aV_x) = \pi_u(a) W_x$ for $a \in A$ and $x \in M$. We call $(\bar{\pi}_u \times \mathcal{L})(\mathcal{A} \rtimes_\alpha M)$ *the reduced crossed product* of $\mathcal{A}$ by the semigroup M under the action α and denote it by $\mathcal{A} \rtimes_{\alpha r} M$. In fact, $\mathcal{A} \rtimes_{\alpha r} M$ is generated by $\{\pi_u(a)\mathcal{L}_x | a \in \mathcal{A}, x \in M\}$. In the case of the trivial C^*-dynamical system, $\mathbb{C} \rtimes_{\alpha r} M$ is generated by the left regular isometric representation $\mathcal{L}$ of M on $l^2(M)$, that is, the closed linear span of $\{\mathcal{L}_{x_1} \mathcal{L}^*_{x_2} \cdots \mathcal{L}^*_{x_{2n}} \mathcal{L}_{x_{2n+1}} \mid x_i \in M\}$. The C^*-algebra generated by the left regular isometric representation has several names. We shall call it the reduced semigroup C^*-algebras and denote it $C^*_{red}(M)$ instead of $\mathbb{C} \rtimes_{\alpha r} M$. If $M = \mathbb{N}$, the semigroup of natural numbers, $C^*_{red}(M)$ is the Toeplitz algebra. So the reduced semigroup C^*-algebras can be considered C^*-algebras like the Toeplitz algebras.

3. C^*-algebras like the Toeplitz algebra

We can give an order on the semigroup M as follows: if an element x in M is contained in yM for some element $y \in M$, then x and y are comparable and we denote this by $y \leq x$. This relation makes M a pre-order semigroup. If the unit of M is the only invertible element of M, the above relation on M becomes a partial order on M.

The partially ordered group (G, M) is quasi-lattice ordered group if every finite subset of G with an upper bound in M has a least upper bound in M. Equivalently, (G, M) is quasi-lattice ordered if and only if every element of G having an upper bound in M has a least upper bound, and every two elements in M havnig a common upper bound have a least common upper bound. If x and y have a common upper bound in M, their least common upper bound will be denoted by $x \vee y$.

The Toeplitz algebra is the C^*-algebra generated by the left regular representation of $\mathbb{N}$, which is the semigroup generated by only one element. The semigroup $\mathbb{N}$ is unperforated, well-ordered and totally ordered. Furthermore $(\mathbb{Z}, \mathbb{N})$ is clearly a quasi-lattice group. The following theorem shows that semigroups whose generators are more than one can induce the Toeplitz algebra.

Theorem 1. *Let M be a subsemigroup of the integer group $\mathbb{Z}$ which generates $\mathbb{Z}$ and contains $\{n + k \mid \ k \in \mathbb{N}\}$ for some $n \in M$, and $M \cap (-M) = \{0\}$. Then $C^*_{red}(M)$ is isomorphic to the Toeplitz algebra.*

In [8], A. Nica define a covariant representation of a quasi-lattice ordered group as follows; the isometric representation $W : M \to \mathcal{B}(H)$ is covariant if it satisfies

$$W_x W_x^* W_y W_y^* = \begin{cases} W_{x\vee y} W_{x\vee y}^* & \text{if x, y have a common upper bound} \\ 0 & \text{otherwise.} \end{cases}$$

The important example of an covariant isometric representation of the quasi-lattice ordered group M is the left regular isometric representation on $l^2(M)$.

There exists the universal C^*-algebra for covariant isometric representations of M. Here we will denote it $C^*_{cor}(M)$ in order to discriminate from the full semigroup C^*-algebra. $C^*_{cor}(M)$ is the C^*-algebra generated by a canonical covariant isometric representation $i : M \to C^*_{cor}(M)$ with the following property: if V is a covariant isometric representation of M, then there is a homomorphism $\Phi : C^*_{cov}(M) \to C^*(\{V_x : x \in V\})$ such that $\Phi(i(x)) = V_x$. A. Nica introduce the amenability of quasi-lattice ordered groups which is equivalent to that the left regular isometric represenation of $C^*_{cov}(M)$ is faithful. Every abelian quasi-lattice ordered group is amenable.

We have a semigroup $P = \{0, 2, 3, \dots\}$ as an interesting and informative example of Theorem 1. The semigroup P is perforated and not quasi-lattice ordered group. Because 2 and 3 have common upper bound 5 and 6 but not have a least common upper bound. Since P has two generators, so apparently $C^*_{red}(P)$ is generated by two non-unitary isometries. However, the following proposition shows that $C^*_{red}(P)$ is generated by a non-unitary isometry.

Corollary 2. *A single non-unitary isometry generates* $C^*_{red}(P)$.

It is known that $C^*_{red}(\mathbb{N})$ and $C^*(\mathbb{N})$ are isomorphic to the Toeplitz algebra and we have proved that the generating subsemigroups of $\mathbb{Z}$ induce the reduced semigroup C^*-algebras isomorphic to the Toeplitz algebra. But we can also show that the reduced semigroup C^*-algebras are not isomorphic to the semigroup C^*-algebras even when the semigroups of $\mathbb{Z}$ generate $\mathbb{Z}$.

R. G. Douglas got a remarkable result of the C^*-algebras generated by one-parameter semigroup of isometries [4]. This is an interesting and comparative example of Douglas's result.

Proposition 3. *The reduced semigroup C^*-algebra $C^*_{red}(P)$ is not isomorphic to the semigroup C^*-algebra $C^*(P)$.*

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